Urs.

Recall: $(M, \omega = d\lambda)$ exact symp mfld.

$H \in C^\infty(M, \mathbb{R})$.

$\Sigma = H^{-1}(0)$ closed. $\lambda|_\Sigma$ Cont. form. normalize H s.t. $X_H|_\Sigma = R$ Reeb v.F.

Lemma. (period action inequality for $\mathcal{A}^H$)

$\exists$ constants $c > 0$, $\varepsilon > 0$, s.t. $(v, \eta) \in \mathcal{L} \times \mathbb{R}$.

$\|\nabla \mathcal{A}^H(v, \eta)\| < \varepsilon \Rightarrow |\eta| \le c(|\mathcal{A}^H(v,\eta)| + 1)$.

Pf of Lemma.

$\exists \delta > 0$ s.t. $|\lambda(X_H)|_{H^{-1}((-\delta, \delta))}| \geq \frac{1}{2}$.

$\exists C_1 > 0$ s.t.

$\| \lambda|_{H^{-1}([-\delta, \delta])} \| \le C_1$.

$\delta_0 := \min\{\delta, \frac{1}{4}\} > 0$.

Step 1. $(v, \eta) \in \mathcal{L} \times \mathbb{R}$ and

$v(t) \in H^{-1}([-\delta_0, \delta_0]) \quad \forall t \in S^1$

then $\exists$ $C_2 > 0$ s.t.

$|\eta| \le C_2(|\mathcal{A}^H(v,\eta)| + \|\nabla \mathcal{A}^H(v,\eta)\|)$

Proof of Step 1.

$$|\mathcal{A}^H(v,\eta)| = \left| \int v^*\lambda + \eta \int_0^1 H(v)\,dt \right|$$

$$= \left| \int_0^1 \lambda(v(t))(\partial_t v - \eta X_H(v)) + \eta \int_0^1 H(v)\,dt \right.$$

$$+ \left. \eta \int_0^1 \lambda(v(t)) X_H(v)\,dt \right|$$

$$\geq \Big| \underbrace{\eta \int_0^1 \lambda(v) X_H(v)\, dt}_{\geq |\eta| \frac{1}{2}} \Big| - \Big| \underbrace{\int_0^1 \lambda(v)\, \partial_t v - \eta X_H(v)}_{\leq c_1 \| \partial_t v - \eta X_H(v) \|_{L^2}} \Big|$$

$$- \Big| \underbrace{\eta \int_0^1 H(v)\, dt}_{\leq |\eta| \cdot \delta_0 \leq \frac{|\eta|}{4}} \Big|$$

$$\geq \frac{|\eta|}{4} - c_1 \underbrace{\| \partial_t v - \eta X_H(v) \|_{L^2}}_{\leq \| \nabla \mathcal{A}^H(v,\eta) \|}$$

$$\Rightarrow |\eta| \leq 4 \left(|\mathcal{A}^H(v,\eta)| + c_1 \| \nabla \mathcal{A}^H(v,\eta) \| \right).$$

set $c_2 = \max(4, 4 \cdot c_1)$.

Step 2. $\exists\, \varepsilon_0 > 0$ with the following property.

Assume $(v,\eta) \in \mathcal{L} \times \mathbb{R}$,

$\exists\, t_0 \in S^1$ s.t.

$v(t_0) \notin H^{-1}([-\delta_0, \delta_0])$, then

$\| \nabla \mathcal{A}^H(v,\eta) \| \geq \varepsilon_0$.

Case 1: $\exists\, t_1 \in S^1$ s.t. $v(t_1) \in H^{-1}\left(\left[-\frac{\delta_0}{2}, \frac{\delta_0}{2}\right]\right)$

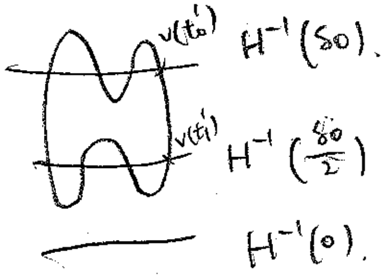
$$K := \max_{\substack{x \in H^{-1}([-\delta_0, \delta_0]) \\ t \in S^1}} \| \nabla_{J_t} H(x) \|_t$$

gradient w.r.t. metric $\omega(\cdot, J_t \cdot)$

normal w.r.t. metric $\omega(\cdot, J_t \cdot)$

$\Rightarrow \exists\ t_0', t_1' \in S^1$ such that

$$|H(v(t_0'))| = \delta_0$$
$$|H(v(t_1'))| = \delta_0/2$$
$$\frac{\delta_0}{2} \le |H(v(t))| \le \delta_0, \quad \forall t \in [t_0', t_1'],$$



$$\begin{aligned}
\frac{\delta_0}{2} &= |H(v(t_0'))| - |H(v(t_1'))| \\
&\le |H(v(t_0')) - H(v(t_1'))| \\
&= \left| \int_{t_0'}^{t_1'} \frac{d}{dt} H(v(t))\, dt \right| \\
&= \left| \int_{t_0'}^{t_1'} dH(v(t))\, \partial_t v(t)\, dt \right| \\
&= \left| \int_{t_0'}^{t_1'} dH(v(t)) \big(\partial_t v(t) - \eta X_H(v(t))\big) dt \right| \\
&= \left| \int_{t_0'}^{t_1'} \langle \nabla_t H(v(t)), \partial_t v(t) - \eta X_H(v(t)) \rangle_t\, dt \right| \\
&\le \int_{t_0'}^{t_1'} \|\nabla_t H(v(t))\|_t \|\partial_t v(t) - \eta X_H(v)\|_t\, dt \\
&\le K \int_{t_0'}^{t_1'} \|\partial_t v - \eta X_H(v)\|_t\, dt \\
&\le K \|\partial_t v - \eta X_H(v)\|_{L^1} \\
&\le K \|\partial_t v - \eta X_H(v)\|_{L^2} \\
&\le K \|\nabla \mathcal{A}^H(v, \eta)\|
\end{aligned}$$

$$\Rightarrow \|\nabla \mathcal{A}^H(v,\eta)\| \ge \frac{\delta_0}{2K}.$$

Case 2: $v(t) \notin H^{-1}([-\frac{\delta_0}{2}, \frac{\delta_0}{2}])$ $\forall t \in S^1$.

$\Rightarrow \|\nabla \mathcal{A}^H(v,\eta)\| \geq \left| \int_0^1 H(v)\,dt \right| \geq \frac{\delta_0}{2}$

Step 2 follows with $\varepsilon_0 = \min\left\{\frac{\delta_0}{2}, \frac{\delta_0}{2K}\right\}$.

Step 3. Proof of Lemma.

Choose $\varepsilon = \min\{\varepsilon_0, 1\}$.

$(v,\eta) \in \mathcal{L} \times \mathbb{R}$

$\|\nabla \mathcal{A}^H(v,\eta)\| < \varepsilon$.

$\overset{\text{Step 2}}{\Rightarrow}$ $v(t) \in H^{-1}([-\delta_0, \delta_0])$ $\forall t \in S^1$

$\overset{\Rightarrow}{\text{Step 1.}}$ $|\eta| \leq c_2 \left(|\mathcal{A}^H(v,\eta)| + \|\nabla \mathcal{A}^H(v,\eta)\| \right)$

$\leq c_2 \left(|\mathcal{A}^H(v,\eta)| + 1 \right)$

Set $c = c_2$.

Computations of RFH.

Thm 1: Assume Σ is displaceable.

then $RFH(\Sigma, M) = \{0\}$.

(L, g) Riem mfd.

unit cotang. bdle $SL = \{v \in TL : \|v\|_g = 1\}$

$\subset TL \underset{\substack{\text{identify} \\ \text{with } g}}{\cong} T^*L$

$\mathcal{L}_L \subset C^\infty(S^1, L)$ component of contractible loops in L.

Thm 2.

$$RFH_*(SL, TL) = \begin{cases} H_*(\mathcal{L}_L) & * > 1 \\ H^{-*+1}(\mathcal{L}_L) & * < 0 \\ H_0(\mathcal{L}_L) \oplus H^1(\mathcal{L}_L) & * = 0,\ e(TL) = 0 \text{ (Euler class)} \\ H^1(\mathcal{L}_L) & * = 0,\ e(TL) \neq 0 \\ H_1(\mathcal{L}_L) \oplus H^0(\mathcal{L}_L) & * = 1,\ e(TL) = 0 \\ H_1(\mathcal{L}_L) & * = 1,\ e(TL) \neq 0 \end{cases}$$

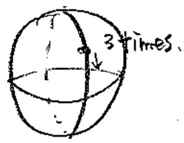
proof of thm 2 for some special cases.

$L = S^n$, $n \geq 4$.

On unit tang bdl

Reeb flow $\longleftrightarrow$ geod. flow

geodesics on S^n = great circles



Critical mfd of $\mathcal{A}^H$

$\mathrm{Crit}\,\mathcal{A}^H = \bigsqcup_{m \in \mathbb{Z}} SS^n_m \cong SS^n$ unit tang. bundle

m = number of times geod. goes around

Grading given by Conley-Zehnder index.

Fact: c geodesic going m times around.

$\mu_{CZ}(c) = (2n-2) \cdot m$.

$\mathcal{A}^H$ is not Morse but Morse-Bott
$\rightsquigarrow$ have to choose an additional Morse fn f on $\mathrm{Crit}\,\mathcal{A}^H$.

Choose Morse fn f on SS^n with 4 crit pts

- $2n-1$
- n
- $n-1$
- 0

$c \in \mathrm{Crit} f \subset \mathrm{Crit}\,\mathcal{A}^H$.

grading $: \mu(c) = \mu_{CZ}(c) + \mu_{Morse}(c)$,

$m=-2$	$m=-1$	$m=0$	$m=1$	$m=2$
			$4n-3$	$4n-4$
		$2n-1$		
			$3n-2$	
			$3n-3$	
		$2n-1$	$2n-2$	
		n		
		$n-1$		
	1	0		
	$2-n$			
	$1-n$			
$3-2n$	$2-2n$			

(Action = period in restricted contact case).

excluded since cascades have to flow downhill

On same component Gysin seq. $\exists$ 2 grad flow lines (i.e. 0 in $\mathbb{Z}_2$).

excluded if $n \geq 4$. (index difference is too large)

$\therefore$ In $\mathbb{Z}_2$, no grad. flow lines.

$\Rightarrow$ $RFH(\Sigma, M) = CFH$ chain gps.
(This equals the loop space homology/cohom.)

Cor 1. (obstructions to exact contact embeddings)
Assume $(M, d\lambda = \omega)$ is subcritical Stein mfd, then $\nexists$ $\iota : SS^n \to M$
s.t. $\iota^* \lambda$ gives contact structure on SS^n.

proof: Fact: In subcritical ($<n$) Stein mfd each cpt set is displaceable.
$\Rightarrow RFH(\iota(SS^n), M) = \{0\}$.
Computation before $\Rightarrow RFH(\iota(SS^n), M) \neq \{0\}$
$\times$

Cor 2. Assume N simply conn. closed mfd.
For generic
$F \in C_c^\infty(T^*N \times [0,1], \mathbb{R})$
$\exists$ ∞ many leafwise intersection pts on SN

proof: Chas-Sullivan $\underset{N \text{ simply conn.}}{\Rightarrow}$ $\dim H_*(\mathcal{L}N) = \infty$

$\Rightarrow \dim RFH(SN, TN) = \infty$.