Urs.

Rabinowitz action fct.

(M,ω) symp mfld

$\omega|_{\pi_2(M)} = 0.$

$\mathcal{L} \subset C^\infty(S^1, M).$

component of contractible loops.

$H \in C^\infty(M, \mathbb{R})$

$$\mathcal{A}^H : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$$
$$\quad (v, \eta)$$

choose filling disk $\bar{v}$ for v.

$$\mathcal{A}^H(v,\eta) = -\int \bar{v}^*\omega - \eta \int_0^1 H(v)\,dt$$

crit. pts

$$\left.\begin{array}{l} \partial_t v = \eta X_H(v) \\ \int_0^1 H(v)\,dt = 0. \end{array}\right\} (*)$$

preservation of energy.

$(*) \Leftrightarrow (**)$

$$\left.\begin{array}{l} \partial_t v = \eta X_H(v) \\ H(v(t)) = 0. \quad t \in S^1 \end{array}\right\} (**).$$

periodic orbits of period η on energy hypersurface $\Sigma = H^{-1}(0)$.

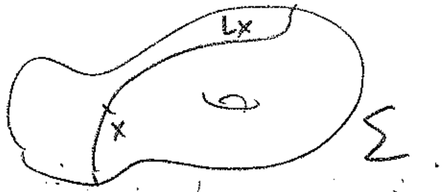
perturbations of $\mathcal{A}^H$ and leafwise intersection pts

preservation of energy

$\Rightarrow \Sigma$ is foliated.

$\underset{x}{\overset{\Sigma}{\cup}}$

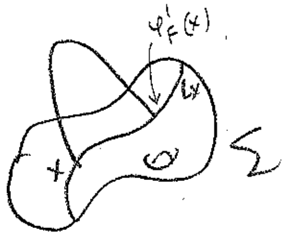
leaf thru x : $L_x = \{\varphi_H^t(x) : t \in \mathbb{R}\}$



$F \in C^\infty(M \times [0,1], \mathbb{R})$. time-dep Hamil.

Def: $x \in \Sigma$ is called a leafwise intersection pt.

for F if $\varphi_F^1(x) \in L_x$



Rmk. leafwise intersection pts of higher codim

$\Sigma \subset (M, \omega)$ coisotropic

(if $\forall x \in \Sigma$, $(T_x\Sigma)^\omega \subset T_x\Sigma$)

Fact: If Σ is coisotropic $\rightsquigarrow$ foliation s.t.

$x \in \Sigma$, L_x leaf thru x

then $T_x L_x = (T_x\Sigma)^\omega$

$\mathrm{codim}(\Sigma, M) = k \Rightarrow \dim L = k$.

Leaf intersection pt.
$x \in \Sigma$ s.t. $\varphi_F^1(x) \in L_x$.

Special cases: $k=0$: 1-periodic orbits of F.
$k=1$: our case.
$k=\frac{1}{2}\dim M$: Σ is Lagr.
$x \in \Sigma \Rightarrow L_x = \Sigma$.
leafwise intersection pt.
$\leftrightarrow$ Lagr int. pts of
Σ and $\varphi_F^1(\Sigma)$.

Existence of leafwise intersection pts
for $k=0, n$ = Arnold conjectures.
Existence of leafwise int. pts
in general = interpolation between Arnold
conj's.

Back to Rabinowitz action fct =
weakly time-dep. case:
$\chi \in C^\infty(S^1, [0,\infty))$
$\int_0^1 \chi(t)\,dt = 1$.
$H^\chi \in C^\infty(M \times S^1, \mathbb{R})$.
(weakly) time-dep Hamiltonian
$H^\chi(x,t) = \chi(t) H(x)$, $t \in S^1$, $x \in M$.

$\mathcal{A}^{H^\chi} : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$

$\mathcal{A}^{H^\chi}(v, \eta) = -\int \bar{v}\,\omega - \eta \int \chi(t) H(v(t))\, dt$.

Critical pts : Reparametrized period orbits on $\Sigma = H^{-1}(0)$.

Def : The (H^χ, F) is called a Moser pair

if $\mathrm{supp}(\chi) \subset (\frac{1}{2}, 1)$

$F_t = 0$, $t \in [\frac{1}{2}, 1]$.

If (H^χ, F) is a Moser pair.

then time supports of H^χ and F are disjoint.

Prop : Assume (H^χ, F) is a Moser pair.

$(v, \eta) \in \mathrm{Crit}\, \mathcal{A}^{H^\chi}_F$

$\mathcal{A}^{H^\chi}_F : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$

$\mathcal{A}^{H^\chi}_F(v, \eta) = \mathcal{A}^{H^\chi}(v, \eta) - \int_0^1 F(v(t), t)\, dt$

Then $v(0)$ is a leafwise intersection pt.

Pf : $(v, \eta) \in \mathrm{crit}\, \mathcal{A}^{H^\chi}_F$

crit. pt eq.

$$\left.\begin{aligned} \partial_t v(t) &= \eta \chi(t) X_H(v(t)) + X_{F_t}(v(t)) \\ \int_0^1 \chi(t) H(v(t))\, dt &= 0. \end{aligned}\right\} (*)$$

Step 1. Assume (v, η) sol. of $(*)$.

then $v(0) \in \Sigma = H^{-1}(0)$.

Pf of Step 1: $t \in [1/2, 1]$.

$$\Rightarrow \frac{d}{dt} H(v(t)) = dH(v(t))\, \partial_t v(t)$$

$$\underset{(*)_1}{=} dH(v(t))\Big(\eta \chi(t) X_H(v(t)) + \underbrace{X_{F_t}(v(t))}_{=0}\Big)$$

time support of $F_t \subset (0, 1/2)$

$$= \eta \chi(t)\, \underbrace{\omega\big(X_H(v(t)), X_H(v(t))\big)}_{=0}$$

$$= 0$$

$$\Rightarrow H(v)\big|_{[1/2,1]} = c = \text{Const}$$

Computation of c using $(*)_2$

$$0 = \int_0^1 \chi(t) H(v(t))\, dt$$

$$\underset{\substack{\text{supp } \chi(t) \\ \subset (1/2, 1)}}{=} \int_{1/2}^1 \chi(t)\, c\, dt$$

$$= c \int_{1/2}^1 \chi(t)\, dt$$

$$\underset{\substack{\text{supp } \chi(t) \\ \subset (1/2, 1)}}{=} c \underbrace{\int_0^1 \chi(t)\, dt}_{=1}$$

$$= c.$$

$\Rightarrow c = 0.$

$\Rightarrow v\big|_{[1/2,1]} \in \Sigma = H^{-1}(0).$

Since v is periodic. $v(0) = v(1) \in \Sigma$.

Step 2 : proof of Prop.

$\operatorname{supp} \mathcal{X} \subset (1/2, 1)$

$\Rightarrow \partial_t v|_{[0,1/2]} = X_{F_t}(v)|_{[0,1/2]}$.

$\Rightarrow v(1/2) = \varphi_F^{1/2}(v(0))$

$\underset{\text{time support of } F \text{ in } [0,1/2]}{=} \varphi_F^1(v(0))$ ———— (1)

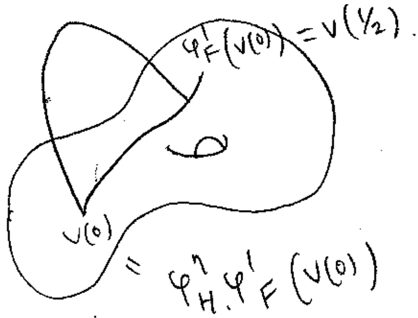
Time support of F in $[0, 1/2]$.

$\Rightarrow \partial_t v|_{[1/2,1]} = \eta\, \mathcal{X}(t)\, X_H(v)|_{[1/2,1]}$.

$\Rightarrow v(0) = v(1) = \varphi_H^{\eta}(v(1/2))$.

$\Rightarrow v(1/2) = \varphi_H^{-\eta}(v(0)) \in L_{v(0)}$.

(1) & (2) $\Rightarrow$ Prop.



Computation of Rabinowitz Floer homology before Definition.

Case 1 : Assume there are no nontrivial periodic orbits of X_H on Σ. (i.e. periodic orbits of period $\neq 0$)

Then $RFH(\Sigma, M) = H(\Sigma)$
singular homology of Σ

Reason : No nontrivial per. orbits

$\Rightarrow \operatorname{crit} \mathcal{A}^H \cong \Sigma$ (constants - periodic orbits of per $= 0$)

$\Rightarrow RFH(\Sigma, M) = H(\Sigma)$.

Def : $\Sigma \subset M$ is called Hamil. displaceable if $\exists\ F \in C_c^\infty(M \times [0,1], \mathbb{R})$
cptly supp.

s.t. $\varphi_F^1(\Sigma) \cap \Sigma = \{0\}$.

Case 2 : Σ is Hamil. displaceable, then $RFH(\Sigma, M) = \{0\}$.

Reason : no leafwise int. pts for F.

☹ This does not make sense, since
$\exists$ displaceable energy hypersurfaces
with no nontrivial periodic orbit
(Counterexamples to Hamiltonian Seifert
Conj : Herman, Ginzburg, Gürel, ...)

Problem : Moduli space of grad flow lines
are in general not cpt up to breaking

way out : Need to assume additional conditions
on Σ to make sure the necessary
compactness holds.

Gradient flow lines for $\mathcal{A}^H$.

Metric on $\mathcal{L} \times \mathbb{R}$:

J_t family of ω-cptible almost cpx str.

$(v, \eta) \in \mathcal{L} \times \mathbb{R}$

$$(\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2) \in T_{(v,\eta)}(\mathcal{L} \times \mathbb{R})$$
$$= \Gamma(v^* TM) \times \mathbb{R}$$

$$m_J\left((\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2)\right) = m\left((\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2)\right)$$
$$= \int_0^1 \omega\left(\hat{v}_1(t), J_t(v(t))\,\hat{v}_2(t)\right) dt + \hat{\eta}_1 \cdot \hat{\eta}_2$$

$$\nabla_m \mathcal{A}^H(v,\eta) = \nabla \mathcal{A}^H(v,\eta) = \begin{pmatrix} J_t(v(t))\left(\partial_t v - \eta X_H(v(t))\right) \\ -\int_0^1 H(v(t))\, dt \end{pmatrix}$$

Gradient flow eq:

$$(v,\eta) \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty(\mathbb{R},\mathbb{R})$$

$$\left.\begin{aligned} \partial_s v + J_t(v)\big(\partial_t v - \eta X_H(v)\big) &= 0 \\ \partial_s \eta &= \int_0^1 H(v(t))\,dt \end{aligned}\right\} (**)$$

Three problems:

(i) C^0 - bound on v. ← usual Floer theory

(ii) C^0 - bound on derivatives of v ← $\omega|_{\pi_2} = 0$. no bubbling

(iii) C^0 - bound on η.

curve sitaat (pseudoholom

Problem (i) & (ii):

standard problems in Floer theory:

(i) OK if M is closed, convex at infinity or geom. bounded.

(ii) Since $\omega|_{\pi_2(M)} = 0 \Rightarrow$ no bubbling $\Rightarrow$ derivatives can not explode

(iii) new issue in RFH.

Assumption which guarantees bound on η:

Σ is of restricted contact type

i.e. $\exists\, \lambda \in \Omega^1(M)$

s.t. $\omega = d\lambda$.

$\lambda|_\Sigma$ contact form on Σ.

$\Rightarrow \exists$ Reeb vector R on Σ.

Why is this assumption useful ?

Normalize H s.t. $X_H|_\Sigma = R$

Assume $(v,\eta) \in \mathrm{Crit}\,\mathcal{A}^H$, i.e. sol of

$$\left.\begin{aligned} \partial_t v &= \eta X_H(v) \\ H(v(t)) &= 0 \quad t \in S^1 \end{aligned}\right\}$$

$$\mathcal{A}^H(v,\eta) = -\int \bar{v}^*\omega - \eta \int_0^1 \underbrace{H(v)}_{=0} dt$$

$\parallel$ stokes

$$= -\int v^*\lambda$$

$$= -\int_0^1 \lambda(v(t)) \underbrace{\partial_t v}_{= \eta X_H = \eta R} dt$$

$$= -\eta \int_0^1 \underbrace{\lambda(v(t)) R(v(t))}_{=1} dt$$

$$= -\eta$$

η is asymptotically bdd in terms of action

Aim : Find an uniform bound on η along gradient flow lines which only depends on asymptotic action values.

Crucial step

Fundamental Lemma (Period-Action inequality for almost Reeb orbits)

$\exists$ constants $c>0$, $\varepsilon>0$ s.t.

$\|\nabla \mathcal{A}^H(v,\eta)\| < \varepsilon$

$\Rightarrow |\eta| = c(|\mathcal{A}^H(v,\eta)| + 1)$ proof later.

How does F. Lemma imply bound on η.

Assume $\underset{w=}{(v,\eta)} \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty(\mathbb{R}\times\mathbb{R})$.

grad. flow of $\mathcal{A}^H$.

$\lim_{s\to\pm\infty} w(s) = w_\pm \in \mathrm{Crit}\,\mathcal{A}^H$.

$\sigma \in \mathbb{R}$, ε as in Fundamental Lemma.

$\tau(\sigma) := \inf\{\tau \geq 0 : \|\nabla\mathcal{A}^H(w(\sigma+\tau))\| \leq \varepsilon\}$.

Claim: $\tau(\sigma) \leq \dfrac{\mathcal{A}^H(w_-) - \mathcal{A}^H(w_+)}{\varepsilon^2}$

Pf: (Gradient flow equation).

$$\mathcal{A}^H(w_-) - \mathcal{A}^H(w_+) = -\int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}^H(w(s))\,ds$$

$$= -\int_{-\infty}^{\infty} d\mathcal{A}^H(w(s))\,\partial_s w(s)\,ds$$

grad. flow eq. $\quad = + \int_{-\infty}^{\infty} \underbrace{m\left(\nabla\mathcal{A}^H(w(s)), \nabla\mathcal{A}^H(w(s))\right)}_{\geq 0} ds$

$$\geq \int_\sigma^{\sigma+\tau(\sigma)} \underbrace{\|\nabla\mathcal{A}^H(w(s))\|^2}_{\geq \varepsilon^2} ds.$$

$$\geq \varepsilon^2 \tau(\sigma).$$

H constant outside cpt set.

$\Rightarrow \exists\, c' > 0$ s.t.

$|H(x)| \leq c', \forall x \in M$.

$$\Rightarrow \; |\eta(\sigma)| \le \underbrace{|\eta(\sigma+\tau(\sigma))|}_{\substack{\wedge\!\! \mid \\ C\left(\left|\mathcal{A}^H(w(\sigma+\tau(\sigma)))\right|+1\right)}} + \int_\sigma^{\sigma+\tau(\sigma)} \underbrace{|\partial_s \eta(s)|}_{\substack{\wedge\!\!\mid \text{ grad flow eq.} \\ C'}}\, ds$$

Fundamental lemma

$$\underset{\substack{\text{action}\\\text{decreasing}\\\text{along gr. flow lines.}}}{\le} C\left(\max\left\{|\mathcal{A}^H(w_-)|, |\mathcal{A}^H(w_+)|\right\}+1\right)$$

$$\le C\left(\max\left\{|\mathcal{A}^H(w_-)|, |\mathcal{A}^H(w_+)|\right\}+1\right) + C'\,\tau(\sigma)$$

$$\underset{\text{Claim}}{\le} C\left(\max\left\{|\mathcal{A}^H(w_-)|, |\mathcal{A}^H(w_+)|\right\}+1\right) + \underbrace{C'\,\frac{\mathcal{A}^H(w_-)-\mathcal{A}^H(w_+)}{\varepsilon^2}}_{\text{only dep. on asymp. action values}}$$