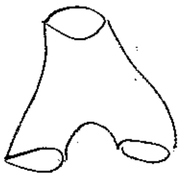
Otto:

Def. $(M, \xi = \ker\alpha)$ is called Stein-fillable if (M, α) arises as level set of a psh fct on (W, J). (Stein mfd)

Rmk: Stein-fillable $\Rightarrow$ strongly fillable $\Rightarrow$ weakly fillable $\Rightarrow$ tight

Thm of Eliashberg, Gromov.

Def: Two oriented manifolds M_1, M_2 are called (oriented) cobordant if $\exists$ cpt mfd W s.t. $\partial W = \overset{\text{opposite orientation}}{\overline{M_1}} \dot\cup M_2$



$(S^1 \dot\cup S^1) \xrightarrow[\text{cob.}]{} S^1$

Rmk.
Cobordance defines equivalence relation.
(can glue cobordisms)

Def: Let (W, ω) be a symp. mfd with bdry. A bdry component $M \subset \partial W$ is called
- convex if $\exists$ Liouville vf pointing outward
- concave ——— inward.

Ex: $([0,1] \times M, d(e^t\alpha))$, (M, α) cont.

$\{0\} \times M$ is concave, $X = \frac{\partial}{\partial t}$ is Liouville. $\{1\} \times M$ is convex.

Def: Let (W,ω) be a symp mfd whose bdry comp are all concave or convex.

Then (W,ω) is called cpt symp. cobordism.

Def: We can complete these cobordisms by attaching
$$\begin{cases} \left((-\infty,0]\times M_i,\ d(e^t\alpha_i)\right) \text{ to concave ends} \\ \left([0,\infty)\times M_j,\ d(e^t\alpha_j)\right) \text{ to convex ends} \end{cases}$$

Rem: symp cobordisms cannot be turned upside-down $\Rightarrow$ symp cobordism only defines a reflexive and transitive relation.
$\uparrow$
can still glue.

Thm $(T^3, \alpha_n = \cos n\theta\, d\varphi_1 + \sin n\theta\, d\varphi_2)$ is not strongly fillable.

idea: $\mathbb{C}^2 \leftarrow$ symp cob. from ϕ to (S^3, α_0)

$L := \{|z_1| = |z_2| = 1\}$ $\leftarrow$ Clifford torus.
(Lag. submfd)

Rmk: $L \subset (W,\omega)$ Lag $\Rightarrow \exists$ nbhd $V(L)$ that is symp to $(T^*L, d\lambda_{can})$.

Put $W := (\mathbb{C}^2 \setminus L)$.

$\rightsquigarrow$ symp cob. from (T^3, α_1) to (S^3, α_0)

λ_{can}.

Take n-fold branched cover (cover θ coord n-times)

$\rightsquigarrow$ get symp from (T^3, α_n) to n copies (S^3, α_0).

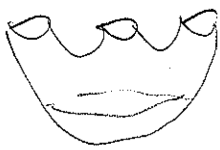
(T^3, α_n)

Now assume (T^3, α_n) is strongly fillable

$\Rightarrow$ attach this filling to W_n

cover.

$\Rightarrow$ get strong filling for n copies of (S^3, α_0)



Thm (Gromov)

Thm (McDuff).

Let (Z, ω) be a symp mfd w/ convex ends, If one end $\equiv (S^3, \alpha_0)$ then ∂Z is connected.

pf uses holomorphic curves

max. prin.

Hdl body thm:

Construct mfds with simple building blocks:

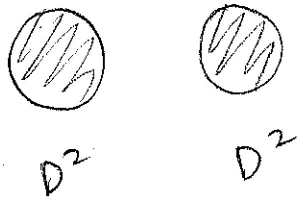
Def: An n-dim k-hdl is $D^k \times D^{n-k}$.

construct mfds inductively:

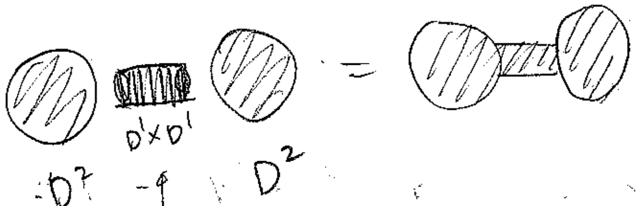
$M_0 = \mathring{\bigcup}$ 0-hdls $(D^n \mathring{\cup} D^n \cup \cdots \mathring{\cup} D^n)$

$M_k = M_{k-1} \cup \mathring{\bigcup} D^k \times D^{n-k} \,/\, \sim$ gluing

M_0



M_1



to glue, choose $\varphi_i^k : (\partial D^k) \times D^{n-k} \hookrightarrow \partial M_{k-1}$

1-hdl

Ex: $\varphi_1^1 : S^0 \times D^1 \hookrightarrow \partial M_0$

$S^0 \times D^1$ = two intervals.

Define $\sim$ by

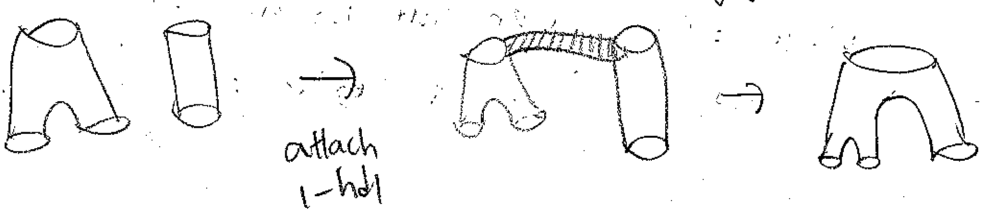
$x \in \partial M_{k-1} \sim y \in (\partial D^k) \times D^{n-k} \subset D^k \times D^{n-k}$ k-handle

$x \Leftrightarrow \varphi_i^k(y)$.

Rmk: can also attach hdls to cobordisms.

choose the gluing maps $\varphi_i^k : \partial D^k \times D^{n-k} \hookrightarrow \partial W$.

where W is the cobordism.

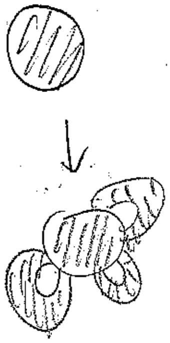


Rmk. Attaching 1-hdls induce conn. sum on bdry, in general attaching k-hdls induces k-surgery on bdry.

Applications. can construct mfds with any given finitely presented gp as fundamental gp.

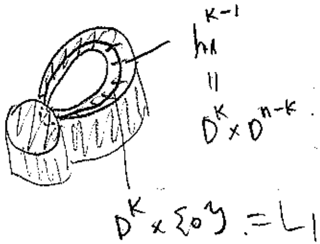
Start with D^n, $n \geq 4$.
$G = \langle g_1, \dots, g_n \mid \underbrace{r_1, \dots, r_m}_{\text{words in } g} \rangle$

First: get $\tilde{G} \cong \mathbb{Z} * \cdots * \mathbb{Z} = \langle g_1, \dots, g_n \rangle$
to get $M_{\tilde{G}}$: attach n 1-hdls to D^n.



observe $\pi_1(M_{\tilde{G}}) = \mathbb{Z} * \cdots * \mathbb{Z}$.

Rmk. hdl cancellation
M_{k-1} be hdl body



choose $i: D^k \hookrightarrow M_{k-1}$
$i(D^k) = L_0$.
can glue L_0 and L_1 to get an embedded $S^k \subset \partial(M_{k-1})$

If this S^k has trivial normal, then we can choose an emb. $\varphi^{k+1}_{L_0 \cup L_1} : (S^k \times D^{n-k-1}) \hookrightarrow \partial M_{k-1}$

$\downarrow$

gluing map from $(k+1)$-hdl $D^{k+1} \times D^{n-k-1}$

Define : $M_{k+1} := M_k \cup D^{k+1} \times D^{m-k-1} / \varphi^{k+1}_{L_0 \cup L_1}$

then we „cancel" h_1^{k-1}

$W \xrightarrow[\text{k-hdl}]{\text{attach}} \tilde{W} \xrightarrow[\text{(k+1)-hdl according to the above}]{\text{attach}} \tilde{\tilde{W}}$. Claim $W \cong \tilde{\tilde{W}}$

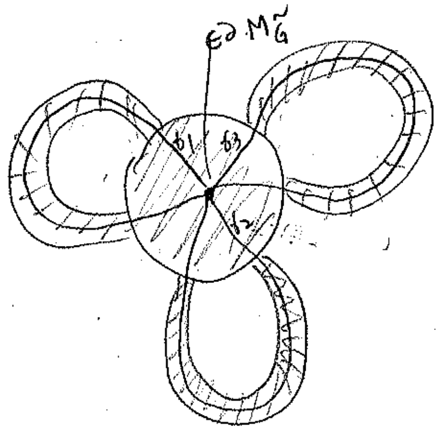
---

back : $\pi_1 = \langle g_1 \ldots g_n \rangle = \tilde{G}$

Ex : suppose we want the relation $r_1 = g_1$

choose an embedding : $\gamma_1 : S^1 \hookrightarrow \partial M_{\tilde{G}}$ s.t.

$[\gamma_i] = g_i$



If we attach a 2-hdl $D^2 \times D^{n-2}$ along γ_i

then we add the relation $r_i = [\gamma_i] = g_1$

To get a relation $r = g_{i_1} \cdots g_{i_p}$.

Choose $\gamma_{i_1} * \cdots * \gamma_{i_p} \subset \partial(M_{\tilde G})$

n-1 dim.

not embedding.

But if $n \geq 4$, can get rid of self-intersection

$\rightsquigarrow$ get smooth emb $\gamma : S^1 \hookrightarrow \partial(M_{\tilde G})$

s.t. $[\gamma] = g_{i_1} \cdots g_{i_p}$.

Attaching a 2-handle along φ_γ

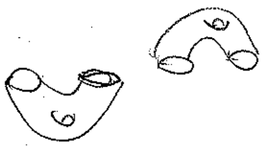
extend to $S^1 \times D^{n-1} \hookrightarrow M_{\tilde G}$

induces the required relation.

M_G is mfd with boundary.

Take the double

$M_G \cup_\partial \overline{M_G}$ is closed. $\pi_1(M_G \cup_\partial \overline{M_G}) = G$.



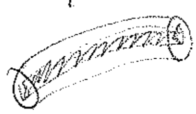
Surgery: Let M^n be a smooth mfd.

Choose an embedding $\varphi : S^k \times D_1^{n-k} \hookrightarrow M$

$$\tilde M := M \setminus \varphi(S^k \times D_{1/2}^{n-k}) \mathring{\cup} (D_1^{k+1} \times S^{n-k-1}) / \sim .$$

radius 1

observes



$$\partial(S^k \times D_1^{n-k}) = S^k \times S^{n-k-1}$$
$$= \partial(D^{k+1} \times S^{n-k-1})$$

where: $x \in \varphi(S^k \times S^{n-k-1} \times [\tfrac{1}{2}, 1]) \subset \varphi(S^k \times D_1^{n-k})$

$\tilde{y} \in S^k \times S^{n-k-1} \times [\tfrac{1}{2}, 1] \subset D_1^{k+1} \times S^{n-k-1}$

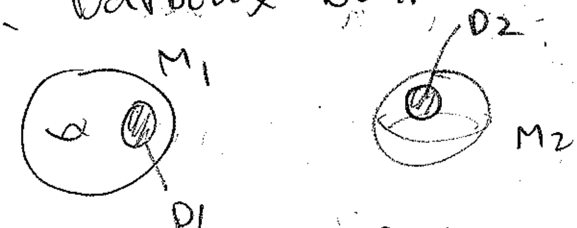
$\Leftrightarrow \varphi^{-1}(x) = y$.

Claim: Let W^n be a cobordism. k-hdl attachment to W induces surgery along $(S^{k-1}\times D^{n-k})$ to $\partial(W^n)$.

Contact surgery: (M_1^{2n-1}, α_1), (M_2^{2n-1}, α_2) cont.

Goal: construct cont. str. on $M_1 \# M_2$.

Take $D_i \subset M_i$. Darboux ball

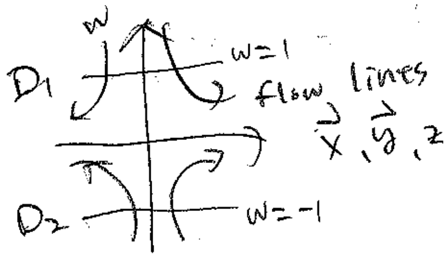


Enough to take conn. sum $D_1 \# D_2$.

Put $W := \Big(\mathbb{R}^{2n}, \omega = \underbrace{d\vec{x}\wedge d\vec{y}}_{(n-1)\text{ pairs}} + dz\wedge dw\Big)$

Embed $D_1 \hookrightarrow W$ at $w = +1$

$D_2 \hookrightarrow W$ at $w = -1$



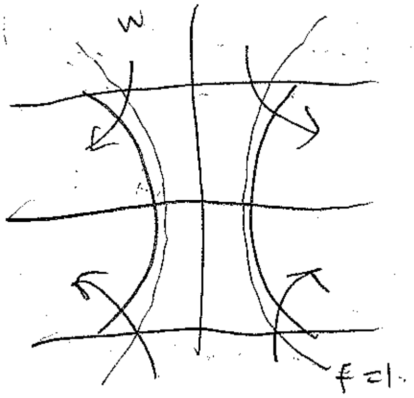
Put $X := \frac{1}{2}\left(\vec{x}\,\partial_{\vec{x}} + \vec{y}\,\partial_{\vec{y}}\right) + 2z\partial_z - w\partial_w$

observe $X \pitchfork D_i$.

$\iota_X \omega = \frac{1}{2}\left(\vec{x}\,d\vec{y} - \vec{y}\,d\vec{x}\right) + w\,dz$

This is std form on D_1, but gives the opposite orientation on D_2.

Now, Conn. D_1 to D_2 via a tube.



$\vec{x}, \vec{y}, z$

To obtain this tube,
Consider $f = \vec{x}^2 + \vec{y}^2 + z^2 - w^2$

$\mathcal{L}_X f \neq 0$ if $f \neq 0$.

Now modify f to $\tilde{f}$ s.t. level sets look like the tube.

$\rightsquigarrow$ get tube = topologically $\cong D_1 \# D_2$.

observe. since X is transverse to tube, we get a cont. str. on $D_1 \# D_2$.

If induces the above std forms as way from the conn. sum part

glue to $M_1 \setminus D_1^{small}$ and $M_2 \setminus D_2^{small}$

Rmk. If (M_1, α_1), (M_2, α_2) are $(--)$-fillable then $(M_1 \# M_2, \alpha_1 \# \alpha_2)$ is also $(-)$ fillable.

Def: $(M^{2n+1}, \xi \overset{\text{ker}\alpha}{=})$ contact

$L \subset M$ is called isotropic if $TL \subset \xi$

Legendrian if L is isotropic and $\dim L = n$.

Let L be isotropic in (M, ξ).

choose $J: \xi \to \xi$ compatible cpx str with $d\alpha$.

Fact $\nu(L) \cong \mathbb{R}R_\alpha \oplus JTL \oplus CSN(L)$

Def: $TL \subset TL^{d\alpha} \subset \xi$

$TL^{d\alpha}/TL = CSN(L)$

conformal sym normal bundle of L.

Thm. Let (M_i, α_i) be contact. $i=1,2$

suppose $L_i \subset M_i$ is an isotropic submfd.

If $\exists$ bdl map $\Psi: CSN(L_1) \to CSN(L_2)$ covering a diffeo. $\Psi_0: L_1 \to L_2$,

then $\exists$ nbhds $\nu(L_1)$, $\nu(L_2)$ and contactomorphism $\tilde{\Psi}: \nu(L_1) \to \nu(L_2)$.