Otto

<u>Contact vector field:</u>

<u>Ex:</u> $(\mathbb{R}^{2n+1}, \alpha_r = dz + \frac{1}{2}(xdy - ydx))$

Then $X = (x, y, 2z) \rightsquigarrow$ Hamiltonian $H = -2z$

<u>Rem</u> Flow increase size of ball $\Rightarrow$ size has no meaning in a contact sense
different from sympl / Riem

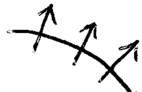
<u>Relation with symplectic manifolds:</u>

Let (W, ω) be a sympl mfd

Let $M \subset W$ be a hypersurface

<u>Def</u>: X is called a Liouvill v.f. if $\mathcal{L}_X \omega = \omega$

<u>Prop</u> If $X \pitchfork M$ then $(M, \xi = \ker(i_X \omega)|_M)$



M

<u>Observation</u> $\mathcal{L}_X \omega = \omega$

$\Leftrightarrow d(i_X \omega) + i_X \underbrace{d\omega}_{0} = 0$

Put $\alpha = i_X \omega$, $\dim W = n$

$$\alpha \wedge d\alpha^{n-1} = (i_X \omega) \wedge d(i_X \omega)^{n-1}$$
$$= i_X \omega \wedge \omega^{n-1}$$
$$= \frac{1}{n} i_X(\omega^n)$$

Restriction to get volume form on M

Def Let (W,ω) be a sympl mfd with $\partial W = M$ s.t. $\exists$ Liouville v.f. pointing outward, then (W,ω) is called a <u>strong symplectic filling</u> of M

Ex $(W = D^{2n},\ \omega = dx\wedge dy)$

$X = \frac{1}{2}(x \frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$

$\iota_X \omega = x\,dy - y\,dx$ is standard contact form on S^{2n-1}

Ex M smooth mfd

$(W = T^*M,\ d\lambda_{can})$ $\quad d\lambda_{can} = dp\wedge dq$

coordinate free definition of λ_{can}

$\pi : T^*M \to M$

$a \in T^*_{\pi(a)}M = \mathrm{Hom}(T_{\pi(a)}M, \mathbb{R})$

$a : T_{\pi(a)}M \to \mathbb{R}$ define canonical 1-form

$\vartheta_a = a\circ T\pi : T_aT^*M \to \mathbb{R}$ $\Rightarrow T_a^*(TM)$

$\rightsquigarrow$ combine them to get $\vartheta = \lambda_{can}$ $\qquad$ <u>Remark</u> $\lambda_{can} = p\,dq$

to define we need a Riem metric g

$(W_{\le 1} = T^*_{\le 1}M,\ d\lambda_{can})$ has Liouville vector field $p\frac{\partial}{\partial p}$

$\Rightarrow (\partial W_{\le 1} = ST^*M,\ \lambda_{can})$ is a contact mfd

Rem The contact structure does not depends on g because of Gray stability

Rem Fix metric get natural contact form α_g. The Reeb flow of α_g corresponds to the geodesic flow

Ex $(T^3,\ \alpha_1 = \cos\theta\, d\varphi_1 + \sin\theta\, d\varphi_2)$ is strong fillable

$(\varphi_1,\varphi_2,\theta)$ $\quad (S^1\times D \times S^1\times D,\ dr_1\wedge d\varphi_1 + dr_2\wedge d\varphi_2)$

$((\varphi_1,r_1),(\varphi_2,r_2))$ $\qquad X = r_1\frac{\partial}{\partial r_1} + r_2\frac{\partial}{\partial r_2}$ is Liouville

Now consider $(T^3,\ \alpha_n = \cos(n\theta)\,d\varphi_1 + \sin(n\theta)\,d\varphi_2)$

Thm (Eliashberg) Only α_1 is strongly fillable.

Rem For strongly ~~fillable~~ fillings W, the symplectic str near ∂W looks like

$((-\varepsilon,0]\times\partial W,\ d(e^t(\iota_X\omega)|_{\partial W}))$

In particular, near ∂W, ω is exact.

Def (W,ω) is called a weak filling for $(M,\xi) = \partial W$

if • $\alpha\wedge d\alpha^{n-1}$ orients M } should coincide

$\iota_\nu\omega^n$ orients M

(ν: outwards normal)

• $\omega|_\xi > 0$, i.e., ω induces same orientation on ξ as $d\alpha$

Ex $(W = T^*M,\ d(\lambda_{can}) + \varepsilon\pi^*\sigma)$ is a weak filling

$\pi : T^*M \to M$

σ closed 2-form on M

If M is compact, then $\exists$ small enough ε s.t. the above is a weak filling for $S(T^*M), \lambda_{can}$

Remark If σ is not exact, then this weak filling is not strong

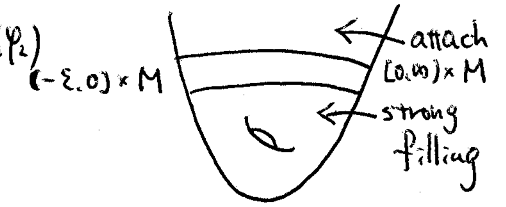
Def (M,α) contact

Then $(\mathbb{R}\times M,\ d(e^t\alpha))$ is symplectic

This sympl mfd is called symplectization

Rem Symplectic field theory studies invariants of symplectic and contact manifold by considering holomorphic curves in symplectizations and cobordisms.

SFT can say something about strong fillings. ~~but nothing about weak filling, because we cannot attach a symplectization.~~



<u>weak fillings</u> for $(T^3,\ \alpha_n = \cos(n\theta)\,d\varphi_1 + \sin(n\theta)\,d\varphi_2)$

$\psi : T^3 \to T^3$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \theta\end{pmatrix} \mapsto \begin{pmatrix}\varphi_1 - \sin(n\theta)\\ \varphi_2 + \cos(n\theta)\\ \theta\end{pmatrix}$$

$\tilde\alpha_n := \psi^*\alpha = \cos(n\theta)\,d\varphi_1 - n\cos^2(n\theta)\,d\theta$
$\quad + \sin(n\theta)\,d\varphi_2 - n\sin^2(n\theta)\,d\theta$
$\quad = \cos(n\theta)\,d\varphi_1 + \sin(n\theta)\,d\varphi_2 - n\,d\theta$

Claim $(T^2\times D^2,\ d\varphi_1\wedge d\varphi_2 - r\,dr\wedge d\theta)$ is a weak filling for $(T^3, \tilde\alpha_n)$

<u>Check</u>: Observe $\xi_n = \ker\tilde\alpha_n = \mathrm{span}\big(X = \cos(n\theta)\frac{\partial}{\partial\varphi_1} + \sin(n\theta)\frac{\partial}{\partial\varphi_2} + \frac{1}{n}\frac{\partial}{\partial\theta},$
$\qquad Y = -\sin(n\theta)\frac{\partial}{\partial\varphi_1} + \cos(n\theta)\frac{\partial}{\partial\varphi_2}\big)$

<u>Check orientation</u>: $\tilde\alpha_n\wedge d\tilde\alpha_n$
$= n\,d\theta\wedge d\varphi_1\wedge d\varphi_2$

Filling gives
$-2r\,d\varphi_1\wedge d\varphi_2\wedge dr\wedge d\theta \to$ restrict to $r=1 \rightsquigarrow -2\,d\theta\wedge d\varphi_1\wedge d\varphi_2$
same orientation

orientation on ξ

$\omega|_\xi > 0$? $\quad d\tilde\alpha_n(X,Y) = 1,\ \omega(X,Y) = 1 \rightsquigarrow \omega|_\xi > 0$

Rem: So called overtwisted contact structure are not even weak fillable

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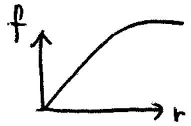
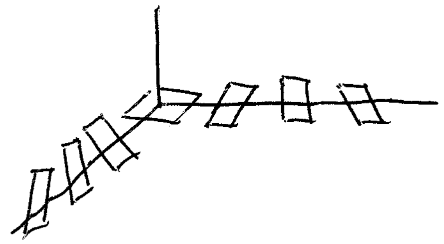
Def (M^3, ξ) is called <u>overtwisted</u> if
$\exists D^2 \overset{i}{\hookrightarrow} M^3$ st the foliation induced by ξ on $T(iD^2)$
looks

$(T(iD^2) \cap \xi)$

Ex $(\mathbb{R}^3, \alpha = \cos(f(r))\,dz + r\sin(f(r))\,d\theta)$





Remark $\exists$ contact structures that are <u>not overtwisted</u> $\underset{= \text{tight}}{}$ ~~⇏~~, but not even weak fillable

<u>Special fillings</u>

<u>Stein mfds</u>:

Def A complex mfd (W, J) is called Stein if it admits a proper holomorphic embedding into $\mathbb{C}^n$

Ex $\mathbb{C}^n$, regular 0-set of polynomial

Def (W, J) almost cpx mfd
Then $f: W \to \mathbb{R}$ is called strictly plurisubharmonic (psh)
if $-d(d^c f)(-, J-)$ is a metric
$(d^c f = (df) \circ J)$

In particular, a psh fct gives as a sympl form $-d(d^c f)$ that is compatible with J

Prop Suppose (W, J) is almost cpx that admits a psh f.
Then regular level sets of f are contact manifold

$\omega := -d(d^c f)$
$i_X \omega = -d^c f$
observe X is Liouville $\quad d(i_X\omega) = -d(d^c f) = \omega$
$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{\mathcal{L}_X \omega}$

$\omega(X, JX) = -d^c f(JX)$
$\vee$
$0 \qquad = -df \circ J(JX) = df(X)$

$\Rightarrow$ X is positively transverse to regular level set.
$\Rightarrow$ use earlier prop to get contact structure.

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Ex $(W = D^{2n} \subset \mathbb{C}^n, J = i)$
$f = \frac{1}{4}(|x|^2 + |y|^2)$ is psh $\qquad df = \frac{1}{2}(x\,dx + y\,dy)$
$= \frac{1}{4}|z|^2$
$\Rightarrow df \circ i = (-x\,dy + y\,dx)\frac{1}{2}$
$\Rightarrow -d(d^c f) = dx \wedge dy$
$\Rightarrow -d(d^c f)(-, i-)$ std metric

Thm (Lefschetz)
Stein mfds (W^{2n}, J) admit psh $\overset{\text{(Morse)}}{\vee}$ fcts whose critical pts have index $\leq n$

Cor: Stein mfds of cpx dim n have the homotopy type of an n-dim CW-cpx
(Very special)

Rem: $\exists$ exotic Stein structures on $\mathbb{R}^{2n}$ whose psh fcts always have critical pts of index n (cf Kaliman)

Thm: (W, J) almost cpx mfd
(Grauert) If W admits a psh fct, then W is Stein.

~~Thm (Eliashberg)~~

Remark Stein manifolds $\overset{\text{of } \dim_{\mathbb{R}} = 2n}{}$ admit a handle body decomposition whose handle bodys have index $\leq n$

Thm (Eliashberg)
(W, J) almost cpx mfd that admits a hdl body decomposition of hdl with index $\leq n$ then J can be deformed into a Stein str.

<u>Singularities</u>:
$f: \mathbb{C}^n \to \mathbb{C}$, $f(0) = 0$
Suppose that 0 is an isolated singularity, $df(0) = 0$ (no other pts in nbd)

Def $\Sigma_f = \{f^{-1}(0)\} \cap S^{2n-1}_\varepsilon$ is called the link of the singularity $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$

Claim Links of singularities carry natural contact structure
basically $h: \mathbb{C}^n \to \mathbb{R}$ is psh
$\qquad z \mapsto \frac{1}{4}|z|^2$

Ex: $f = \sum_{i=1}^n z_i^{a_i}$
Then Σ_f is called a Brieskorn mfd
$\Rightarrow$ Brieskorn mfds are contact

Rem: All exotic spheres in dim 7 are realized by Brieskorn mfds.