Geometric Evolution Problems in a Closed Pseudohermitian 3-Manifold
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1 Geometrization problem of contact 3-manifolds

1.1 Geometrization problem of 3-manifolds via Hamilton Ricci flow

- *Decomposition of* 3*-manifolds*: Assume that M is closed and orientable.

1. *Cutting along Spheres (The Sphere Decomposition)* : Connected sum decomposition of M into prime pieces.
   (a) (Kneser, 1929; Milnor, 1962) Every M has a prime decomposition
   $$M \cong M_1 \# ... \# M_k,$$
   and
   $$M \cong (K_1 \# ... \# K_p) \# (L_1 \# ... \# L_q) \# (\#_1^r S^2 \times S^1).$$
   where M_i are prime and K_i, L_i are irreducible. $\pi_1(L_i) < \infty, \pi_1(K_i) = \infty$ and $K(\pi, 1)$.
   (b) Moreover
   $$L_i = \Sigma_i / G_i$$
   is a finite quotient of a homotopy 3-sphere.
   $$M \cong (K_1 \# ... \# K_p) \# (\Sigma_1 / G_1 \# ... \# \Sigma_q / G_q) \# (\#_1^r S^2 \times S^1).$$

2. *Cutting along the incompressible tori (Torus Decomposition)* :

   (a) (Jaco-Shalen, 1979; Johannson, 1979) M is closed, orientable, irreducible

   $$\exists\{T_i^2\}(\text{ if any}):\ \text{finite disjoint incompressible tori}$$

   such that

   $$M\backslash \cup T_i^2 = \text{Seifert fiber space or torus-irreducible.}$$

- Geometrization Conjecture:

  1. Thurston; Topology : Every closed irreducible 3-manifold has either geometric structure or its simple pieces have geometric structure. More precisely, there is a finite collection of disjoint, embedded 2-spheres and incompressible 2-tori such that after cutting M^3 along these surfaces and capping the boundary 2-spheres by 3-balls, the interior of each component of the resulting 3-manifold admits a complete locally homogeneous metric.
  2. Structure of Three Dimensional Manifolds

     $$M \cong (K_1\#...\#K_p)\#(\Sigma_1/G_1\#...\#\Sigma_q/G_q)\#(\#_1^r S^2 \times S^1).$$

     (a) The Poincare Conjecture $(\Sigma_1/G_1\#...\#\Sigma_q/G_q)$ ::three dimensional space where every closed loop can be shrunk to a point; the space is conjecture to be the three-sphere. *If M is a homotopic 3-sphere, then it is diffeomorphic to the sphere S^3.Equivalently, if M is closed simply-connected, then M=S^3.*

     (b) The space-form problem .

     (c) Seifert spaces and their quotients.

     (d) Hyperbolic Conjecture $(K_1\#...\#K_p)$.

- *The Flow method :*

  1. Riemannian geometry aspects: Existence of a " best possible " metric on closed 3-manifolds. Generically, one must allow the optimal metric to have degenerate region. Then the topology decomposition suggests that " The degeneration should be via the pinching off 2-spheres (sphere decomposition) and collapse of graph manifolds along the circles and tori (torus decomposition)

2. Hamilton Ricci flow : A solution $(M^3, g(t))$, $t \in [0, T)$ to the Ricci Flow (RF)
$$\frac{\partial}{\partial t} g = -2Rc.$$
3. Hamilton and Perelman : Every closed irreducible 3-manifold has either geometric structure or it splits along disjoint incompressible tori as
$$(S^3/G_1 \# ... \# S^3/G_q) \# (M_{thick} \cup M_{thin}),$$
where M_{thick} is a disjoint union of hyperbolic manifolds, and $M_{thin}$is a graph manifold, a manifold obtained by gluing along boundary tori of geometric 3-manifolds which are not modeled on $\mathbf{H}^3$.

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1.2 Geometrization problem of contact 3-manifolds

- Overview :

1. The CR analogue of Thurston's geometrization conjecture on contact 3-manifolds.
2. Classify tight contact structures on all closed irreducible 3-manifolds.
   (a) Do all hyperbolic manifolds admit a tight contact structure ?
   (b) Which Seifert fibred spaces admit a tight contact structure ?
   (c) Do all rational homology spheres admit a tight contact structure ?
3. A tight contact structure $\leftrightarrow$ geometry of the underlying 3-manifold

- Overtwisted or Tight Contact Struction for a contact 3-manifold :

1. The characteristic foliation Σ_ξ of Σ in M : For a generic surface $\Sigma \subset M$, the intersection $\xi \cap T\Sigma$ is a line field except at finite many points where $\xi \cap T\Sigma = \xi = T\Sigma$. Consider the integral curve for the intersection $\xi \cap T\Sigma$, we get a characteristic foliation Σ_ξ of Σ with singularities.
2. An embedded disk $D^2 \subset M$ is called an overtwisted disk for ξ $\Leftrightarrow$ $TD^2 = \xi$ along ∂D^2 $\Leftrightarrow$ D_ξ contains a closed circle leaf.

3. ξ is overtwisted contact structure if M contains an overtwisted disk. Otherwise is called tight contact structure.

- Existence

1. Contact Topology on 3-manifolds :
   (a) There is a contact sphere decomposition (1929, 1962).
   (b) Is there a contact JSJ(Jaco-Shalen-Johannson, 1979) decomposition?

$$\mathrm{M}\backslash \cup \mathrm{T}_i^2 = \text{Seifert fiber space or torus-irreducible.}$$

   (c) *Eliashberg, Giroux, Honda, Lisca, Gompf; Contact structure* : Existence and classify tight contact structures on all closed irreducible 3-manifolds.

$$\text{tight} \supsetneqq \text{symplectic fillable} \supsetneqq \text{Stein fillable}$$

   and

$$\text{tight} \supsetneqq \text{embedded CR structure.}$$

   (d) (Kamishima and Tsuboi, 1991) : If M admits a *CR structure with vanishing torsion, then it is a Seifert manifold.* Classified a *spherical CR structure with vanishing torsion.*
   (e) (Chang-Chiu-Wu, 2009) : *spherical CR structure with positive Webster scalar curvature and vanishing torsion* $\Longrightarrow \exists$ *constant Tanaka-Webster curvature and vanishing torsion.*
   (f) (Lisca, 2007) M : closed oriented Serfert fiber 3-manifold. Then either M is orientation-preserving diffeomorphism to M_n for some $n \geq 1$ or M carries a positive tight contact structure
      i. $S_r^3(T_{p,q})$: oriented 3-manifold obtained by performing rational r-surgery along torus knot $T_{p,q} \subset S^3$. By Kirby calculus :

$$S^3_{p^2n-pn-1}(T_{p,pn+1}) = M(-\frac{1}{p}, \frac{n}{pn+1}, \frac{1}{p(n+1)+1})$$

      ii. M_n : $p = 2, r = 2n-1, M_1 = M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$.

iii. Method : Heegaard Floer theory and contact Ozsváth-Szabó invariant

(g) $\Omega \subset S^3$

$M = \Omega/\Gamma$
$\Rightarrow$ CR spherical
$\Rightarrow$ symplectic fillable (ounds the symplectic orbifold CH^2/Γ)
$\Rightarrow$ tight

- Classification

1. (Eliashberg): Any tight contact structure on S^3 is isotopic to the standard one
2. Any diffeomorphism of S^3 can extend to a diffeomorphism of 4-ball D^4 by holomorphic fillable
3. $S^3, R^3, S^1 \times S^2$
4. (Giroux) Any tight contact structure on T^3 is contactomorphism to one of
$$\xi_n = \ker(\cos(n\theta_1)d\theta_2 + \sin(n\theta_1)d\theta_3)$$
5. (Giroux, Honda): $S^1 \times D^2,\ T^2 \times [0,1], L(p,q)$ and T^2-bundle over S^1
6. (Honda) : S^1-bundle over closed surfaces

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2 Pseudohermitian 3-manifold

- Let M be a closed 3-manifold with an oriented contact structure ξ. There always exists a global contact form θ, obtained by patching together local ones with a partition of unity. The characteristic vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. A CR-structure compatible with ξ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -identity$. A pseudohermitian structure compatible with ξ is a CR-structure J compatible with ξ together with a global contact form θ.

- Given a pseudohermitian structure (J, θ), we can choose a complex vector field Z_1, an eigenvector of J with eigenvalue i, and a complex 1-form θ^1 such that $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is dual to $\{T, Z_1, Z_{\bar{1}}\}$. It follows that $d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$ for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call such a pseudohermitian structure (J, θ) positive, and we can choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}}.$$

- We'll always assume our pseudohermitian structure (J, θ) is positive and $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of (J, θ) is the connection $\nabla^{\psi.h.}$ on $TM \otimes C$ (and extended to tensors) given by

$$\nabla^{\psi.h.} Z_1 = \omega_1{}^1 \otimes Z_1, \nabla^{\psi.h.} Z_{\bar{1}} = \omega_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \nabla^{\psi.h.} T = 0$$

in which the 1-form $\omega_1{}^1$ is uniquely determined by the following equation with a normalization condition:

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \omega_1{}^1 + A^1{}_{\bar{1}} \theta \wedge \theta^{\bar{1}} \\ \omega_1{}^1 &+ \omega_{\bar{1}}{}^{\bar{1}} = 0. \end{aligned} \tag{1}$$

The coefficient $A^1{}_{\bar{1}}$ is called the (pseudohermitian) torsion. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}} A^1{}_{\bar{1}} = A^1{}_{\bar{1}}$. And A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Differentiating $\omega_1{}^1$ gives

$$d\omega_1{}^1 = W\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A_{11,\bar{1}}\theta^1 \wedge \theta)$$

where W is the Tanaka-Webster curvature.

- We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_{,1} = Z_1 f$, $f_{1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1{}^1(Z_{\bar{1}}) Z_1 f$ for a (smooth) function f. We define the subgradient operator ∇_b and the sublaplacian operator Δ_b by

$$\nabla_b f = f_{,\bar{1}} Z_1 + f_{,1} Z_{\bar{1}},$$
$$\Delta_b f = f_{,1\bar{1}} + f_{,\bar{1}1},$$

respectively. Moreover *we first define the Levi metric h on* $\ker\theta$ *by*

$$h(X,Y) = d\theta(X, JY).$$

3 CR Geometric Evolution Equations

- References :

  1. . (with J.-H. Cheng) The Harnack Estimate for the Yamabe Flow on CR Manifolds of Dimension 3, Annals of Global Analysis and Geometry Vol. 21, No. 2 (2002), 111-121.
  2. (with H.-L. Chiu and C.-T. Wu) The Li-Yau-Hamilton inequality for Yamabe flow on a closed CR 3-manifold, Transactions of AMS, Vol 362 (2010), 1681–1698.
  3. (with C.-Q. Hu and C.-T. Wu) Li-Yau-Hamilton Inequality for Yamabe Flow on CR 3-Manifolds with Tanaka-Webster Curvature of Change Sign, submitted, 2009.
  4. (with J.-H. Cheng and C.-T. Wu) The Cartan Flow in a Closed Pseudohermitian 3-Manifold with Vanishing Torsion, in preparation.
  5. (with J.-H. Cheng and C.-T. Wu) The Entropy Formulas and its Monotonicity Properties under Coupled Torsion Flow in a Closed Pseudohermitian 3-Manifold, in preparation.

3.1 The Cartan Flow

- Existence of a spherical CR structure :

  1. *Definition :* We call a CR structure J spherical if Cartan curvature tensor Q_{11} vanishes identically. Here

$$Q_{11} = \frac{1}{6} W_{11} + \frac{i}{2} W A_{11} - A_{11,0} - \frac{2i}{3} A_{11,\bar{1}1}.$$

Note that (M, J, θ) is called a spherical pseudohermitian 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant.

2. A closed spherical pseudohermitian 3-manifold (M, J, θ) is locally CR equivalent to the standard pseudohermitian 3-sphere $(S^3, \widehat{J}, \widehat{\theta})$.
3. *The Cartan flow* :
$$\frac{\partial J}{\partial t} = 2Q_J. \tag{2}$$
   (a) i. *Problem* : Existence of spherical CR structure if $A_{11} = 0$?.
       ii. *Problem* : All hyperbolic manifolds admit a spherical CR structure?
   (a) Chang-Cheng-Wu : The long-time existence and asymptotic convergence problrm.
   (b) *Conjecture* : A closed spherical CR 3-manifold with positive Tanaka-Webster curvature is CR equivalent to
$$(S^3/G_1\#...\#S^3/G_q)\#(\#_1^r S^2 \times S^1).$$
   (c) In case of a closed Riemannian 3-manifold with positive scalar curvature. As a consequence of Perelman's result on Ricci flow, M is isomorphic to
$$(S^3/G_1\#...\#S^3/G_q)\#(\#_1^r S^2 \times S^1).$$

3.2 The CR Yamabe Flow

- Classification of a closed spherical CR 3-manifold with positive Tanaka-Webster curvature :

1. Given a contact 3-manifold (M, J, θ), we define the Webster metric $g_\lambda = d\theta + \lambda^{-2}\theta^2$. W : the Tanaka-Webster curvature. R : scalar curvature of g_λ :
$$R^\lambda = 4W - 2\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - 2\lambda^{-2}$$
and if the pseudohermitian torsion A_{11} is vanishing, then
$$(R_{ij}^\lambda) = \begin{pmatrix} 2W - 2\lambda^{-2} & 0 & 0 \\ 0 & 2W - 2\lambda^{-2} & 0 \\ 0 & 0 & 2\lambda^{-2} \end{pmatrix}.$$

2. *The CR Yamabe Flow :*

   (a) i. $A_{11} = 0$, then $W > 0 \Longrightarrow R_{ij}^{\lambda} > 0$ for some λ. Then in case of normalize Ricc flow (NRF) if $g(0)$ has positive Ricci curvature, then the NRF has a solution for a long time and the solution converges to a constant curvature metric. In particular, M has geometric structure. That is

   $$M = (S^3/G_1\#...\#S^3/G_q).$$

   ii. (Chang-Chiu-Wu, 2009) : *spherical CRstructure with positive Webster scalar c and vanishing torsion* $\Longrightarrow \exists$ *constant Tanaka-Webster curvature and vanishing torsion.*($W = C > 0$ and $|A_{\bar{1}\bar{1}}|^2 = 0$. It is spherical).

   $$\partial_t \theta_{(t)} = -2(W - r)\theta_{(t)}.$$

   iii. Conjecture : A closed spherical CR 3-manifold with positive Tanaka-Webster curvature is CR equivalent to

   $$(S^3/G_1\#...\#S^3/G_q)\#(\#_1^r S^2 \times S^1)?$$

   iv. *Problem* : the formation of singularity in a closed spherical CR 3-manifold with positive Tanaka-Webster curvature : *Problem* : Spherical and $W > 0$: The CR Yamabe flow (*Yes :* Riemannian Yamabe flow, Ye).

   v. Harnack-type estimates for the CR Yamabe flow in a closed spherical CR 3-manifold w*ith positive Webster scalar curvatur* and vanishing torsion : Chang-Cheng (2002), Chang-Chiu-Wu (2009).

   vi. Hamilton-Perelman program :

3.3 The Torsion Flow

- 1. (a) The CR Einstein-Hilbert Action

  $$\frac{d}{dt}\int_M W_{J,\theta}\theta\wedge d\theta = -2\int_M |A_{\bar{1}\bar{1}}|^2\,\theta\wedge d\theta - 2\int_M (W - \widehat{W})^2\theta\wedge d\theta$$

  The negative gradient flow :

  $$\begin{cases} \partial_t J_{(t)} = -2JA_{J,\theta} \\ \partial_t \theta_{(t)} = -2(W - \widehat{W})\theta_{(t)}. \end{cases}$$

(b) *The torsion flow* :

i. If $W > |A_{\bar{1}\bar{1}}|^2 \geq 0$, then R^λ is positive for some λ. As a consequence of Perelman's result

$$M = (S^3/G_1\#...\#S^3/G_q)\#(\#_1^r S^2 \times S^1).$$

Problem : (M, J, θ) with positive Tanaka-Webster curvature and

$$W > \ |A_{\bar{1}\bar{1}}|^2 \geq 0.$$

Is the torsion flow

$$\partial_t J_{(t)} = 2JA_{J,\theta},$$

converges to a CR structure with vanishing torsion?

ii. The coupled torsion flow : *The CR analogue of the coupled Ricci flow*

$$\begin{cases} \partial_t J_{(t)} = 2E \ ; \ \partial_t \theta_{(t)} = -2\mu(t)\theta_{(t)} \\ \partial_t \varphi(t) = -2\Delta_b \varphi + |\nabla_b \varphi|^2 - W \end{cases} .$$

(c) *The torsion flow for the geometrization problem of contact* 3-*manifold :*

i. We proposed to deform any fixed CR structure under *the torsion flow* on a three dimensional space which shall break up the space eventually. It should lead to the contact topological decomposition according to *(???)*. The asymptoic state (singularity formation) of *the torsion flow* is expected to be broken up into pieces which will either collapse or produce metrics which satisfy the *spherical CR structure with vanishing torsion*. However, the deformation will encounter singularities. The major question is to find a way to describe *all possible singularities*.