Symplectic Construction of Moduli Spaces

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X is a compact Kähler manifold of (complex) dimension k with form ω.

G a (complex) reductive group with algebra $\mathfrak{g}$.
$K \subseteq G$ a real (compact) form with algebra $\mathfrak{k}$.

B a non-degenerate $\mathrm{Ad}(G)$-invariant (resp. $\mathrm{Ad}(K)$-invariant) bi-linear form on $\mathfrak{g}$ (resp. $\mathfrak{k}$).

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. (Example: $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus \mathfrak{p}$.)

Example: If $\mathfrak{g}$ is semi-simple, then B is the Killing form.

The Betti moduli $\mathcal{M}_B$:

Assume $\pi = \pi_1(X)$ is finitely generated with n generators and finitely many relations.

$\mathrm{Hom}(\pi_1, G)$ is an affine algebraic variety over $\mathbb{C}$, in fact a subvariety of G^n.

There is an algebraic adjoint G-action on $\mathrm{Hom}(\pi_1, G)$,

$$G \times \mathrm{Hom}(\pi_1, G) \longrightarrow \mathrm{Hom}(\pi_1, G)$$

$$(g, \rho) \mapsto \mathrm{Ad}(g)(\rho)$$

Categorical quotient:

$\mathrm{Hom}(\pi_1, G)$ is an affine variety, so is defined by a ring R.

Hence the $\mathrm{Ad}(G)$-action on $\mathrm{Hom}(\pi_1, G)$ induces a G-action on R. Denote by R^G the invariant subring.

The Betti moduli

$$\mathcal{M}_B(G) = \mathrm{Hom}(\pi_1, G)/G$$

is the variety defined by R^G.

Geometric explanation:

The orbits $\mathrm{Ad}(G)$-action may not be closed in $\mathrm{Hom}(\pi_1, G)$.

This implies that the geometric quotient of the $\mathrm{Ad}(G)$-action may not be Hausdorff.

Must identify each orbit with the orbits in its closure.

$\mathcal{M}_B(G)$ parameterizes the reductive representations.

Remark: $\mathcal{M}_B(G)$ has a variety structure coming from the variety structure on G.

Example: X a compact Riemann surface of genus g.

$G = \mathbb{C}^\times$.

Set $[A, B] = ABA^{-1}B^{-1}$

$\pi_1 = \langle A_i, B_i, 1 \leq i \leq g \mid \prod_{i=1}^{g} [A_i, B_i] \rangle$

$\mathrm{Hom}(\pi_1, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{2g}$.

$\mathcal{M}_B(\mathbb{C}^\times) \cong (\mathbb{C}^\times)^{2g}$.

Example: $G = \mathrm{SL}(2, \mathbb{C})$.

$\mathrm{Hom}(\pi_1, G) \cong$
$\{a_i, b_i : 1 \leq i \leq g, \prod_{i=1}^{g}[a_i, b_i] = e \in G\}$.

Consider the representation ρ corresponding to

$$a_i, b_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad 1 \leq i \leq g.$$

The G-orbit of ρ is not closed because it does not contain the trivial representation.

Symplectic structures

$\rho \in \mathrm{Hom}(\pi, G)$ together with the $\mathrm{Ad}(G)$-action on $\mathfrak{g}$ defines a π-module structure on $\mathfrak{g}$.

The tangent space at $[\rho] \in \mathcal{M}_B(G)$ is $\mathrm{H}^1(\pi, \mathrm{Ad}_\rho \mathfrak{g})$.

Suppose X is a compact Riemann surface.

Then there is a symplectic structure $\Omega_{\mathbb{J}}$ on $\mathcal{M}_B(G)$:
$\mathrm{H}^1(\pi, \mathrm{Ad}_\rho \mathfrak{g}) \times \mathrm{H}^1(\pi, \mathrm{Ad}_\rho \mathfrak{g}) \longrightarrow \mathrm{H}^2(\pi, \mathbb{C}) \cong \mathbb{C}$.

$\mathcal{M}_B(K)$ is symplectic while $\mathcal{M}_B(G)$ is complex symplectic, i.e. $\mathcal{M}_B(G)$ has two symplectic structures.

The de Rham construction
We begin with bilinear forms
$\Omega(\eta_1, \eta_2) = \int_X B(\eta_1, \eta_2)\omega^i$.

Notice that multiplication on the form is the wedge.

For each $\eta \in \Omega^1(\mathfrak{g})$, the curvature is
$F(\eta) = d\eta + \frac{1}{2}[\eta, \eta]$.

$\Omega^2(\mathfrak{g})$ is canonically duel to $\Omega^0(\mathfrak{g})$:
$\Omega^0(\mathfrak{g}) \times \Omega^2(\mathfrak{g}) \longrightarrow \mathbb{C}, \quad (\eta, w) = \int_X B(\eta, w)\omega^{k-1}$.

Complex symplectic quotient

There is a (gauge) action
$\Omega^0(G)\times\Omega^1(\mathfrak{g})\longrightarrow\Omega^1(\mathfrak{g}),\ \ (g,\eta)\mapsto(dg)g^{-1}+\mathrm{Ad}(g)(\eta).$

Define the (moment) map by curvature
$\mu:\Omega^1(\mathfrak{g})\longrightarrow\Omega^2(\mathfrak{g})\cong\Omega^0(\mathfrak{g}),\quad \mu(\eta)=F(\eta).$

The de Rham moduli $\mathcal{M}_{dR}\cong\mu^{-1}(0)//\Omega^0(G)$.

$\mathcal{M}_{dR}$ acquires a natural symplectic form $\Omega_{\mathbb{J}}$ from the reduction.

Suppose that $\rho \in \mathrm{Hom}(\pi, G)$ is a representation.

Let $\tilde{d} : \Omega^0(\tilde{X}, \mathfrak{g}) \longrightarrow \Omega^1(\tilde{X}, \mathfrak{g})$,
be the exterior derivative operator.

Form the ρ-twisted bundle $\tilde{X} \times_{\mathrm{Ad}(\rho)} \mathfrak{g}$.

$\tilde{d}$ descents to a flat connection $d + \eta$ on X, where $\eta \in \Omega^1(\mathfrak{g})$.

This defines $\mathcal{M}_B \xrightarrow{\simeq} \mathcal{M}_{dR}$.

Example: $G = \mathbb{C}$. (a not so great example)

This is simply the de Rham cohomology:

Gauge group is $\Omega^0(\mathbb{C})$ and $\mu(A) = dA$

$$0 \longrightarrow \Omega^0(\mathbb{C}) \xrightarrow{d} \Omega^1(\mathbb{C}) \xrightarrow{d} \Omega^2(\mathbb{C}) \cdots.$$

$\mathcal{M}_{dR} \cong \mathrm{H}^1_{dR}(\Omega^\bullet(\mathbb{C}))$.

Suppose X is a Riemann surface of genus g.

Then $\mathcal{M}_{dR} \cong (\mathbb{C})^{2g}$.

Example: $G = \mathbb{C}^\times$ and $G = \mathrm{SL}(2, \mathbb{C})$.

Gauge group is $\Omega^0(\mathbb{C}^\times)$ and $\mu(A) = dA$.

$$\Omega^0(\mathbb{C}^\times) \xrightarrow{d\log} \Omega^1(\mathbb{C}) \xrightarrow{d} \Omega^2(\mathbb{C}) \longrightarrow \cdots$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{exp} \mathbb{C}^\times \longrightarrow 0$$

$$\cdots \to \mathrm{H}^1(\mathbb{Z}) \to \mathrm{H}^1(\mathbb{C}) \to \mathrm{H}^1(\mathbb{C}^\times) \to \mathrm{H}^2(\mathbb{Z}) \to \cdots$$

$$\mathcal{M}_{dR} \cong \mathrm{H}^1(\mathbb{C}^\times)^0 \cong \mathrm{H}^1(\mathbb{C}) / \operatorname{im}(\mathrm{H}^1(\mathbb{Z})).$$

Suppose X is a Riemann surface of genus g.

Then $\mathcal{M}_{dR} \cong (\mathbb{C}^\times)^{2g}$.

The Dolbeault construction

Let h be the Hermitian metric $h(X,Y) = B(X,\bar{Y})$.

Let $K \subset G$ be the compact form preserving h.

The Kähler metric on X together with h defines a Hermitian metric h_j on $\Omega^j(\mathfrak{g})$:

$h_j(\eta_1,\eta_2) = \int_X h(\eta_1,\eta_2)\omega^k$.

For $A \in \Omega^1(\mathfrak{k})$, let
$D_A : \Omega^0(\mathfrak{g}) \longrightarrow \Omega^1(\mathfrak{g}), \quad D_A(f) = df + [A,f]$.

Let $D_A^* : \Omega^1(\mathfrak{g}) \longrightarrow \Omega^0(\mathfrak{g}), \; h_0(D_A^*\eta, f) = h_1(\eta, D_A f)$.

HyperKähler structure on $\Omega^1(\mathfrak{g}) \cong \mathfrak{g} \otimes \Omega^1(X)$

$\mathbb{I} = i \otimes 1, \quad \mathbb{J} = \tau \otimes j, \quad \mathbb{K} = \mathbb{I}\mathbb{J}$, where
j is the complex structure on X,
$\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the adjoint with respect to B.

The Riemannian metric from h is compatible with $\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$. Together, they form a HyperKähler structure on $\Omega^1(\mathfrak{g})$.

The following action is symplectic with respect to $\Omega_{\mathbb{I}}$, $\Omega_{\mathbb{J}}$, $\Omega_{\mathbb{K}}$:
$\Omega^0(K) \times \Omega^1(\mathfrak{g}) \longrightarrow \Omega^1(\mathfrak{g}), \; (g, \eta) \mapsto (dg)g^{-1} + \mathrm{Ad}(g)(\eta)$.

HyperKähler reduction

$\eta \in \Omega^1(\mathfrak{g})$, $\eta = A + \Psi$,
where $A \in \Omega^1(\mathfrak{k})$, $\Psi \in \Omega^1(\mathfrak{p})$

The action is also Hamiltonian with respect to the three symplectic structures, resulting in the HyperKähler quotient $\mathcal{M}$ with moment maps

$$\begin{cases} \mu_{\mathbb{I}}(\eta) &= D_A^* \Psi \\ \mu_{\mathbb{J}}(\eta) &= \mathrm{Im}(F(\eta)) \\ \mu_{\mathbb{K}}(\eta) &= \mathrm{Re}(F(\eta)) \end{cases}$$

The smooth part of $\mathcal{M}_{Dol} \cong \Omega^1(\mathfrak{g})///\Omega^0(K)$ has a HyperKähler structure.

The twister space.

Let $a \in S^2 \subset \mathbb{R}^3$. Then

$a_1\mathbb{I} + a_2\mathbb{J} + a_3\mathbb{K}$ is a complex structure on $\mathcal{M}$ compatible with g.

This gives a family of Kähler structures on $\mathcal{M}$, parameterized by S^2.

All these structures are isomorphic to each other, except $\mathbb{J}$.

The Abelian case (Hodge Theory):

X a compact Riemann surface, $G = \mathbb{C}$
(Bad example)

Fix the trivial Hermitian metric h on $X \times \mathbb{C}$.

$0 \longrightarrow \Omega^0(\mathbb{C}) \xrightarrow{d} \Omega^1(\mathbb{C}) \xrightarrow{d} \Omega^2(\mathbb{C}) \cdots$.

$\eta \in \Omega^1(\mathbb{C}),\ \eta = 0 + \Psi,\ \Psi \in \Omega^1(\mathbb{C})$

$$dA = d\Psi = 0, \quad d^*\Psi = 0.$$

$\Psi \in \mathcal{H}^1(\mathbb{C})$ is harmonic.

X a compact Riemann surface, $G = \mathbb{C}^{\times}$:

$\eta \in \Omega^1(\mathbb{C}),\ \eta = A + \Psi,\ A \in \Omega^1(i\mathbb{R}),\ \Psi \in \Omega^1(\mathbb{R})$

$$dA = d\Psi = 0, \quad d^*\Psi = 0.$$

$dA = 0 \ \Rightarrow A$ is closed.

$d\Psi = 0, \quad d^*\Psi = 0 \Rightarrow \Psi$ is harmonic.

$0 \longrightarrow \Omega^0(\mathrm{U}(1)) \xrightarrow{dlog} Z^1(i\mathbb{R}) \times \mathcal{H}^1(\mathbb{R}) \xrightarrow{d} \Omega^2(\mathbb{C}) \cdots.$

The complex structure $\mathbb{J}$ gives the complex structure on $\mathcal{M}_{Dol} = T^*Jac(X)$.

Real forms:

If $K \subset G$ is a real form, then $\mathbb{I}$ and $\Omega_{\mathbb{I}}$ disappear, but $\mathbb{J}$ and $\Omega_{\mathbb{J}}$ still exist. Hence $\mathcal{M}_{Dol}(K)$ is a Kähler quotient with respect to $\mathbb{J}$.

Example: $K = \mathrm{U}(1)$, $\mathcal{M}_{Dol}(K) \cong Jac(X)$.

Example: $K = \mathbb{R}^{\times}$, $\mathcal{M}_{Dol}(K) = \mathbb{J}_2(X) \times \mathrm{H}^0(\mathbf{\Omega}^1)$, where $\mathbf{\Omega}^1$ is the sheaf of holomorphic 1-forms on X.

The complex structure $\mathbb{J}$ depends on the complex structure of X.

The symplectic structure $\Omega_{\mathbb{J}}$ depends on the symplectic structure ω on X only.

In general, there are more symplectomorphisms than there are complex automorphisms.

Example: If X is a compact Riemann surface, then the space of complex automorphism is finite when $g > 1$. By the $\mathcal{M}_B$ construction, $\Omega_{\mathbb{J}}$ is topological.

Dynamics of the mapping class group: X a Riemann surface of genus $g > 0$.

$$MCG = Homeo(X)/Homeo(X)_0.$$

There are actions

$$Homeo(X) \times \pi \longrightarrow \pi.$$

$$MCG \times \pi \longrightarrow \pi.$$

$$MCG \times \mathrm{Hom}(\pi, G) \longrightarrow \mathrm{Hom}(\pi, G)$$

$$MCG \times \mathcal{M}_B \longrightarrow \mathcal{M}_B.$$

Hamiltonian actions on $\mathcal{M}_B$

Let X be of genus $g > 1$ and $\gamma \subset X$ a based simple closed curve.

$\mathrm{Tr}_\gamma : \mathcal{M}_B \longrightarrow \mathbb{C}, \quad \mathrm{Tr}_\gamma([\rho]) = \mathrm{Tr}(\rho(\gamma)).$

There are $3g - 3$ mutually non-intersecting simple closed curves.

There is a $(\mathbb{C}^\times)^{3g-3}$-action on $\mathcal{M}_B(G)$.

Suppose $K \subset G$ is a compact form.

Then $\mathcal{M}_B(K)$ is compact.

There is a $\mathrm{U}(1)^{3g-3}$-action on $\mathcal{M}_B(K)$.

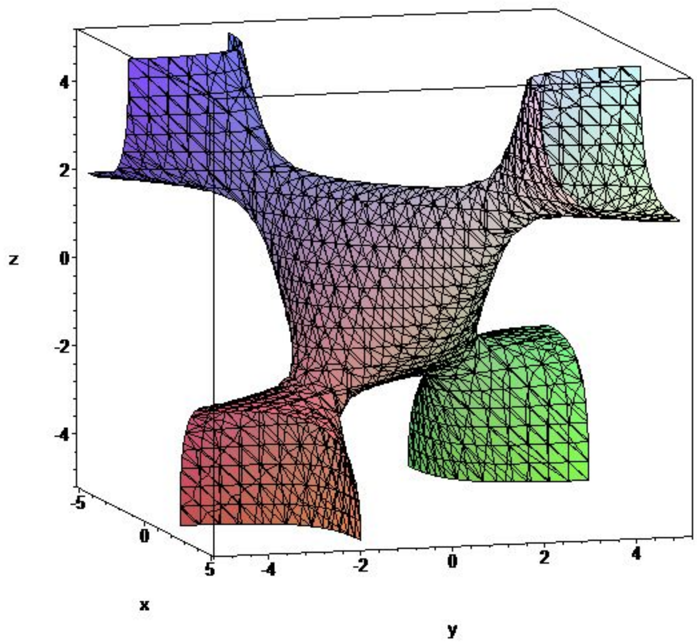
If $K = \mathrm{SU}(2)$, then $\mathcal{M}_B(K)$ is integrable.

X is a (punctured) torus; $K = \mathrm{SU}(2)$.

$\pi = \langle A, B | [A, B] \rangle$

$\mathcal{M}_B$ is defined by $x^2 + y^2 + z^2 - xyz - 2$.

$k = x^2 + y^2 + z^2 - xyz - 2$ is the boundary trace function.



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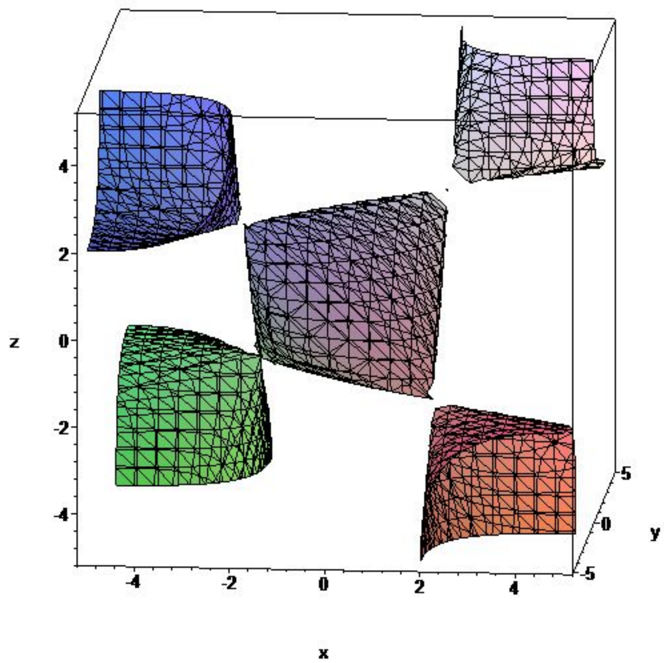

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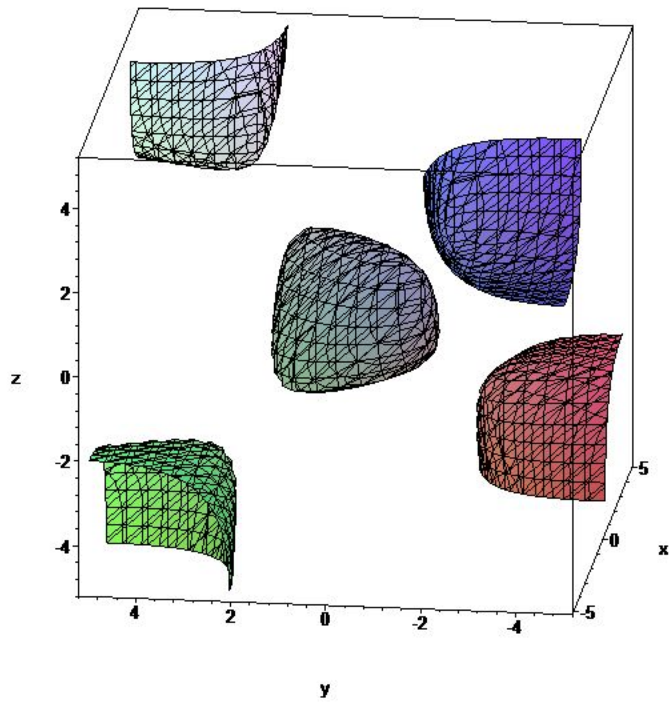

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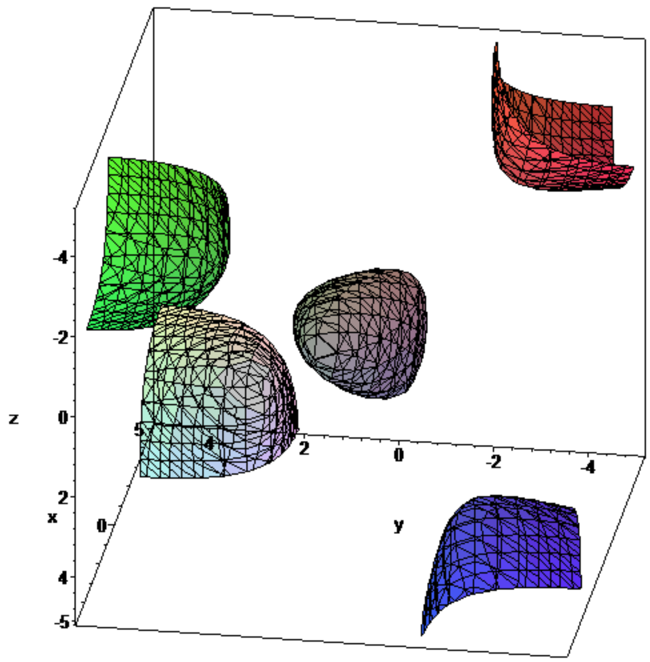
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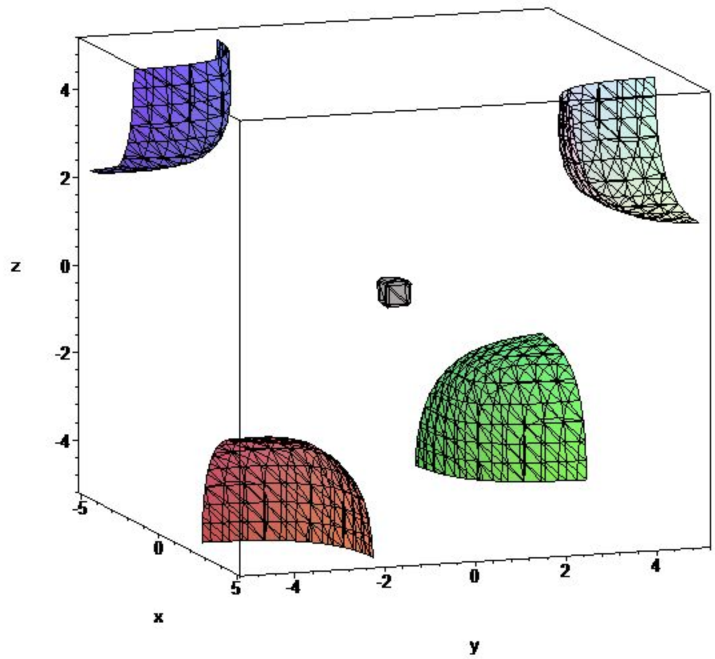

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