Morse homology

M closed mfd

Def $f \in C^\infty(M, \mathbb{R})$ is called Morse

$\Leftrightarrow \forall\, x \in \mathrm{crit} f = \{x \in M : df(x) = 0\}$

$\mathrm{Ker}\ \underset{\uparrow \text{Hessian}}{H_f(x)} = \{0\}$

Def f : Morse fct

$x \in \mathrm{crit} f$

$\mu(x) :=$ number of negative EV of $H_f(x)$ counted with multiplicity

Morse index

closed ~~Mor~~ mfd

(M, f) ← Morse

$CM_*(f) := \mathrm{crit} f \otimes \mathbb{Z}_2$

↑ grading given by Morse index

$\mathbb{Z}_2$-vector space with basis critical points of f

Boundary operator:

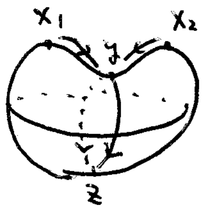
$\partial : CM_*(f) \circlearrowleft$

∂ linear

$x \in \mathrm{crit} f$

$$\partial(x) = \sum_{\substack{y \in \mathrm{crit} f \\ \mu(y) = \mu(x) - 1}} \underset{\uparrow\ \mathbb{Z}_2\text{-count}}{\#_2} \left\{ \begin{array}{c} \text{gradi flow} \\ \text{lines of } f \\ \text{from } x \text{ to } y \end{array} \right\} y$$

Example



$\updownarrow f$ hight fct

$\mu(x_1) = \mu(x_2) = 2$
$\mu(y) = 1$
$\mu(z) = 0$

$\partial x_1 = y = \partial x_2$
$\partial y = 2z = 0$
$\partial z = 0$
In particular $\partial^2 = 0$.

Morse homology
$$HM_*(f) = \frac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}$$

$HM_2(f) = \langle [x_1 + x_2] \rangle \cong \mathbb{Z}_2$
$HM_1(f) = \{0\}$
$HM_0(f) = \langle [z] \rangle \cong \mathbb{Z}_2$

$\Rightarrow HM_*(f) = H_*(S^2, \mathbb{Z}_2)$ singular homology

Fact 1 $\partial^2 = 0$

Fact 2 M closed $HM_*(f) = H_*(M)$

Corollary (Morse inequality) $\leftarrow$ Betti number
$$\# \operatorname{crit} f \geq \sum_{k=0}^{\dim M} b_k(M)$$

proof
$$\begin{aligned}
\# \operatorname{crit} f &= \dim CM_*(f) \\
&\geq \dim HM_*(f) \\
&\overset{\text{Fact 2}}{=} \dim H_*(M) \\
&= \sum_{k=0}^{\dim M} b_k(M)
\end{aligned}$$

Gradient flow equation:

g Riemanian metric on M

Gradient : $x \in M$
$$df(x)\xi = g_x(\nabla_g f(x), \xi) \quad \forall \xi \in T_x M$$

Gradient flow line $x \in C^\infty(\mathbb{R}, M)$
$$\partial_s x(s) + \nabla f(x(s)) = 0 \quad \forall s \in \mathbb{R} \quad (*)$$

Remark1 : Gradient flow lines flow downhill, i.e.,
f is decreasing along gradient flow lines

proof x sol of $(*)$
$$\begin{aligned}
\frac{d}{ds} f(x(s)) &= df(x(s))\,\partial_s x(s) \\
&\overset{(*)}{=} -df(x(s))\,\nabla f(x(s)) \\
&\overset{\text{def of } \nabla}{=} -g_{x(s)}(\nabla f(x(s)), \nabla f(x(s))) \\
&\leq 0 \qquad \blacksquare
\end{aligned}$$

Remark 2 : (Timeshift)

x sol of $(*)$
$r \in \mathbb{R}$ $\quad r_* x(s) = x(r+s) \quad s \in \mathbb{R}$
$\Rightarrow r_* x$ sol of $(*)$

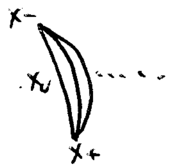
Counting gradient flow lines means counting unparametrized gradient flow lines, i.e., equivalence class of sol of (*) modulo time shift.

$\partial^2 = 0 \rightsquigarrow$ compactness of moduli spaces of gradient flow lines

x_ν $\nu \in \mathbb{N}$ solutions of (*)

Arzelà-Ascoli + (*)

$\Rightarrow \exists\, \nu_j$ subsequence x sol of (*) s.t. $x_{\nu_j} \xrightarrow[j\to\infty]{C^\infty_{loc}} x$

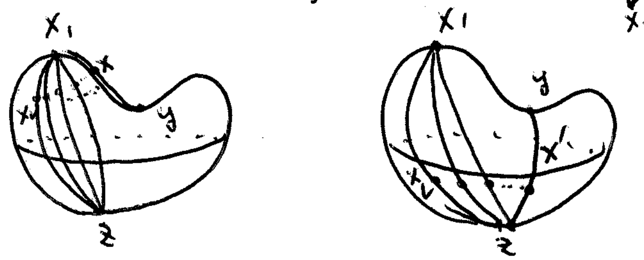


Q: if $\lim_{s\to\pm\infty} x_\nu = x_\pm \in \mathrm{Crit} f$ (fixed asymptotic)

does it follow that $\lim_{s\to\infty} x = x_\pm$

A: No

Example



Observe: Changing parametrization of x_ν might give rise to different limit solution

Way out Consider all possible limit gradient flow lines

Def: Let $x_\pm \in \mathrm{crit} f$

A broken gradient flow line from x_- to x_+ is a tuple $y = \{x^k\}_{1\le k\le n}$ $n \in \mathbb{N}$ s.t.

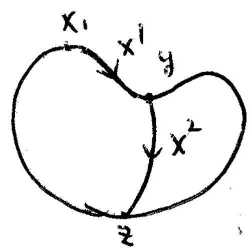
(i) $k \in \{1,\dots,n\}$ x^k is a nonconstant gradient flow line

(ii) $\lim_{s\to-\infty} x^1(s) = x_-$

$\lim_{s\to\infty} x^k(s) = \lim_{s\to-\infty} x^{k+1}(s)$ $k \in \{1,\dots,n-1\}$

$\lim_{s\to\infty} x^n(s) = x_+$

Ex:



Def: Let $x_\nu \in C^\infty(\mathbb{R}, M)$ sequence of gradient flow lines s.t. $\lim_{s\to\pm\infty} x_\nu(s) = x_\pm \in \mathrm{Crit} f$.

$y = \{x^k\}_{1\le k\le n}$ a broken gradient flow line from x_- to x_+.

x_ν Floer-Gromov converges to y if $1\le k\le n$ $\exists$ sequence $r_\nu^k \in \mathbb{R}$ s.t. $(r_\nu^k)_* x_\nu \xrightarrow{C^\infty_{loc}} x^k$

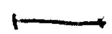
Fact: M closed, f Morse fct on M

x_ν sequence of gradient flow lines s.t. $\lim_{s\to\pm\infty} x_\nu(s) = x_\pm \in \mathrm{crit} f$ $x_- \neq x_+$

Then $\exists$ subsequence ν_j and a broken gradient flow line $y = \{x^k\}_{1\le k\le n}$ from x_- to x_+ s.t. $x_{\nu_j} \xrightarrow{\text{Floer-Gromov}} y$

Reason why $\partial^2 = 0$

Recall classification of compact 1-dim mfds with bdry

-  circle
-  interval

finite disjoint unions of these 

Fact these are all examples

Cor: N cpt 1-dim mfd with bdry $\Rightarrow \#\partial N$ even $(\#_2 \partial N = 0)$

M closed mfd, f Morse function, g Riem metric

$x_-, x_+ \in \mathrm{crit} f$

$\widetilde{\mathcal{M}}(x_-, x_+) = \widetilde{\mathcal{M}}(x_-, x_+; f, g)$

$= \{ x$ gradient flow lines : $\lim_{s\to\pm\infty} x(s) = x_\pm \}$

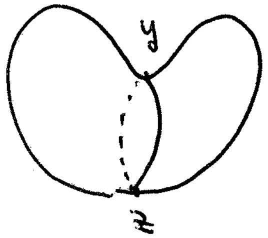
$\mathcal{M}(x_-, x_+) = \widetilde{\mathcal{M}}(x_-, x_+)/\mathbb{R}$ ← acting by time shift

Fact g generic Riem metric

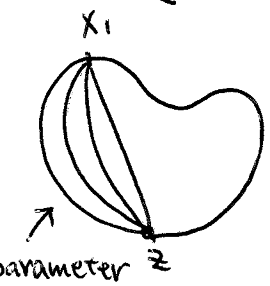
$\widetilde{\mathcal{M}}(x_-, x_+)$ mfd

$\dim \widetilde{\mathcal{M}}(x_-, x_+) = \mu(x_-) - \mu(x_+)$

$x_- \neq x_+ \Rightarrow \mathbb{R}$ acts freely on $\widetilde{\mathcal{M}} \Rightarrow \dim \mathcal{M}(x_-, x_+) = \mu(x_-) - \mu(x_+) - 1$



$\dim \mathcal{M}(y,z) = \underbrace{\mu(y)}_{=1} - \underbrace{\mu(z)}_{=0} - 1 = 0$

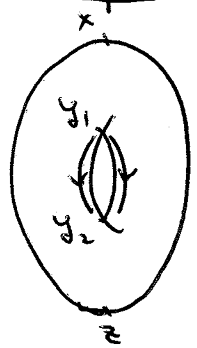


$\dim \mathcal{M}(x_1,z) = 1$

1-parameter family of gradient lines

Non example:

Torus



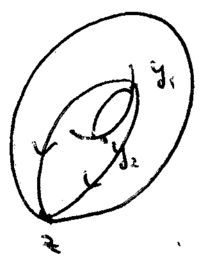
f hight fct

gradient flow lines between saddle points but

$\dim \mathcal{M}(y_1,y_2) = \mu(y_1) - \mu(y_2) - 1$
$= -1$

ungeneric

Tilt torus slightly



no gradient flow lines between y_1 and y_2 anymore

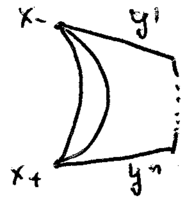
M closed, f Morse, g generic Riem metric

Case 1 $\mu(x_-) = \mu(x_+) + 1$

$\Rightarrow \mathcal{M}(x_-,x_+)$ compact 0-dim mfd (i.e., finite number of pts)

$\mathcal{M}(x_-,x_+)$ 0-dim mfd by previous fact. Why is $\mathcal{M}(x_-,x_+)$ cpt

Only obstruction to compactness breaking



$n \geq 2$

$\Rightarrow \exists j \in \{1 \ldots n\}$ s.t. y^j lies in moduli space of neg-dim ↯

$\rightarrow$ No braking ($n=1$) $\Rightarrow$ compactness

~~Counting~~

Conclusion: Counting gradient flow lines is well-defined

∂ makes sense as linear operator $CM_*(f) \circlearrowleft$

Case 2: $x_-, x_+ \in \operatorname{Crit} f$

$\mu(x_-) = \mu(x_+) + 2$

$\Rightarrow \dim \mathcal{M}(x_-,x_+) = 1$

$\mathcal{M}(x_-,x_+)$ need not be compact

$\partial\mathcal{M}(x_-,x_+)$ 1-times broken gradient flow lines.

$$\Rightarrow \partial\mathcal{M}(x_-,x_+) = \bigsqcup_{\substack{y \in \operatorname{Crit} f \\ \mu(y) = \mu(x_-) - 1 \\ = \mu(x_+) + 1}} \mathcal{M}(x_-,y) \times \mathcal{M}(y,x_+)$$

Classification of cpt 1-dim mfd with bdry

$$\Rightarrow \#_2 \bigsqcup \mathcal{M}(x_-,y) \times \mathcal{M}(y,x_+) = 0$$

$$\Rightarrow \partial^2 x_- = \sum_{\substack{x_+ \\ \mu(x_+) = \mu(x_-) - 2}} \underbrace{\#_2\Big(\bigsqcup \mathcal{M}(x_-,y) \times \mathcal{M}(y,x_+)\Big)}_{=0} x_+ = 0 \quad \blacksquare$$

Urs

Floer homology

Floer's motivation: Arnold conjecture

$\updownarrow$

Morse inequalities in infinite dimensional set-up

(M,ω) closed sympl mfd

For simplicity assume

$\omega|_{\pi_2(M)} = 0$ (i.e., $\int_{S^2} u^*\omega = 0 \ \forall u \in C^\infty(S^2,M)$)

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$H \in C^\infty(M \times S^1, \mathbb{R})$ Time-dep Hamiltonian

Variational approach to periodic orbits via action fct of classical mechanics

$\mathcal{L} \subset C^\infty(S^1, M)$

↑ component of contractible loop in free loop space

Action fct of class mechanics

$$\mathcal{A}_H : \underset{\substack{\Downarrow \\ v}}{\mathcal{L}} \longrightarrow \mathbb{R}$$

Since v is contractible, $\exists$ filling disc $\tilde{v} \in C^\infty(D, M)$, $D = \{z \in \mathbb{C} : \|z\| \le 1\}$

$$\mathcal{A}_H(v) = -\int_D \tilde{v}^*\omega - \int_0^1 H(v(t), t)\,dt \qquad \tilde{v}(e^{2\pi i t}) = v(t)$$

Since $\omega|_{\pi_2(M)} = 0$, $\mathcal{A}_H$ is well-defined i.e., independent of choice of filling disk

<u>Critical points of $\mathcal{A}_H$</u>

$v \in \mathcal{L}$

$T_v\mathcal{L} = \Gamma(v^*TM)$ vector field along v

$$d\mathcal{A}_H(v)\hat{v} = -\int_{S^1} \omega(\hat{v}, \partial_t v)\,dt - \int_0^1 dH_t(v)\hat{v}, \quad H_t = H(\cdot, t)$$

$$= -\int_{S^1} \omega(\hat{v}, \partial_t v)\,dt - \int_0^1 \omega(X_{H_t}(v), \hat{v})\,dt$$

$$= \int_0^1 \omega(\partial_t v - X_{H_t}(v), \hat{v})\,dt$$

$\Rightarrow$ $v \in \operatorname{crit} \mathcal{A}_H$

$\Leftrightarrow$ $\partial_t v - X_{H_t}(v) = 0$ i.e., v contractible periodic orbit of X_{H_t}

<u>Aim</u>: Do Morse homology for $\mathcal{A}_H$.

For gradient flow equation need to choose metric on $\mathcal{L}$
- Conley-Zehnder: $W^{1/2,2}$-metric if $M = T^{2n}$ torus
- Floer: L^2-metric

Pick family of ω compatible almost cpx str J_t

$\omega(\cdot, J_t \cdot)$ Riemannian metric

$v \in \mathcal{L}$

$\hat{v}_1, \hat{v}_2 \in T_v\mathcal{L}$ $\quad m_J(\hat{v}_1, \hat{v}_2) = \int_0^1 \omega_{v(t)}(\hat{v}_1(t), J_t(v(t))\hat{v}_2(t))\,dt$

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Computation of

$$\nabla_{m_J}\mathcal{A}_H = \nabla\mathcal{A}_H$$

$v \in \mathcal{L}$, $\hat{v} \in T_v\mathcal{L}$

$$d\mathcal{A}_H(v)\hat{v} = \int_0^1 \omega(\partial_t v - X_{H_t}(v), \hat{v})\,dt$$

$$= -\int_0^1 \omega(J_t(v)^2(\partial_t v - X_{H_t}(v)), \hat{v})\,dt$$

$$= \int_0^1 \omega(\hat{v}, J_t(v)(J_t(v)(\partial_t v - X_{H_t}(v))))\,dt$$

$$d\mathcal{A}_H(v)\hat{v} = m_J(\hat{v}, \nabla\mathcal{A}_H(v))$$

$$\Rightarrow \nabla\mathcal{A}_H(v) = J_t(v)(\partial_t v - X_{H_t}(v))$$

Gradient flow line:

Formally "$v \in C^\infty(\mathbb{R}, \mathcal{L})$" satisfing $\partial_t v(s) + \nabla\mathcal{A}_H(v(s)) = 0$ ("O.D.E." on $\mathcal{L}$)

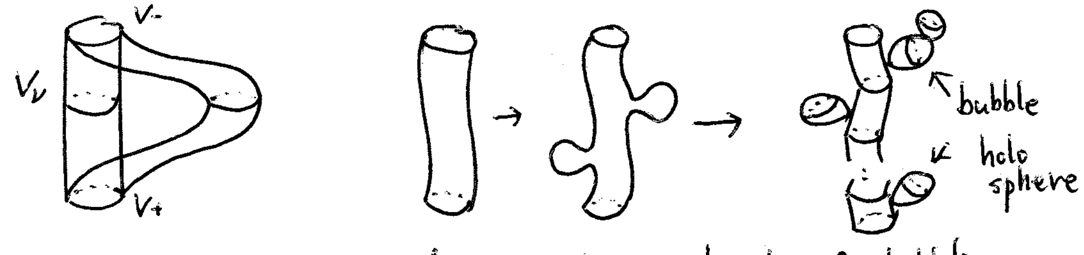
<u>Floer</u>: Interpret $v \in C^\infty(\mathbb{R} \times S^1, M)$

satisfing $\partial_s v(s,t) + J_t(v(s,t))(\partial_t v(s,t) - X_{H_t}(v(s,t))) = 0$ perturbed holomorphic equation $(*)$

<u>Compactness of gradient flow lines up to breaking.</u>

M closed, but derivatives could explode ($\mathcal{L}$ not compact)

v_ν sol of $(*)$ $\quad \lim_{s \to \pm\infty} v_\nu(s) = v_\pm \in \operatorname{crit} \mathcal{A}_H$



A priori: Limit breaking & bubbling

Hypothesis: $\omega|_{\pi_2(M)} = 0$

exclude bubbling since $u \in C^\infty(S^2; M)$ holomorphic sphere

$$\Rightarrow \int_{S^2} u^*\omega \underset{\substack{\uparrow \\ \text{holomorphic equation}}}{=} \int_{S^2} |du|^2\,d\mathrm{vol} \underset{\substack{\uparrow \\ u \text{ non const}}}{>} 0 \quad ↯$$

$\Rightarrow$ no bubbling

Only obstruction to compactness breaking (as in finite dim case)

Floer homology can be defined as Morse homology

$$HF_*(H) := HM_*(\mathcal{A}_H)$$

Floer homology $\qquad$ Morse homology of $\mathcal{A}_H$

Fact: Up to isomorphism $HF_*(H)$ is independent of H.

Computation of Floer homology

By previous fact can choose our favorite Hamiltonian H

Choose: $H \equiv 0$

⚠ ☹ For $H \equiv 0$ nondegeneracy (Lemma of Akaho) is not satisfied i.e.,

$\mathcal{A} = \mathcal{A}_0$ is not Morse.

$\operatorname{crit} \mathcal{A}_0 = \{ v \in \mathcal{L} : \partial_t v = 0 \}$ ← constant loops

$\cong M$

i.e., critical points of $\mathcal{A}$ are not isolated

$\mathcal{A}$ is not Morse but still Morse-Bott

Intermezzo: Morse-Bott homology

X mfd

$f \in C^\infty(X;\mathbb{R})$ is called Morse-Bott if

$\operatorname{crit} f$ mfd and $\forall x \in \operatorname{crit} f$ $\; T_x \operatorname{crit} f = \ker H_f(x)$ ← Hessian.

Auxiliary choice:

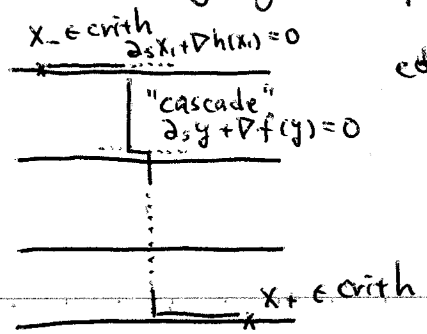
Choose $h \in C^\infty(\operatorname{crit} f, \mathbb{R})$ Morse

Chain groups: $CM(f,h) = \operatorname{crit} h \otimes \mathbb{Z}_2$

$\mathbb{Z}_2$-vector space generated by critical points of h

Boundary operator:

Defined by counting "gradient flow lines with cascades"



connected components of $\operatorname{crit} f$

Apply this to $\mathcal{A}_0$:

$\operatorname{crit} \mathcal{A}_0 \cong M \qquad \mathcal{A}_0|_{\operatorname{crit} \mathcal{A}_0} \equiv 0.$

Since cascades flow downhill

$\Rightarrow$ no cascades

Need only count "gradient flow lines with zero cascades" i.e., Morse gradient flow lines of h

$$\Rightarrow HF_*(0) = HM_*(\mathcal{A}, h) = HM_*(h) = H_*(M)$$

Proof of the Arnold conjecture

(Morse inequality for $\mathcal{A}_H$)

$$\#\{\text{periodic orbits of } H\} = \dim CM(\mathcal{A}_H) \geq \dim HM(\mathcal{A}_H) = \dim HF(H) = \dim HF(0) = \dim H_*(M) = \sum_{k=0}^{\dim M} b_k(M) \quad \blacksquare$$

Rabinowitz action fct.

(M,ω) sympl mfd, $\omega|_{\pi_2(M)} = 0$

$H \in C^\infty(M,\mathbb{R})$ time independent Hamiltonian (autonomous Hamiltonian)

$\mathcal{A}^H : \mathcal{L} \times \mathbb{R} \longrightarrow \mathbb{R}$

$\qquad\qquad v \quad \eta$

$\bar{v}$ filling disk for v

$$\mathcal{A}^H(v,\eta) = -\int_D \bar{v}^*\omega - \eta \int_0^1 H(v)\,dt$$

Lagrangian multiplier $\qquad$ constraint by mean value of H

Critical points:

$(v,\eta) \in \operatorname{crit} \mathcal{A}^H$

$$\left.\begin{aligned} &\partial_t v = \eta X_H(v) \\ &\int_0^1 H(v)\,dt = 0 \end{aligned}\right\} (*)$$

Rem

Since H autonomous → preservation of energy i.e. H preserved along flow lines of X_H

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<u>Proof</u> v : solution of $(*)$

$$\frac{d}{dt} H(v(t)) = dH(v(t))\partial_t v$$
$$= dH(v(t))(\eta X_H(v(t)))$$
$$\overset{(*)_1}{=} \eta\, \omega(X_H(v(t)), X_H(v(t)))$$
$$= 0$$

■

Apply Rmk $(*)$

Conclude $(*) \Leftrightarrow (**)$ $\left.\begin{array}{l}\partial_t v = \eta X_H(v) \\ H(v(t)) = 0\end{array}\right\} (**)$

Mean value constraint replaced by point wise constraint.

Sol of $(**)$ are in 1-1 corespondence with periodic orbit on energy surface $\Sigma = H^{-1}(0)$ of period η

<u>Reason</u> Reparametrize

$$w_v(t) = v\left(\frac{t}{\eta}\right)$$

Since v 1-per $\Rightarrow$ w_v η-per

$\overset{(**)_1}{\Rightarrow} \partial_t w(t) = X_H(w_v(t))$

<u>Rmk</u> Period η is allowed to be negative

$\longleftrightarrow$ periodic orbit trased backwards

Period η is allowed to be 0

$\longleftrightarrow$ constant on Σ

<u>Comparison</u> between $\mathcal{A}^H$ and $\mathcal{A}_H$

Crit $\mathcal{A}_H$: Period fixed to 1
Energy arbitrary

Crit $\mathcal{A}^H$: Period arbitrary
Energy fixed

<u>Perturbations of $\mathcal{A}^H$ and leafwise intersection points</u>

Assume $F \in C^\infty(M \times S^1)$ time-dependent Hamiltonian "perturbation"

Perturbed Rabinowitz action fct

$$\mathcal{A}^H_F : \mathcal{L} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(v, \eta)$$

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$$\mathcal{A}^H_F(v,\eta) = \mathcal{A}^H(v,\eta) - \int_0^1 F(v(t),t)\,dt$$

Reparametrize flow of X_H and X_F such that Time support of H & F are disjoint.

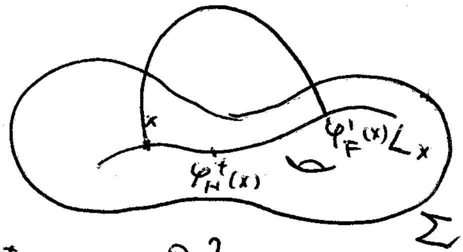
$\longrightarrow$ critical points of $\mathcal{A}^H_F$ give rise to leafwise intersection points

<u>Leafwise intersection point:</u>

$\Sigma = H^{-1}(0)$

By preservation of energy, Σ is foliated by flow φ^t_H of X_H

$x \in \Sigma$ L_x leaf through x $L_x = \{\varphi^t_H(x) : t \in \mathbb{R}\}$



Leafwise intersection point for F

$\Longleftrightarrow$ $x \in \Sigma$ s.t.

$\varphi^1_F(x) \in L_x$

<u>Q</u>: Do leafwise intersection points exist.