Manabu Akaho : Lagrangian Floer theory

Thm (Gromov)

$L \subset \mathbb{C}^n$ a closed (cpt without boundary) Lagrangian submfd

$D := \{ z \in \mathbb{C} \mid |z| \le 1 \}$

$\Rightarrow \exists$ non constant holomorphic map

$u : D \longrightarrow \mathbb{C}^n , \quad u(\partial D) \subset L .$

Def : (M, ω) a symplectic mfd, $L \subset M$ a submfd

L Lagrangian $\overset{\text{def}}{\Longleftrightarrow}$ · $\dim L = \frac{1}{2} \dim M$

· $\omega|_L = 0$

Cor : $L \subset \mathbb{C}^n$ a closed Lag $\Rightarrow H^1(L; \mathbb{R}) \neq 0$

Pf : Remark u holomorphic

$\Rightarrow u^*\omega = |du|^2 \, vol_D$

$\lambda = x_1 dy_1 + \cdots + x_n dy_n \Rightarrow d\lambda|_L = \omega|_L = 0$

$\Rightarrow [\lambda|_L] \in H^1(L; \mathbb{R})$

also, $\int_D u^*\omega > 0$ where $\left(\begin{array}{l} u : D \xrightarrow[\text{hol}]{\text{non const}} \mathbb{C}^n \\ u(\partial D) \subset L \end{array} \right)$

but $\int_D u^*\omega \overset{\text{Stoke}}{=} \int_{\partial D} u^*\lambda$

$\therefore [\lambda|_L] \neq 0 \in H^1(L; \mathbb{R})$ Q.E.D.

Cauchy-Riemann equations:

Def : M a smooth mfd. $J \in End(TM)$.

J is called an almost cpx str. if $\boxed{J \circ J \overset{\text{def}}{=} -\mathbb{1}}$

(M, J) almost cpx mfd.

Ex: $z_1, \cdots, z_n$ local cpx coordinate.

$$J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$$

Def: (M_1, J_1), (M_2, J_2) almost cpx mfds.

$f: M_1 \longrightarrow M_2$ a smooth map

f: pseudoholomorphic map $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $df \circ J_1 = J_2 \circ df$

In particular, if $\dim_{\mathbb{R}} M_1 = 2$, then f is called a pseudoholomorphic curve.

Ex: (M_1, J_1), (M_2, J_2) cpx mfds.

f pseudoholomorphic $\Rightarrow$ f holomorphic.

(M, J) an almost cpx mfd, (Σ, j) a Riem. surface

$L \subset M$ a Lagrangian.

For a smooth map $u: \Sigma \to M$,

$\bar{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j)$ anti complex linear part of du.

or a section of $\Lambda^{0,1}\Sigma \otimes u^*TM$

Def: (C-R eqs.) $\bar{\partial}_J(u) = 0$

Remark: • $\bar{\partial}_J(u) = 0 \Longleftrightarrow u$: pseudoholomorphic

• $\bar{\partial}_J(u) = 0$ a nonlinear elliptic partial diff eqs.

Remark: (M,ω) sym $\Rightarrow \exists J$ almost cpx s.t. $\omega(v,Jv)>0$; $v\neq 0$
$\omega(Jv,Ju)=\omega(v,u)$

a: a section of $\Lambda^{0,1}\Sigma \otimes TM$
$\downarrow$
$\Sigma\times M$

Def (perturbed C-R eq)

$$\bar\partial_J(u)=a \leftarrow a\big|_{\{(z,u(z))\in\Sigma\times M \mid z\in\Sigma\}}$$

Bubble:

(M,J) closed sym mfd, $L\subset M$ closed Lag submfd.
(Σ,j) cpt Riemann surface. $C<\infty$

$$\bar\partial_J(u)=a,\quad u(\partial\Sigma)\subset L,\quad \int_\Sigma |du|^2\le C \quad (*)$$

Let $\{u_i\}_{i=1,2,\dots}$ be a sequence of sol's of $(*)$

Case 1 suppose that $\exists B<\infty$ s.t. $|du_i|<B$

$\Rightarrow \exists$ subsequence $\{u_{i_j}\}_{j=1,\dots}$ s.t.

$u_{i_j}\xrightarrow{C^\infty} u_\infty$ & u_∞ is a sol of $(*)$

$\Rightarrow$ No bubble (Some kind of compactness)

or

Case 2 $\max|du_i|\xrightarrow{i\to\infty}\infty$

$\Rightarrow \exists z_i\in\Sigma$ s.t. $z_i\to z_\infty$; $R_i=|du_i(z_i)|$
$=\max\limits_{z\in\Sigma}|du_i|$

case (a) $z_i\longrightarrow z_\infty$ an interior pt of Σ, or

$z_i\xrightarrow{\text{"slowly"}} z_\infty\in\partial\Sigma$.

$\Rightarrow \exists \{u_{i_j}\}_{j=1,2,\dots}$ s.t. $v_j : \mathbb{C} \longrightarrow M$

$$v_j(z) := u_{i_j}\left(\frac{z + z_{i_j}}{R_i}\right)$$

$v_j \xrightarrow{C^\infty_{loc}} v_\infty$, $v_\infty : \mathbb{C} \to M$ s.t. $\overline{\partial_J}(v_\infty) = 0$

$\int |dv_\infty|^2 \leq C$

v_∞ is non constant

$\xrightarrow[\text{thm}]{\text{removable singularity}}$ $\exists\, \tilde{v} : \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow M$ non constant pseudo hol.

and $\tilde{v}|_{\mathbb{C}} = v_\infty$

case (b) $z_j \xrightarrow{\text{"rapidly"}} z_\infty \in \partial\Sigma$

$\Rightarrow \exists \{u_{i_j}\}_{j=1,2,\dots}$ s.t. $v_j : \mathbb{H} = \{x + iy \mid y \geq 0\} \longrightarrow M$

$$v_j(z) \text{ "}:=\text{" } u_{i_j}\left(\frac{z + z_{i_j}}{R_i}\right)$$

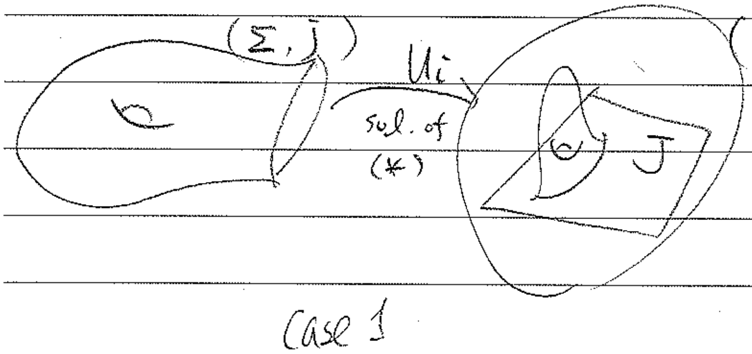
$v_j \xrightarrow{C^\infty_{loc}} v_\infty$, $v_\infty : \mathbb{H} \to M$ s.t. v_∞ is non-const.

$\overline{\partial_J}(v_\infty) = 0$, $v_\infty(\partial\mathbb{H}) \subset L$

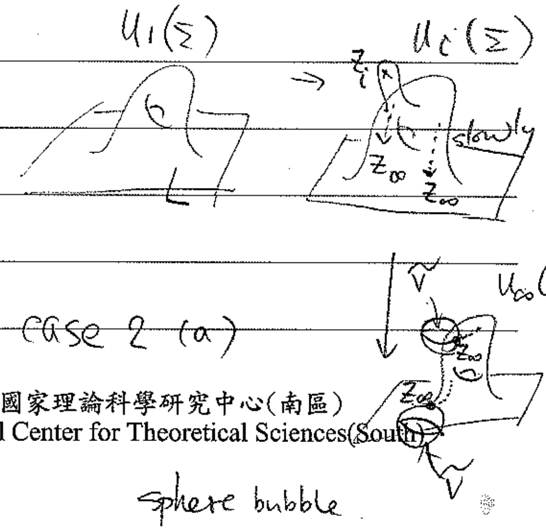
$\int_{\mathbb{H}} |dv_\infty|^2 \leq C$.

$\xrightarrow[\text{Thm}]{\text{removable singularity}}$ $\exists\, \tilde{v} : D = \mathbb{H} \cup \{\infty\} \to M$ non const. pseudo hol

and $\tilde{v}|_{\mathbb{H}} = v_\infty$

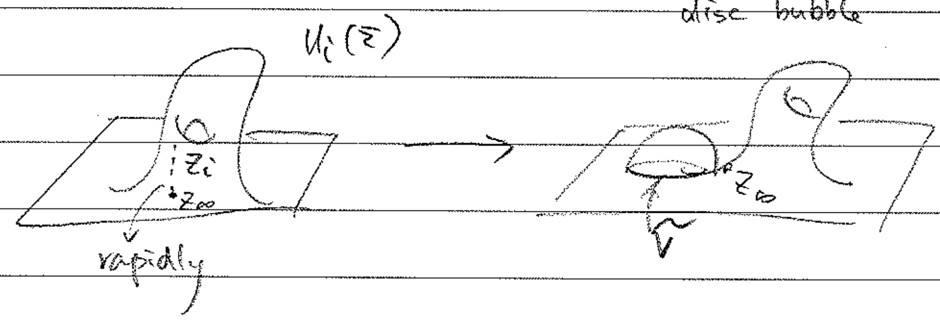


Case 1



case 2 (a)

case (2) b



Moduli spaces

(M, ω, J) a closed symplectic mfd

(Σ, j) cpt Riemann Surface

$L \subset M$ a closed Lag sub.

$\beta \in H_2(M, L; \mathbb{Z})$

$\widetilde{\mathcal{M}}(M, L, \beta, a) := \{ u : \Sigma \to M \mid \bar{\partial}_J(u) = a, \ u(\partial\Sigma) \subset L, \ u_*[\Sigma] = \beta \}$

$\mathcal{M}(M, L, \beta, a) := \widetilde{\mathcal{M}}(M, L, \beta, a) / \sim$

where $u_1 \sim u_2 \overset{def}{\iff} \varphi : \Sigma \longrightarrow \Sigma$ bihol s.t. $u_1 \circ \varphi = u_2$

Let $\alpha = \{a_t\}_{t \in [0,1]}$, ~~$a_0 = 0$, $|a_1| \gg 1$~~,

$\mathcal{M}(M, L, \beta, \alpha) = \bigcup_{t \in [0,1]} \mathcal{M}(M, L, \beta, a_t)$

Remarks: $\widetilde{\mathcal{M}}(M, L, \beta, a)$ is a "smooth mfd of dim $< \infty$".

proof of Gromov's thm ⊛.

Let $L \subset \mathbb{C}^n$ closed Lag. $\Sigma = D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$

$a \in \mathbb{C}^n$, for $u: \Sigma \longrightarrow \mathbb{C}^n$, consider $\frac{\partial u}{\partial \bar{z}} = a$.

(i.e. $\frac{1}{2}\left(\frac{\partial u}{\partial x} + \sqrt{-1}\frac{\partial u}{\partial y}\right) = a$)

Remarks:

① $|a| \gg 0$

$\Rightarrow \nexists_{sol}$ of $\frac{\partial u}{\partial \bar{z}} = a$

② $\exists\, C(L, a) < \infty$ s.t.

$\int |du|^2 \leq C(L, a)$

for a sol u of $\frac{\partial u}{\partial \bar{z}} = a$.

Let $\alpha = \{a_t \in \mathbb{C}^n\}_{t\in[0,1]}$, $a_0 = 0$, $|a_1| \gg 1$

$\beta = 0 \in H_2(M, L; \mathbb{Z})$

$\mathcal{M}(M, L, \beta = 0, \alpha) = \bigcup_{t\in[0,1]} \mathcal{M}(M, L, 0, a_t)$

Remarks:

1° $\mathcal{M}(M, L, 0, a_1) = \phi$

2° $\mathcal{M}(M, L, 0, a_0 = 0) =$ set of pseudo hol. from disc

$\cong \{u: D \longrightarrow L \text{ const. map}\} \cong L$

3° $\mathcal{M}(M, L, 0, \alpha)$ is a smooth mfd of dim $n+1$.

4° Suppose that there is no bubble, then $\mathcal{M}(M, L, 0, \alpha)$ is compact!

5° $\partial\mathcal{M}(M, L, 0, \alpha) = \mathcal{M}(M, L, 0, a_0) \amalg \mathcal{M}(M, L, 0, a_1)$

$\cong L \amalg \phi$

6° Define $ev: \mathcal{M}(M, L, 0, \alpha) \longrightarrow L$

$u \longmapsto u(1)$, $1 \in \partial D$

$\therefore ev_*[\partial\mathcal{M}(M, L, 0, \alpha)] = ev_*[L] = [L] \in H_n(L; \mathbb{Z}_2)$

but $ev_*[\partial\mathcal{M}(M, L, 0, \alpha)] = 0$ $\times$ contradiction.

$\Rightarrow$ There must be bubbles!

7° $\nexists\, u: \mathbb{CP}^1 \longrightarrow \mathbb{C}^n$ non const. holo.

$\Rightarrow \exists\, u: D \longrightarrow \mathbb{C}^n$ non const. hol.

s.t. $u(\partial D) \subset L$.

Q.E.D.