2/25 ①

Morse hom | Floer hom

M a closed mfd

g a Riem metric

$f: M \to \mathbb{R}$ a Morse fun. $\longleftrightarrow$?

grad f the gradient v.f.

critical pt of f

gradient flow line

(M, ω) a closed symplectic mfd. $L \subset M$

$L \subset M$ a closed Lag submfd.

$H: \underset{t}{\mathbb{R}} \times M \to \mathbb{R}$ a Ham. function.

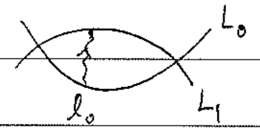
$\rightsquigarrow \phi_t^H: M \to M$ Ham isotopy

Suppose that $\cdot\ \pi_2(M, L) = 0$

$\cdot\ L \pitchfork \phi_1^H(L) \qquad (L_0 := L,\ L_1 := \phi_1^H(L))$

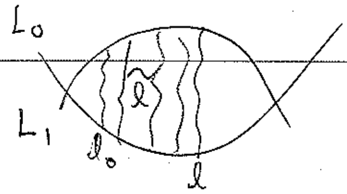
① mfds :

$$l_0(t) := \phi_t^H(p), \quad p \in L$$



$$\Omega_{l_0} := \left\{ l: [0,1] \to M \;\middle|\; \begin{matrix} l(0) \in L_0 \\ l(1) \in L_1 \end{matrix},\ l \overset{\text{homotopic}}{\sim} l_0 \right\}$$

where $l \sim l_0 \iff \exists\, \hat{l}: \underset{s}{[0,1]} \times \underset{t}{[0,1]} \to M$



$\hat{l}(0,t) = l_0(t), \quad \hat{l}(s,0) \in L_0$

$\hat{l}(1,t) = l(t), \quad \hat{l}(s,1) \in L_1$

$\Rightarrow \Omega_{l_0}$ is a "∞-dim mfd".

Tangent:

$l \in \Omega_{l_0}$, $T_l \Omega_{l_0} = \left\{ \xi \text{ sections of } l^* TM \text{ s.t. } \begin{array}{l} \xi(0) \in T_{l(0)} L_0 \\ \xi(1) \in T_{l(1)} L_1 \end{array} \right\}$

② metric

$\exists J$ an almost cpx str s.t.

$\cdot\ \omega(V, JV) > 0 \quad V \neq 0$

$\cdot\ \omega(Ju, Jv) = \omega(u, v)$

$\{J_t\}_{t \in [0,1]}$ a family of such almost cpx strs

$\Rightarrow g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ Riema. metric.

$\xi, \eta \in T_l \Omega_{l_0}$.

$$G_l(\xi, \eta) := \int_0^1 g_{l(t)}(\xi(t), \eta(t))\, dt$$

③ Morse fct

$$F: \Omega_{l_0} \to \mathbb{R}$$

$$F(l) := -\int_{[0,1]\times[0,1]} \tilde{l}^* \omega$$

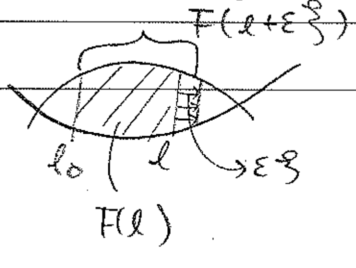
Remark: $\pi_2(M, L) = 0 \Rightarrow F(l)$ is indep of $\tilde{l}$

④ gradient v.f. of $F(l)$:

$$dF_l(\xi) = G_l(\xi, \operatorname{Grad} F)$$

$$\| $$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(l+\varepsilon\xi)$$



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$$\therefore\ dF_\ell(\xi) = -\int_0^1 \omega(\xi, \frac{d\ell}{dt}) = -\int_0^1 \omega(\xi, -J_t J_t \frac{d\ell}{dt})$$

$$\left(\text{where } G(\xi, h) = \int_0^1 g_t(\xi, h)\,dt = \int_0^1 \omega(\xi, J_t h)\,dt \right)$$

$$= \int_0^1 g_t(\xi, J_t \frac{d\ell}{dt})\,dt = G(\xi, J_t \frac{d\ell}{dt})$$

$$\Rightarrow \mathrm{Grad}_\ell F = J_t \frac{d\ell}{dt}$$

⑤ Crit pts:

ℓ critical pt of $F \iff \mathrm{Grad}_\ell F = 0$

$\iff J_t \frac{d\ell}{dt} = 0 \iff \frac{d\ell}{dt} = 0$

i.e. ℓ is constant

$\Rightarrow \ell : [0,1] \longrightarrow L_0 \cap L_1$

back to

③ $\therefore$ F: Morse $\longleftrightarrow L_0 \pitchfork L_1$

⑥ gradient flow lines

$u : \underset{s\in}{\mathbb{R}} \longrightarrow \Omega_{\ell_0}$ is grad flow line if " $\frac{du}{ds} = -\mathrm{Grad}_u F$ "

$\Rightarrow u : \underset{s\in}{\mathbb{R}} \times \underset{t\in}{[0,1]} \longrightarrow M$ s.t. $\frac{\partial u}{\partial s} = -J_t \frac{\partial u}{\partial t}$

$\iff \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0$

"Cauchy − Riem. eqs"

(In the sense that
$\Sigma = \mathbb{R} \times [0,1]$
with stand. cpx str

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Def: $p, q \in \mathrm{Crit}\, F$ $\quad (p, q \in L_0 \cap L_1)$

$$\mathcal{M}_{(p,q)} := \left\{ u : \mathbb{R} \times [0,1] \longrightarrow M \;\middle|\; \begin{array}{l} \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0, \ u(\mathbb{R}, 0) \subset L_0 \\ u(\mathbb{R}, 1) \subset L_1, \ \lim_{s \to -\infty} u(s,t) = p \\ \lim_{s \to +\infty} u(s,t) = q \end{array} \right\} \Big/ \mathbb{R}$$

Thm: for generic "J_t"

(1) $\mathcal{M}(p,q)$ is a smooth mfd of dim $< \infty$

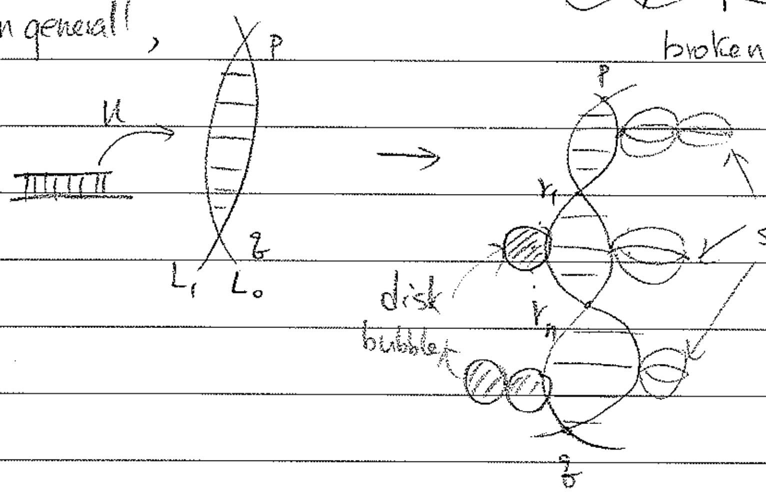
(2) $\underset{\text{assume}}{\overset{\text{Recall}}{)}}$ $\pi_2(M, L) = 0$), then $\mathcal{M}^0(p,q)$ (0-dim component of $\mathcal{M}(p,q)$) is compact.

(3) $\underset{\text{assume}}{\overset{\text{Recall}}{)}}$ ($\pi_2(M, L) = 0$), then $\mathcal{M}^1(p,q)$ can be compactified

s.t. $\partial \mathcal{M}^1(p,q) = \underbrace{\coprod_{r \in L_0 \cap L_1} \mathcal{M}^0(p,r) \times \mathcal{M}^0(r,q)}$

broken gradient flow line

In general,



bubbles appear,

but $\pi_2(M, L) = 0 \Longrightarrow$ no bubbles

Def: $CF := \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2\, p$, $\quad \partial : CF \longrightarrow CF$ is defined:

$$\partial p := \sum_{q \in L_0 \cap L_1} \#_2 \mathcal{M}^0(p,q) \cdot q$$

"Floer chain complex."

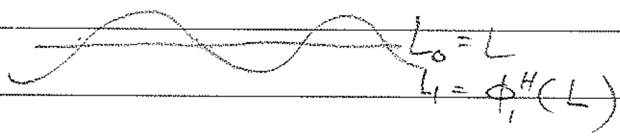
Thm: $\partial^2 = 0$

Floer homology: $HF(L) := \dfrac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}$

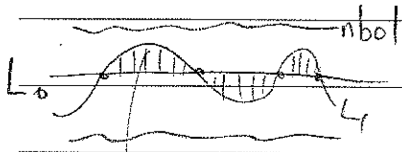
Thm: $HF(L)$ is indep of $\{J_t\}$ and H

Thm: $HF(L) \cong \bigoplus_{i=0}^{\dim L} H_*(L; \mathbb{Z}_2)$

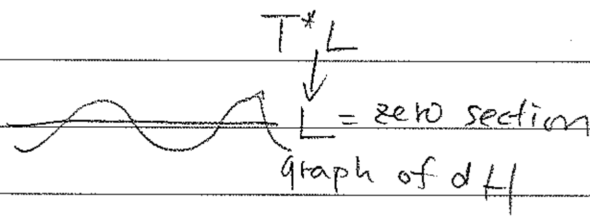
Idea: H "very small Morse fun"



$L_1 = \phi_1^H(L) =$ "graph of dH"

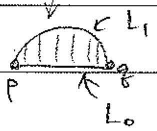


$\xleftrightarrow{\text{iso}}$

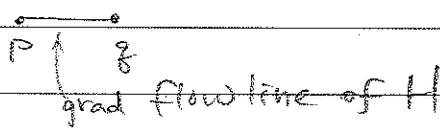


- $\operatorname{Crit} F = L_0 \cap L_1$ $\qquad$ $L \cap$ graph of dH $= \operatorname{Crit} H$

pseudo holomorphic strip

- 

$\longleftrightarrow$



under this identification

$CF \xleftrightarrow{=} CM(H)$

$\partial \xleftrightarrow{=} \partial$

$\Downarrow$

$HF(L) \cong HM(H)$ $\qquad$ Q.E.D.

proof of Floer's thm:

$$\# L \cap \varphi_1^H(L) = \dim CF \geq \dim HF(L)$$

$$= \bigoplus_{i=0}^{\dim L} \dim H_i(L; \mathbb{Z}_2)$$

Q.E.D.

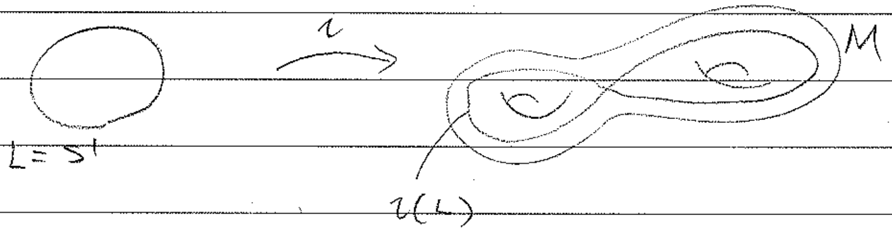
§ Immersed Case:

(M, ω) symp. mfd, L smooth mfd of $\dim = \frac{\dim M}{2}$

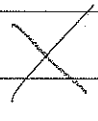
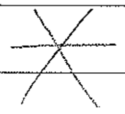
$\iota : L \to M$ immersion.

Def: $\iota : L \to M$ is Lag immersion if $\iota^* \omega = 0$.

Ex:



Thm. (M, ω) a closed symp mfd, L closed mfd. $\iota : L \to M$ Lag imm. Suppose $\pi_2(M, \iota(L)) = 0$, and self-intersection pts are double ones;

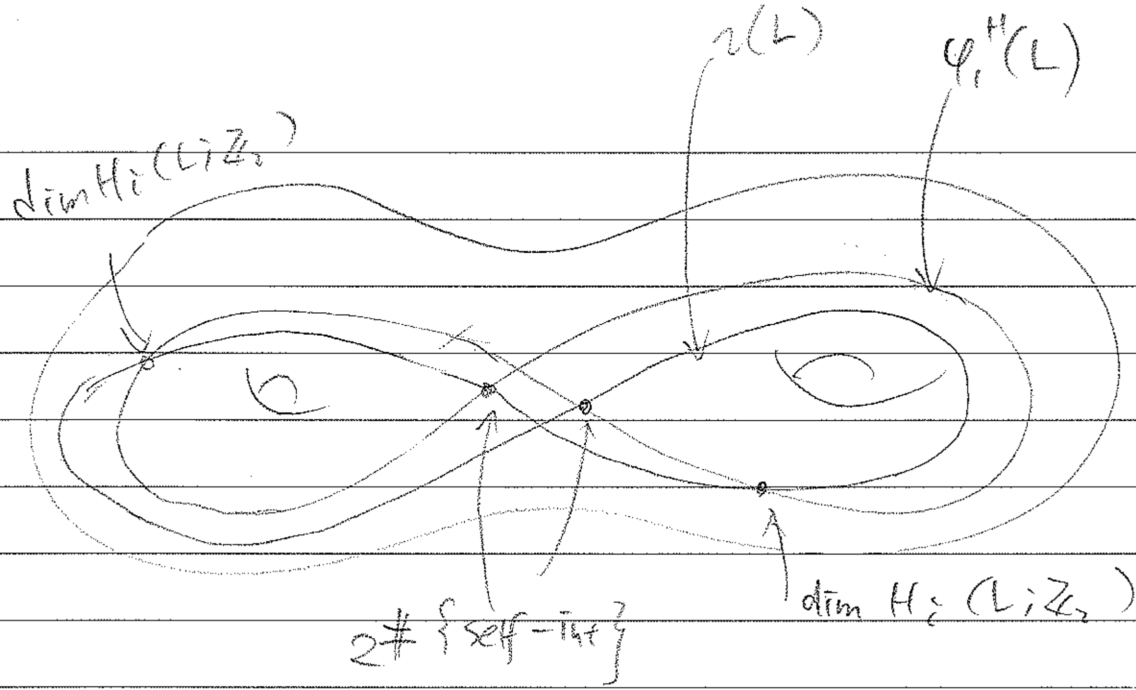
 double, OK,  triple, NO!

and $\iota(L) \pitchfork \varphi_1^H(\iota(L))$

intersections are also double

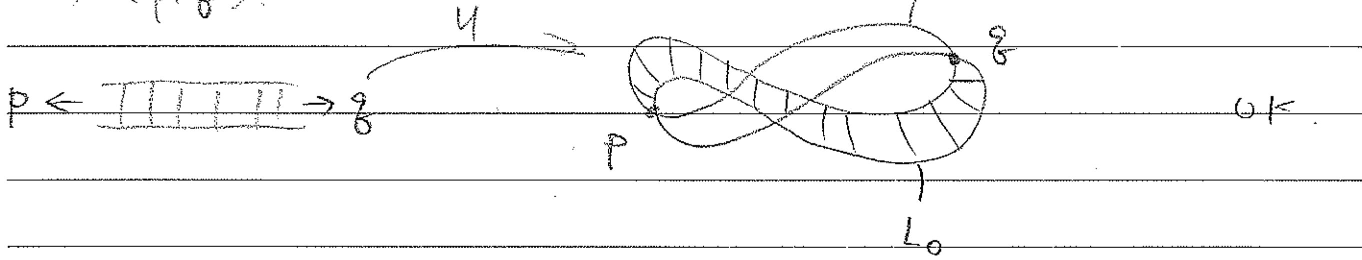
$\Longrightarrow$ Then $\# \iota(L) \cap \varphi_1^H(\iota(L)) \geq \bigoplus_{i=0}^{\dim L} \dim H_i(L; \mathbb{Z}_2)$

$+ 2 \cdot \# \left\{ \begin{matrix} \text{self-intersection} \\ \text{pts of } \iota(L) \end{matrix} \right\}$

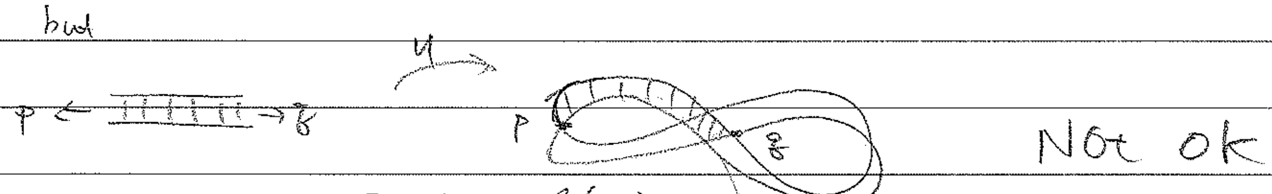


$$CF = \bigoplus_{p\in\tau(L)\cap\varphi_1^H(\tau(L))} \mathbb{Z}_2\, p \quad , \quad \partial p = \sum_{q\in\tau(L)\cap\varphi_1^H(\tau(L))} \#_2 M(p,q)\, q$$

$M(p,q)$:



but



$u(\mathbb{R},0)\subset\tau(L)$

$u(\mathbb{R},1)\subset\varphi_1^H(\tau(L))$

不會是 u的 strip的 image.