Manabu.

Arnold conjecture

(M, ω) a closed symp mfld.

$H : \mathbb{R} \times M \to \mathbb{R}$ a smooth fn

$H(t+1, x) = H(t, x)$

$\rightsquigarrow X_H$ the v.f. on M

s.t. $dH = \omega(\cdot, X_H)$.

$c : \mathbb{R} \to M$ s.t. $\dot{c} = X_H$, $c(t+1) = c(t)$.

Conj : Suppose that all such c are non deg

$$\Rightarrow \# \{ c : \mathbb{R} \to M \mid \underset{\text{non-deg}}{\dot{c} = X_H, c(t+1) = c(t)} \} \geq \sum_{i=0}^{\dim M} \dim H_i(M).$$

Ex : (M, ω) a closed symp mfld

$f : M \to \mathbb{R}$ Morse fn

Consider $H = f \rightsquigarrow X_f$

* $X_f(p) = 0 \Leftrightarrow df(p) = 0$.

* $c : \mathbb{R} \to M$ $c(t) = p \in \mathrm{Crit} f$.

$$\rightsquigarrow \# \{ c : \mathbb{R} \to M \mid \underset{c:\text{ non-deg}}{\dot{c} = X_f, c(t+1) = c(t)} \}$$

$$\geq \# \{ c : \mathbb{R} \to M \mid c(t) = p, X_f(p) = 0 \}$$

$$= \# \{ p \in M \mid df(p) = 0 \} \underset{\text{ineq}}{\overset{\text{Morse}}{\geq}} \sum_{i=0}^{\dim M} \dim H_i(M).$$

Thm. (Floer)

(M,ω) a closed symp mfld,

$L \subset M$ a closed Lag. submfld.

$H : \mathbb{R}\times M \to \mathbb{R}$ a smooth fn.

$\rightsquigarrow X_H$ a Hamil. v.f. i.e. $dH = \omega(\cdot, X_H)$.

Let $\phi_t^H : M \to M$ satisfy

$$\frac{d}{dt}\phi_t^H = X_H.$$

$$\phi_0^H = \mathrm{id}_M.$$

Suppose that

- $\pi_2(M,L) = 0$.
- $L \pitchfork \phi_1^H(L)$ $\quad$ ∵ dim $= \frac{1}{2}$ dim M ∴ intersection pt.

$$\Rightarrow \# L \cap \phi_1^H(L) \geq \sum_{i=0}^{\dim L} \dim H_i(L;\mathbb{Z}_2).$$

Remark (relation of the Conj & the thm)

(M,ω) closed symp mfld.

$$\begin{tikzcd} & M\times M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & M \end{tikzcd}$$

$\tilde{\omega} = p_1^*\omega - p_2^*\omega$.

$(M\times M, \tilde{\omega})$: symp mfld.

$\Delta := \{(x,x) \in M\times M\}$.

$\Delta \subset M\times M$ a Lagr. sbmfld

$\tilde{H} : \mathbb{R}\times(M\times M) \to \mathbb{R}$ a smooth fn.

$\tilde{H}(t,x_1,x_2) := H(t, x_1)$. $\qquad H(t+1,x) = H(t,x)$.

$\rightsquigarrow \phi_t^{\tilde H} : M\times M \to M\times M$ the isotopy gen. by $\tilde H$, $\tilde\omega$.

$$\phi_t^{\tilde H}(x_1,x_2) = (\phi_t^H(x_1), x_2)$$

$$\exists\ \Delta \cap \phi_1^{\tilde H}(\Delta) \ni p = (x_1,x_2)$$

$$\Rightarrow x_1 = x_2$$

$$p = (\phi_1^H(y), y)$$

$\leftrightarrow$ $x\in M$ s.t. $x = \phi_1^H(x)$.

because $(x_1,x_2) = (y, \phi_1^H(y))$.

$\Rightarrow$ put $c(t) = \phi_t^H(x)$, $x = \phi_1^H(x)$.

$$\Rightarrow \dot c = X_H,\ c(t+1) = c(t).$$

On the other hand,

$$\begin{aligned}\dot c &= X_H \\ c(t+1) &= c(t)\end{aligned} \rightsquigarrow x = c(0) \Rightarrow x = \phi_1^H(x).$$

$$\#\{c : \mathbb{R}\to M \mid \dot c = X_H,\ c(t+1) = c(t)\}$$
non-deg

$$= \#\,\Delta \pitchfork \phi_1^{\tilde H}(\Delta).$$

$\pitchfork$

(Rmk. $\pi_2(M) = 0$
$\Rightarrow \pi_2(M\times M, \Delta) = 0$)

$$\underset{\text{Floer}}{\geq} \sum_{i=0}^{\dim\Delta} \dim H_i(\Delta;\mathbb{Z}_2) = \sum_{i=0}^{\dim M} \dim H_i(M;\mathbb{Z}_2).$$

Morse homology.

M : a closed mfd.

g : a Riem. metric on M.

$f \overset{M \to \mathbb{R}}{:}$ Morse fn

$\leadsto \exists X_f$ a v.f. on M

s.t $df = g(\cdot, X_f)$

grad. v.f.

Def: $p, q \in \mathrm{Crit}\, f$.

$$\mathcal{M}(p,q) := \left\{ \ell : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \dot{\ell} = -X_f \\ \lim_{s \to -\infty} \ell(s) = p \\ \lim_{s \to +\infty} \ell(s) = q \end{array} \right\} \Big/ \underset{\text{times}}{\mathbb{R}}$$

Thm. for "generic g"

(1) $\mathcal{M}(p,q)$ is a smooth mfld

$\dim \mathcal{M}(p,q) = \underbrace{\mu(p) - \mu(q)}_{\text{Morse index}} - 1$.

(2) If $\dim \mathcal{M}(p,q) = 0$ (i.e. $\mu(p) - \mu(q) - 1 = 0$).

$\mathcal{M}(p,q)$ is cpt

(3) $\dim \mathcal{M}(p,q) = 1$ ($\mu(p) - \mu(q) - 1 = 1$).

$\Rightarrow \mathcal{M}(p,q)$ can be compactified s.t.

$$\overline{\mathcal{M}}(p,q) = \coprod_{\substack{r \in \mathrm{Crit} f \\ \mu(r) = \mu(p) - 1 \\ = \mu(q) + 1}} \mathcal{M}(p,r) \times \mathcal{M}(r,q).$$

Def: $CM_i(f) := \bigoplus_{\substack{p \in \mathrm{Crit} f \\ \mu(p) = i}} \mathbb{Z}_2 \, p$.

$$\partial : CM_i(f) \to CM_{i-1}(f).$$

$$\partial p = \sum_{\mu(q) = i-1} \#_2 M(p,q) \, q.$$

Thm. $\partial^2 = 0$.

$\rightsquigarrow HM(f) = \dfrac{\mathrm{Ker}\,\partial}{\mathrm{Im}\,\partial}$

Thm. $HM(f) \cong \underset{\text{singular homology}}{H(M)}$.