No.
Date

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Symplectic linear algebra: finite dimensional

Def: Let V be a real vector space
A symplectic linear form is a skew-symmetric bilineer form that is non-degenerate, i.e.
$$\varphi_\omega : V \longrightarrow V^* \quad \text{is an isomorphism.}$$
$$v \longmapsto \omega(v, \cdot)$$

Prop $(V, \omega) \cong (\mathbb{R}^{2n}, \omega_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$

proof Gram-Schmit
Alternatively $e_1, \dots, e_n, e_{n+1}, \dots e_{2n}$ basis of $\mathbb{R}^{2n}$, then
$\omega_0 = e_1^* \wedge e_{n+1}^* + \dots + e_n^* \wedge e_{2n}^*$

Def (V, ω) sympl vector space
$U \subset V$ subspace
$U^\omega = \{ v \in V \mid \omega(v, u) = 0 \quad \forall u \in U \}$

Rem $U \cap U^\omega = \{0\}$ does not always hold

Def
* $U \subset U^\omega$ then U is called isotropic
* $U \supset U^\omega$ then U is called coisotropic
* $U = U^\omega$ then U is called Lagrangian
  (both isotropic and coisotropic)

Rem This can be done for vector bundle as well
$\rightsquigarrow$ symplectic vector bundle (E, ω) with $E \downarrow M$ ← symplectic form on each fiber

Def A symplectic mfd is a pair (M, ω), where
1) M^{2n} is smooth mfd
2) ω is diff 2-form s.t. (T_pM, ω_p) is a symplectic vector space
3) $d\omega = 0$

Def (M, ω), (N, η) sympl mfds
$\psi : M \longrightarrow N$ diffeomorphism is called a symplectomorphism if $\psi^* \eta = \omega$.

3) $\longleftrightarrow$ "Flatness" condition

In fact

Thm (Darboux) (M, ω) sympl mfd $p \in M$
$\Rightarrow \exists$ nbd $U \ni p$ s.t. $(U, \omega) \cong (V \subset \mathbb{R}^{2n}, \omega_0)$

Use: Symplectomorphism group is Large

cf Riemanian geometry: isometry is rather small

(M,ω) sympl mfd

μ closed 1-form

$i_X\omega = \mu \Rightarrow \exists!$ sol X ($\because \omega$ non-deg)

Observe: X generates symplectomorphism i.e. $\mathcal{L}_X\omega = 0$

$$\mathcal{L}_X\omega \overset{\uparrow}{=} d(\underbrace{\iota_X\omega}_{=\mu}) + \iota_X(\underbrace{d\omega}_{=0}) = 0$$

Cartan's formula

$\Rightarrow (\varphi_t^X)^*\omega = \omega$

Special case:

Let $H: M\to\mathbb{R}$ fct

Then $\mu = dH$ is a closed 1-form

$\rightsquigarrow$ The corresponding vector field $X = X_H$ is called Hamiltonian vector field

$(M = \mathbb{R}\oplus\mathbb{R}\ni_{\,}, \omega = dp\wedge dq)$

Let $H: M\to\mathbb{R}$ be a fct

$\iota_{X_H}\omega = a\,dq - b\,dp = \mu = dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq$

$X_H = a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q}$

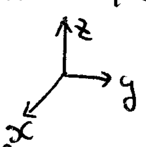
$\Rightarrow X_H = \frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$ Follow flow $\dot p = \frac{\partial H}{\partial q}$, $\dot q = -\frac{\partial H}{\partial p}$ (minus Hamilton equations)

Contact mfds:

Def A contact mfd is a pair (M^{2n+1}, ξ), where

* M^{2n+1} is a smooth mfd
* $\xi \subset TM$ is maximaly non-integrable, i.e., if locally $\xi = \ker\alpha$, then $\alpha\wedge d\alpha^n \neq 0$

Ex $(\mathbb{R}^{2n+1}, \xi_{st} = \ker(dz + \sum_i x_i dy_i))$



Def ξ is called contact structure

The defining form is called contact form

Suppose we have a contact form α

Then

Def The Reeb field is defined by

$\iota_{R_\alpha}d\alpha = 0$, $\iota_{R_\alpha}\alpha = 1$

$d\alpha$ is maximaly non-deg by contact condition, still 1-dim ker

Ex $(\mathbb{R}^{2n+1}, \alpha_0 = dz + x\,dy)$ has Reeb field $R_{\alpha_0} = \frac{\partial}{\partial z}$

Exercise Sympl linear algebra

(V,ω) vector space and ω skew-symm bilinear form ($\omega\in\Lambda^2V$)

Then $\omega^n \neq 0 \iff \omega$ is sympl

Rem Given a contact ~~form~~ structure ξ on M, a defining contact form is not unique: α is a contact form $\Rightarrow f\alpha$ is also $f\neq 0$

Exercise Try to compare Reeb field of $f\alpha$

Existence of contact form

(M,ξ) contact $\xi\subset TM$

$$0\to\xi\to TM\xrightarrow{\alpha} L\to 0$$

"contact form" ξ; some line bdl

Claim: $\alpha\in\Omega^1\otimes L$ defines well-defined $\alpha\wedge d\alpha^n \in \Omega^{2n+1}\otimes L^{n+1}$

The contact condition is then given by $\alpha\wedge d\alpha^{n+1}\neq 0$ as a section of $\Omega^{2n+1}\otimes L^{n+1}$

Existence of contact form $\iff$ coorientable contact structure because L is then trivial

Prop: L is trivial $\iff \exists$ global contact ~~str~~ form

Rem $(M, \xi = \ker\alpha)$

Then $(\xi, d\alpha)$ sympl vb

Prop $L \subset (M^{2n+1}, \xi)$ s.t. $TL \subset \xi$ then $\dim L \leq n$

proof $TL\subset\xi \Rightarrow i: L\hookrightarrow M$ so $i^*\alpha = 0 \Rightarrow i^*d\alpha = 0$

So TL is an isotropic subbdl of $\xi \Rightarrow$ claim

$$\begin{array}{cc} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} & \begin{array}{c} \}n \\ \}n \end{array} \end{array}$$

cf Foliations $\ker\alpha$, $\alpha\wedge d\alpha = 0$

foliations admit integrable submfds of higher dimension

Contact $(\mathbb{R}^3, dz + x\,dy)$ integrable submfd of dim 1

Remark. In foliations, it is not always possible to go from one pt to another.

* (M^{2n+1},ξ) $n\geq 1$ contact mfd, $p,q\in M$, then there is an isotropic path connecting p and q

Def $(M,\xi=\ker\alpha)$, $(N,\eta=\ker\beta)$ are contact

$\psi: M\to N$ diffeomorphism is called contactomorphism if $\psi^*\beta = f\alpha$, $f\neq 0$

ψ is called strict contactomorphism if $f=1$.

Thm (Gray) M closed mfd, α_t a family of contact forms $t\in[0,1]$

Then $\exists$ isotopy $\psi_t: M\to M$ s.t. $\psi_t^*\alpha_t = f_t\alpha_0$ $f_t > 0$.

Thm (Darboux) $(M, \xi = \ker \alpha)$ contact $p \in M$

$$\Rightarrow \exists \text{ nbd } U \ni p \text{ s.t. } (U, \alpha) \underset{\text{strict}}{\cong} (V \subset \mathbb{R}^{2n+1}, \overset{\alpha_0}{\overbrace{dz + \sum_i x_i dy_i}})$$

Rem $\alpha_r = dz + c(x dy - y dx)$ is also called standard form

$c = \frac{1}{2}, 1$

Exercise $(\mathbb{R}^{2n+1}, \alpha_0) \cong (\mathbb{R}^{2n+1}, \alpha_r)$

Idea Group of contactomorphism is large

$H : M \to \mathbb{R} \rightsquigarrow$ can ~~also~~ solve X_H satisfying

$$\iota_{X_H}\alpha = -H \quad \iota_{X_H} d\alpha = dH - (\iota_{R_\alpha} dH)\alpha \Rightarrow \mathcal{L}_{X_H}\alpha = d(i_{X_H}\alpha) + \iota_{X_H} d\alpha$$

i.e., flow preserve contact str.