8.1 Arc Length

Arc Length

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve.

We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for the concepts of area and volume.

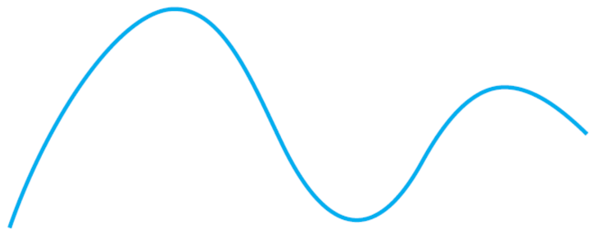

Figure 1

Arc Length

If the curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment).

We are going to define the length of a general curve by first approximating it by a polygon and then taking a limit as the number of segments of the polygon is increased.

Arc Length

This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

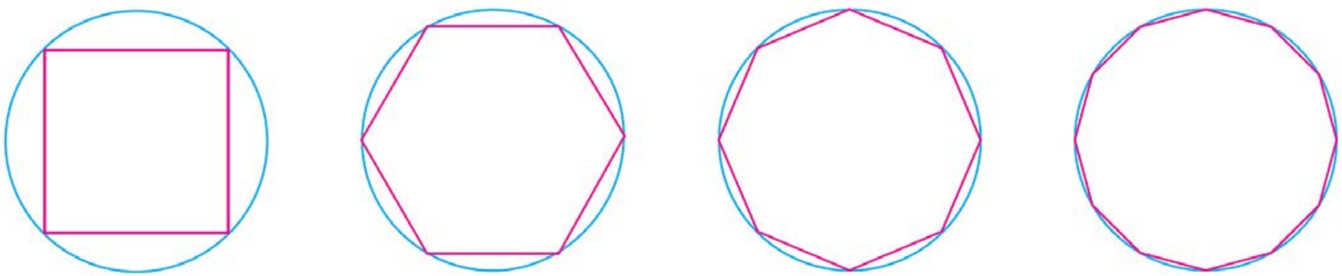

Figure 2

Suppose that a curve C is defined by the equation $y = f(x)$ where f is continuous and $a \leq x \leq b$.

Arc Length

We obtain a polygonal approximation to C by dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$, illustrated in Figure 3, is an approximation to C.

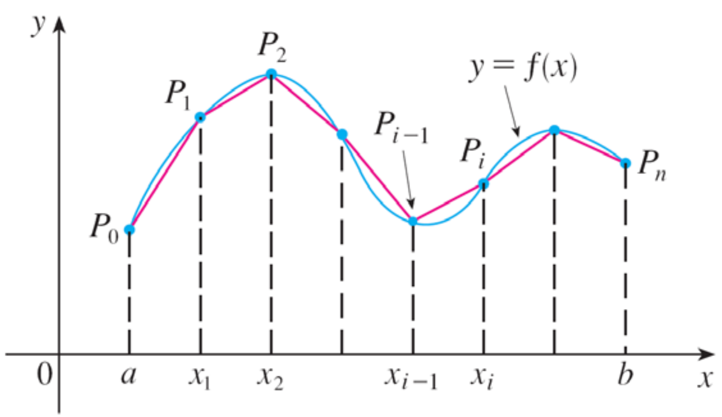

Figure 3

Arc Length

The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. (See Figure 4, where the arc of the curve between P_{i-1} and P_i has been magnified and approximations with successively smaller values of Δx are shown.)

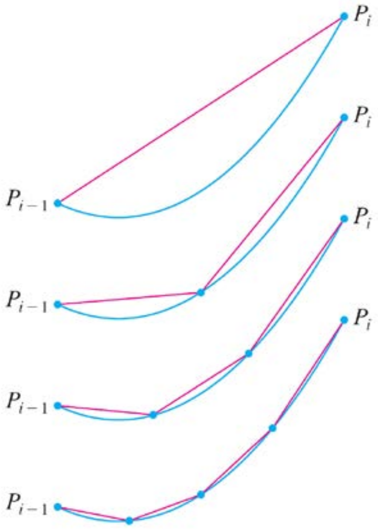

Figure 4

Arc Length

Therefore we define the **length** L of the curve C with equation, $y = f(x)$, $a \le x \le b$ as the limit of the lengths of these inscribed polygons (if the limit exists):

1

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$.

Arc Length

The definition of arc length given by Equation 1 is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. [Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Arc Length

By applying the Mean Value Theorem to f on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is, $$\Delta y_i = f'(x_i^*)\,\Delta x$$

Thus we have

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\,\Delta x]^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\,\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x \quad \text{(since } \Delta x > 0\text{)}$$

Arc Length

Therefore, by Definition 1,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

We recognize this expression as being equal to

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

by the definition of a definite integral. We know that this integral exists because the function $g(x) = \sqrt{1 + [f'(x)]^2}$ is continuous.

Arc Length

Thus we have proved the following theorem:

2 **The Arc Length Formula** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

3

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1

Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)

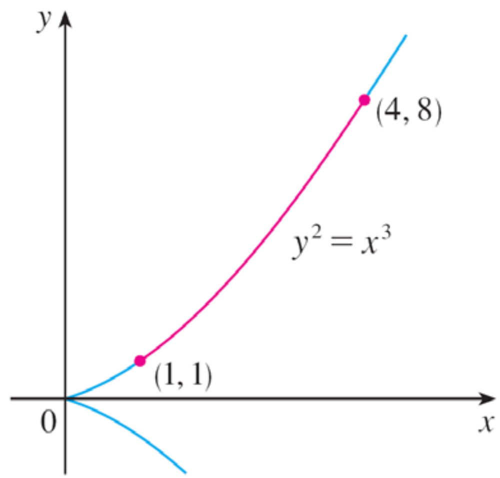

Figure 5

Example 1 – *Solution*

For the top half of the curve we have

$y = x^{3/2}$ $\qquad \dfrac{dy}{dx} = \frac{3}{2}x^{1/2}$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}\, dx$.

When $x = 1$, $u = \frac{13}{4}$; when $x = 4$, $u = 10$.

Example 1 – *Solution*

cont'd

Therefore

$$L = \frac{4}{9}\int_{13/4}^{10}\sqrt{u}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big]_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Arc Length

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in Formula 2 or Equation 3, we obtain the following formula for its length:

4

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The Arc Length Function

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve.

Thus if a smooth curve C has the equation, $y = f(x)$, $a \leq x \leq b$ let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.

Then s is a function, called the **arc length function**, and, by Formula 2,

5 $$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

The Arc Length Function

(We have replaced the variable of integration by *t* so that *x* does not have two meanings.) We can use Part 1 of the Fundamental Theorem of Calculus to differentiate Equation 5 (since the integrand is continuous):

$$\boxed{6} \qquad \frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Equation 6 shows that the rate of change of *s* with respect to *x* is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0.

The Arc Length Function

The differential of arc length is

7
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and this equation is sometimes written in the symmetric form

8
$$(ds)^2 = (dx)^2 + (dy)^2$$

The geometric interpretation of Equation 8 is shown in Figure 7. It can be used as a mnemonic device for remembering both of the Formulas 3 and 4.

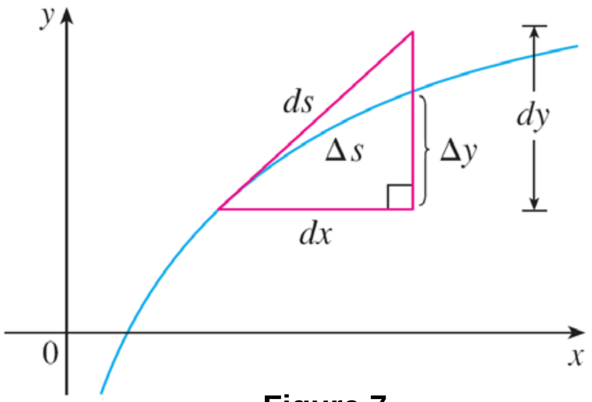

Figure 7

The Arc Length Function

If we write $L = \int ds$, then from Equation 8 either we can solve to get (7), which gives (3), or we can solve to get

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

which gives (4).

Example 4

Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.

Solution:

If $f(x) = x^2 - \frac{1}{8} \ln x$, then $f'(x) = 2x - \frac{1}{8x}$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x} \qquad \text{(since } x > 0\text{)}$$

Example 4 – *Solution*

cont'd

Thus the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \tfrac{1}{8} \ln t \Big]_1^x$$

$$= x^2 + \tfrac{1}{8} \ln x - 1$$

For instance, the arc length along the curve from (1, 1) to (3, *f*(3)) is

$$s(3) = 3^2 + \tfrac{1}{8} \ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$