Otto.

Thm: Let (M_i, α_i) be cont mflds.

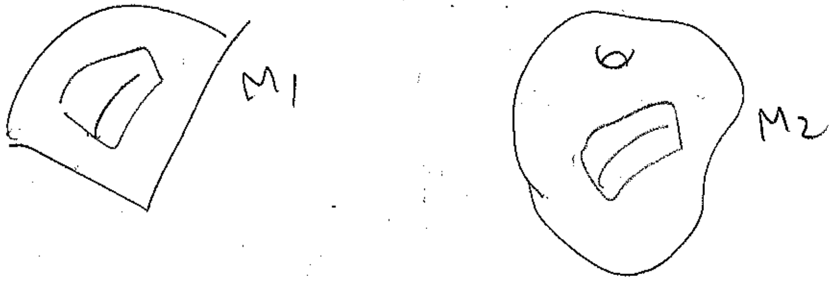
Suppose $L_i \subset M_i$ is an isotropic sbmfld for $i = 1, 2$.

Assume that $\exists$ $\psi_0 : L_1 \to L_2$ diffeo covered by a bdl ma

$$\Psi : CSN(L_1) \to CSN(L_2)$$

Then there is a (strict) contactomorph

$$\tilde{\psi} : \nu(L_1) \to \nu(L_2).$$



Weinstein model:

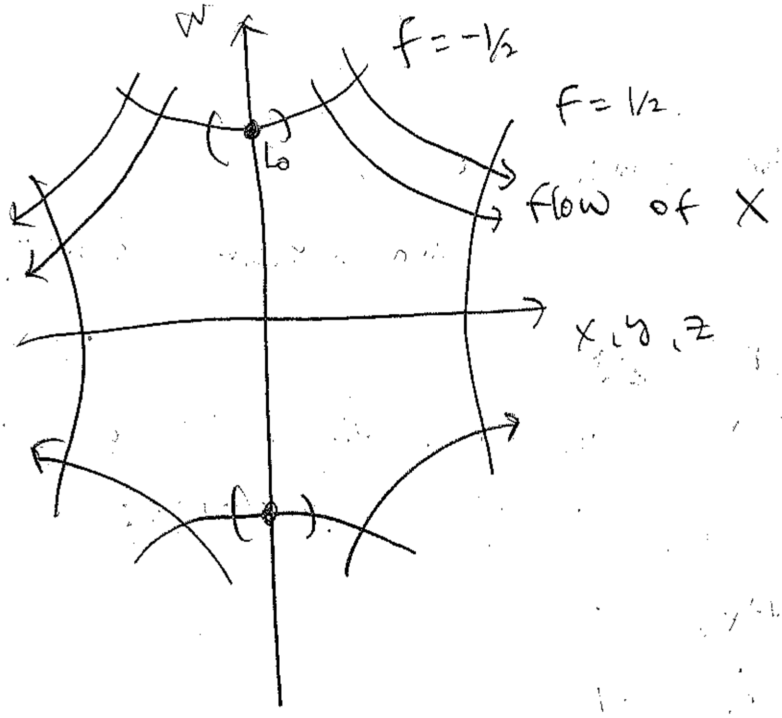
Consider $(\mathbb{R}^{2n}, dx \wedge dy + dz \wedge dw)$

$$\underset{\substack{n-k-1 \;\; n-k-1 \;\; k+1 \;\; k+1}}{(x, y, z, w)}$$

Def: $f := \frac{1}{4}(x^2 + y^2) + z^2 - \frac{1}{2}w^2$

$X := \frac{1}{2}(x\partial_x + y\partial_y) + 2z\partial_z - w\partial_w$

is Liouville and $\pitchfork$ level sets of f $(f \neq 0)$.



Observe: $M_{-1/2} := \{(x, y, z, w) \in \mathbb{R}^{2n} \mid f(x,y,z,w) = -\frac{1}{2}$

is a cont mfd diffeo. to

$S^k \times \mathbb{R}^{2n-k-1}$

$M_{1/2}$ is cont. diffeo to

$\mathbb{R}^{k+1} \times S^{2n-k-2}$

idea: can do surgery by "interpolating" from

$M_{-1/2}$ to $M_{1/2}$.

#

Rmk: L_0 "$\{w^2 = 1\}$" $\subset M_{-1/2}$ is isotropic S^k with

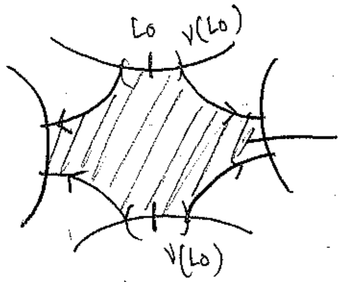
trivial CSN

use $\alpha := 2X^\omega = \frac{1}{2}(\vec{x}\,d\vec{y} - \vec{y}\,d\vec{x}) + 2z\,dw + w\,dz$

Now choose $\overset{\text{isotropic sphere}}{L} \subset (M, \alpha)$ with CSN trivial $\rightsquigarrow$ this tells us: $\exists \psi : \nu(L) \xrightarrow{\text{cont.}} \nu(L_0)$ Weinst

Define a symp. k-hdl.

$$sh^k := \{ (x,y,z,w) \in \mathbb{R}^{2n} \mid -\tfrac{1}{2} \le f(x,y,z,w) \le \tfrac{1}{2} \text{ and } (x,y,z,w) \text{ lies on a flow line thru } \nu(L_0) \}$$

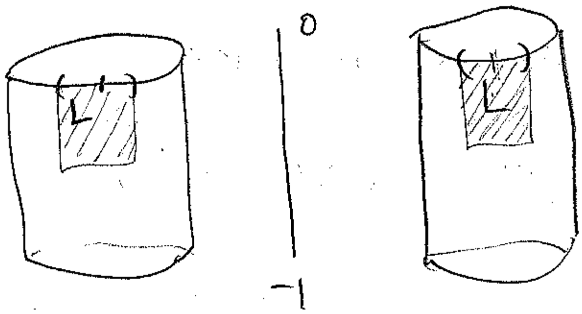


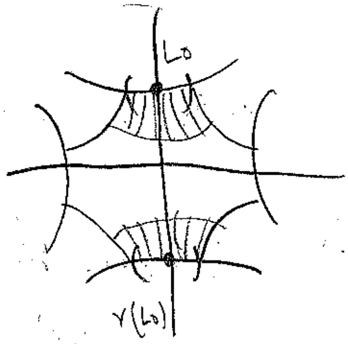
sh^k

(depends on $\nu(L_0)$, i.e. on our choice of $\psi : \nu(L) \to \nu(L_0)$)

Consider the symp. cob.

$$([-1,0] \times M, \ d(e^t \alpha))$$





$M = S^1 \sqcup S^1$ $\quad L = S^0$. $\qquad \psi : \nu(L) \to \nu(L_0)$.

$$\tilde{W} = W \cup sh^k / \sim \qquad (t,p) \in [-\tfrac{1}{2}, 0] \times \nu(L) \sim (x,y,z,w)$$

$$\Leftrightarrow (x,y,z,w) = Fl^X_{t+\frac{1}{2}} \psi(p)$$

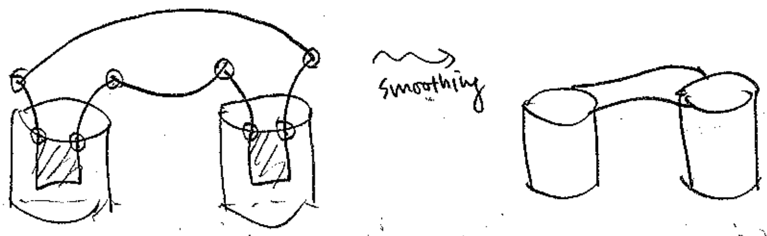
Lemma. $\varphi : (M_0, \alpha_1) \to (M_1, \alpha_1)$ strict contactomorphism.

$$\tilde{\varphi} : \mathbb{R} \times M_0 \to \mathbb{R} \times M_1$$
$$(t,p) \mapsto (t, \varphi(p)) \quad \text{is symplectomorphism.}$$

$$\tilde{\varphi}^* d(e^t \alpha_1) = d(e^t \varphi^* \alpha_1) \underset{\text{strict}}{=} d(e^t \alpha_0).$$

$\Rightarrow$ gluing map is symplectomorphism.

$\Rightarrow \tilde{W}$ is symp. mfd.



$\partial\tilde{W}$ has corners: smooth them by trimming back the boundary.

(watch out near some corners)

keep transversality

$\rightsquigarrow$ on trimmed back bdry. we get induced cont. surg

Rmk: Isotropic submfds satisfy so-called h-principle

Roughly speaking: if topological conditions for finding an isotropic embedding are met, then we can find an isotropic embedding.

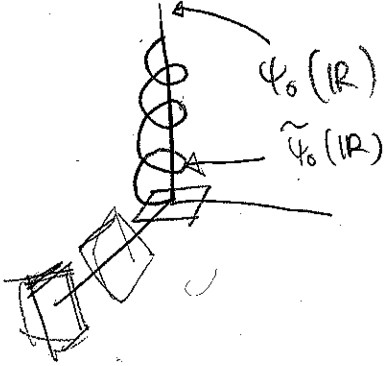
Ex: $\varphi_0 : \overset{L \,\cong}{S^k} \hookrightarrow M$ top. emb.

should be able to cover φ_0 with a bdle map $\Phi : TL \to \xi$. (not the diffe. of φ_0)

$$\begin{tikzcd} TL \arrow[r, "\Psi"] \arrow[d] & \xi \arrow[d] \\ L \arrow[r, "\varphi_0"'] & M \end{tikzcd}$$

h-principle: can homotope φ_0 to $\tilde{\varphi}_0$ C^0-close to φ_0 s.t. $T\tilde{\varphi}_0$ satisfies the bdl map constraint.

Ex:



Application: Let G be finitely presented gp.

$\Rightarrow \exists (M^G, \xi)$ contact s.t. $\pi_1(M^G) \cong G$.

Steps: * Start with S^5.

* Self conn sum (once gives $S^1 \times S^4$)

$\rightsquigarrow$ n times gives M_n, cont. $\pi_1(M_n) = \mathbb{Z} * \cdots * \mathbb{Z}$.

* for relations: choose curves γ s.t $[\gamma] \in \pi_1(M_n)$ to represent a desired relation $g_1, \dots, g_m$.

* realize curves isotropically.
* perform cont. surgery.

Plumbing:

Topologically: Let W_1, W_2 be D^n balls over M_1, M_2

$W_1 =$ 

A fiber of W_1 has nbhd which looks like $D^n \times D^n$

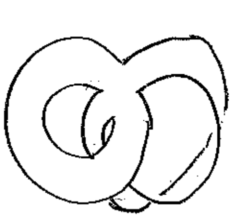
$W_2 =$ 

$\tilde{W} = W_1 \cup W_2 / \sim$

$(x, y) \in (D^n \times D^n) \subset W_1$

$\sim (-y, x) \in D^n \times D^n \subset W_2$.

is called plumbing of W_1, W_2.

 $= T^2 \setminus D^2$

Symplectically:

Let (W_1^{2n}, ω_1), (W_2^{2n}, ω_2) be symp. mfds with convex bdry

Suppose $L_i^n \subset W_i^{2n}$ is Lagr. D^n

s.t. ∂L_i^n is a Legendrian sph. in ∂W_i

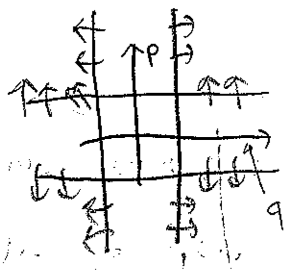
$\overset{\shortparallel}{\text{isotropic of max. dim.}}$

$\Rightarrow \exists\ \nu(L_i) \cong \left(T^* D^n, d(pdq)\right)$, 1-form makes sense.

$T^*D^n \cong \mathring{D}^n \times D^n$

Def $\widetilde{W} := W_1 \mathring{\cup} W_2 / \sim$ $\quad x \sim y$ if $x = (p, q) \in \nu(L_1)$, $y = (-q, p) \in \nu(L_2)$.

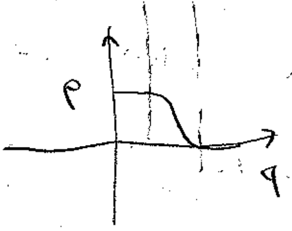
$\Rightarrow$ this is symplectic



Claim:

$X = \left(1 - \rho(|q|^2)\right) q\, \partial_q + \left(\rho(|q|^2) p + 2(q \cdot p) \rho'(|q|^2) q\right) \partial_p$

is Liouville and transverse to ∂



$\Rightarrow \widetilde{W}$ has convex boundary.

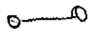
For notation:

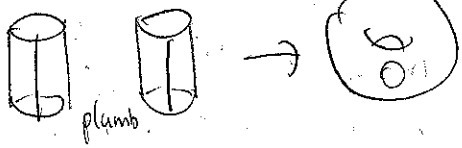
write plumbings as graphs

 $= (T^*S^n, d\lambda_{can})$

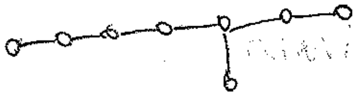
vertex

— = plumbing of vertical fibers

Ex:  $n=1$.



Ex: E_8 plumbing



$n=2 \rightsquigarrow W_{E_8}$

$\partial(W_{E_8})$ = Poincaré homology sphere

n even ≥ 4

$\rightsquigarrow \partial(W_{E_8})$: homotopy sphere which generates the gp of bdry parallelizable exotic spheres.

Open books:

for cont mfds:

Topologically. M smooth mfd.

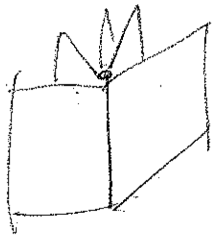
Def: A pair (K, θ) is called an open book if $K \subset M$ is a codim 2 submfd with trivial normal bdl and $\theta : M \setminus K \to S^1$ is fiber bdl s.t. $\theta : \underbrace{K \times D^2}_{\text{trivial normal bdl}} \setminus \{0\} \to \varphi$

$\underset{(r,\varphi)}{\psi}$

Ex: $\mathbb{R}^3$.

$K = z$-axis

$\theta(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$



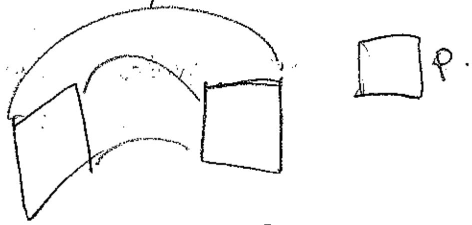
K is called binding

A fiber is called a page of the open book.

Def: An abstract open book is a pair (P, φ) where P is a mfd with boundary and $\varphi: P \to P$ diff that is id near ∂P.

$\rightsquigarrow$ then we can construct

$A = P \times \mathbb{R} / \sim \qquad (x,t) \sim (\varphi(x), t+1)$



A is P-bdl over S^1

$\pi: A \to S^1$.

$B := \partial P \times D^2 \qquad \varphi = id$ near ∂P

$A \cup_\partial B =: X$ has natural open book

Put $K := \partial P \times \{0\}$

$\theta: X \setminus K \to S^1$

$x \mapsto \begin{cases} \varphi & \text{if } x = (p, r, \varphi) \in B \\ \pi(x) & \text{if } x \in A \end{cases}$

Prop. Abstract open book $\Rightarrow$ open book.

Prop. open book $\Rightarrow$ abs. op. book

$\Pi: M \setminus K \to S^1$

$\frac{\partial}{\partial \theta}$ tang. v to S^1

monodromy gives a diffeo $\psi: P = \theta^{-1}(1) \to P = \theta^{-1}(1)$.

Claim: (P, ψ) is the abs. open book.

Abs. cont. op. books:

Let (P, λ) be an exact convex symp mfd. λ is a primit of symp. fo

$\psi: P \to P$ symplecto. that is id near ∂P.

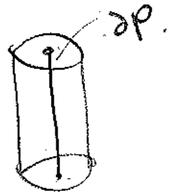
$A := P \times \mathbb{R} / \sim \qquad (x, t) \sim (\psi(x), t + h(x))$

where h comes from

$$\psi^* d\lambda = d\lambda \Rightarrow \psi^* \lambda = \lambda + \underbrace{\mu}_{-dh}$$

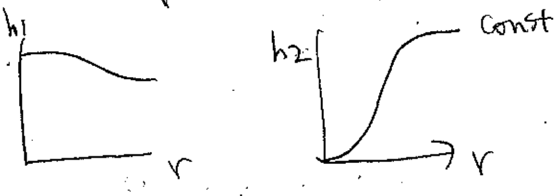
Observe $dt + \lambda$ is preserved under this identificatio

Finally: $B := (\partial P) \times \underset{(r, \varphi)}{D^2}$.



$$\alpha = h_1(r)\, \lambda|_{\partial P} + h_2(r)\, d\varphi.$$

where



$A \cup_{\partial} B$ is abs. o.b. and cont. forms glue to a global cont. form.