1. Irreducible Scheme

A topological space X is irreducible if X is nonempty and can not be a union of two proper closed subsets. A nonempty closed subset Z of X is irreducible if Z equipped with the induced topology from X is an irreducible topological space. A point ξ of an irreducible closed subset Z of X is called a generic point if $Z = \overline{\{\xi\}}$.

Proposition 1.1. A topological space X is irreducible if and only if the intersection of any two nonempty open subsets of X is nonempty.

Proof. Suppose that X is irreducible. Assume that there exist two nonempty open subsets U_1, U_2 so that $U_1 \cap U_2 = \emptyset$. Denote $Z_i = X \setminus U_i$ for $i = 1, 2$. Then Z_1 and Z_2 are proper closed subsets of X. Since $U_1 \cap U_2 = \emptyset$, $Z_1 \cup Z_2 = X$. This leads to a contradiction that X is irreducible. Hence any two nonempty open subsets of X is nonempty.

Conversely, suppose that the intersection of any two nonempty open subsets of X is nonempty. Assume that $X = Z_1 \cup Z_2$ is a union of closed subsets of X. Denote $U_i = X \setminus Z_i$ for $i = 1, 2$. Then $U_1 \cap U_2 = \emptyset$. This shows that either $U_1 = \emptyset$ or $U_2 = \emptyset$. This is equivalent to say that $Z_1 = X$ or $Z_2 = X$. Hence X is irreducible.

□

Definition 1.1. A scheme is irreducible if its underlying topological space is irreducible.

Lemma 1.1. Let $X = \operatorname{Spec} A$ be the spectrum of a ring A and $D(f)$ be the distinguished open set associated with $f \in A$. Then $D(f) = \emptyset$ if and only if f is nilpotent.

Proof. Let f be nilpotent. Then $f^n = 0$ for some $n > 0$. In other words, $f^n \in \mathfrak{p}$ for all $\mathfrak{p} \in X$. Since $\mathfrak{p}$ is prime, $f \in \mathfrak{p}$. We see that $f \in \mathfrak{p}$ for all $\mathfrak{p} \in X$. Hence $\mathfrak{p} \in V(f)$ for all $\mathfrak{p} \in X$. We obtain $X = V(f)$. Hence $D(f) = \emptyset$.

Conversely, suppose $D(f) = \emptyset$. Then $X = V(f)$. This shows that $f \in \mathfrak{p}$ for all $\mathfrak{p}$. Hence $f \in \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = \operatorname{Nil}(A)$. (The nilradical of A.) Hence f is nilpotent.

□

Proposition 1.2. Let $X = \operatorname{Spec} A$ be the spectrum of a ring A. Then X is irreducible if and only if its nil radical $\operatorname{Nil}(A)$ is a prime ideal.

Proof. Let us assume that $\xi = \operatorname{Nil}(A)$ is a prime ideal. Then the $\operatorname{Nil}(A)$ is the intersection of all prime ideals of A. Hence $\mathfrak{p} \in V(\xi)$ for all $\mathfrak{p} \in X$. Hence $X = V(\xi)$. Notice that $V(\xi)$ is a closed subset of X containing ξ. Then $V(\xi)$ contains the closure of ξ. Moreover, if $V(I)$ is a closed subset of X containing ξ, then $\xi \supset I$. Thus $V(\xi) \subset V(I)$. This implies that $V(\xi)$ is in fact the closure of ξ. In other words, ξ is the generic point. Hence all nonempty open subsets of X contain ξ. Hence the intersection of any two nonempty open subsets of X contains ξ.

Suppose that X is irreducible. Let $fg \in \operatorname{Nil}(A)$. To show that $\operatorname{Nil}(A)$ is a prime, we need to show that either $f \in \operatorname{Nil}(A)$ or $g \in \operatorname{Nil}(A)$. Then $D(fg) = D(f) \cap D(g)$. If both $D(f)$ and $D(g)$ are nonempty, then f and g are not nilpotent. Since X is irreducible and $D(f)$ and $D(g)$ are nonempty open subsets of X, $D(f) \cap D(g) = D(fg)$ is nonempty. This implies that fg is not nilpotent, i.e. $fg \notin \operatorname{Nil}(A)$. This leads to the contradiction to the assumption that $fg \in \operatorname{Nil}(A)$. Thus either $D(f) = \emptyset$ or $D(g) = \emptyset$. Hence either f or g is nilpotent, i.e. $f \in \operatorname{Nil}(A)$ or $g \in \operatorname{Nil}(A)$. □

Proposition 1.3. Let $X = \operatorname{Spec} A$. For $\mathfrak{p} \in \operatorname{Spec} A$, $V(\mathfrak{p})$ is an irreducible closed subset. Conversely, every irreducible closed subset of X is of the form $V(\mathfrak{p})$ for some $\mathfrak{p} \in X$.

Proof. Let $Z = V(I)$ be a nonempty closed subset of X where I is a radical ideal of X. Suppose I is not prime. Then we can choose $f, g \in A$ so that $fg \in I$ but $f, g \notin I$. In this case, we consider $(f) + I$ and $(g) + I$. Then $V((f) + I)$ and $V((g) + I)$ are proper closed subsets of $V(I)$. Moreover,

$$V((f) + I) \cup V((g) + I) = V((fg) + I) = V(I) = Z.$$

This implies that Z is not irreducible.

If Z is not irreducible, Z is a union of two proper closed subsets $V(I_1)$ and $V(I_2)$. Assume that I_1 and I_2 are radical ideals of A. Then $I_1 \supset I$ and $I_2 \supset I$. Choose $f_i \in I_i \setminus I$. Then $(f_i) + I \supset I$ for $i = 1, 2$. Then $V((f_i) + I) \supset V(I_i)$. We see that

$$V((f_1 f_2) + I) = V((f_1) + I) \cup V((f_2) + I) = V(I).$$

Then $\sqrt{(f_1 f_2) + I} = I$. Since $(f_1 f_2) \subset (f_1 f_2) + I \subset \sqrt{(f_1 f_2) + I}$, we see that $f_1 f_2 \in I$. This shows that I is not a prime. □

If $\mathfrak{p}$ is a prime ideal of a ring A, then $V(\mathfrak{p})$ is the closure of $\mathfrak{p}$ in $\operatorname{Spec} A$.

Proposition 1.4. Let X be a scheme. Suppose X is irreducible. Then X has a unique generic point.

Proof. Suppose $X = \operatorname{Spec} A$. We have seen that $\xi = \operatorname{Nil}(A)$ is the unique generic point.

Since X is irreducible, any affine open subset U of X is also irreducible. Suppose X (not necessarily affine) has two generic points ξ and η. Then ξ and η are also generic points of U of X. Since U is affine, $\xi = \eta$ by the uniqueness of the generic point of an affine scheme. Hence we proved the uniqueness of generic point of a scheme. Now let us prove the existence.

The affine open subsets of X forms a basis for the Zariski topology of X. Since X is irreducible, the intersection of any two nonempty affine open subsets of X are nonempty. Hence there exists a nonempty affine open set contained in the intersection of any two nonempty affine open subsets of X. Let U, V be nonempty affine open sets. Choose ξ and η so that ξ and η are generic point of U and V respectively. Choose W nonempty affine open so that $W \subset U \cap V$. Then we know both ξ and η are generic point of W. By uniqueness of the generic point of W, $\xi = \eta$. This proves that all the affine open subsets of X share the same generic point ξ. Now let us prove that ξ is the generic point of X.

Let x be any point of X and U be any open neighborhood of x. Since X is a scheme, we can choose an affine open neighborhood V of x contained in U. Since ξ is a generic point of V, U contains ξ. Hence x lies in the closure of $\{\xi\}$. Thus $X = \overline{\{\xi\}}$.

□

Proposition 1.5. Let X be a scheme. Any irreducible closed subset of X has a unique generic point.

Proof. Let Z be an irreducible closed subset of X. For any affine open subset U of X, we consider the intersection $U \cap Z$. Either $U \cap Z$ is empty or $U \cap Z$ is nonempty. Suppose that $U \cap Z$ is nonempty. Then $U \cap Z$ is an irreducible closed subset of U. Write $U = \operatorname{Spec} A$. Then $U \cap Z = V(I)$ for some ideal I. Since $U \cap Z$ is irreducible, $I = \mathfrak{p}$ is a prime ideal. Then $V(\mathfrak{p})$ is the closure of $\mathfrak{p}$ in $U \cap Z$. Hence $\mathfrak{p}$ is the generic point of $U \cap Z$. Let ξ be the generic point of $U \cap Z$. Similar we can show that $U \cap Z$ share the same generic point ξ for all affine open subset U of X. Using the fact that affine open subsets forms a base for the Zariski topology of X, ξ is the generic point of Z. (The detailed proof of these statements is the same as that of the above proposition.)

□

Proposition 1.6. Let X be a scheme. T.F.A.E.

(1) X is irreducible.
(2) There exists an affine open covering $\{U_i\}$ so that U_i is irreducible and $U_i \cap U_j \neq \emptyset$ for all i, j.
(3) X is nonempty and every nonempty affine open $U \subset X$ is irreducible.

Proof. (1) implies (2). Since X is irreducible, it has a unique generic point ξ. Since X is a scheme, we can choose an affine open covering $\{U_i\}$ for X so that U_i are all nonempty. Since U_i is nonempty, U_i contains the generic point ξ for all i. This implies that $U_i \cap U_j$ contains ξ and hence is nonempty. Moreover any subspace of irreducible topological space is also irreducible. Hence U_i is irreducible. Similarly, (1) implies (3).

(2) implies (1). Let $X = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed subsets of X. For any U_i, either $U_i \cap Z_1 = \emptyset$ or $U_i \cap Z_2 = \emptyset$. Hence either $U_i \subset Z_1$ or $U_i \subset Z_2$. We may assume that there exists U_i so that $U_i \subset Z_1$. Since U_i is irreducible, $U_i \cap Z_1 = U_i$ is an irreducible closed subset of U_i. Then $U_i \cap Z_1$ has a unique generic point ξ. For any U_j, we know that $U_i \cap U_j$ is nonempty by assumption and hence contains ξ. Then $U_i \cap U_j$ is densed in U_j, and contained in $Z_1 \cap U_j$. Notice that $Z_1 \cap U_j$ is a closed set containing ξ, then $Z_1 \cap U_j$ contains the closure of ξ in U_j. Since $U_i \cap U_j$ is dense in U_j, the closure of ξ in U_j is U_j. Hence $U_j \subset Z_1 \cap U_j \subset U_j$. We conclude that $U_j \subset Z_1$. Hence $X = \bigcup_i U_i \subset Z_1$ which implies that $X = Z_1$. Thus X is irreducible.

(3) implies (2). Since X is nonempty, we can choose an affine open covering $\{U_i\}$ of X so that U_i are all nonempty. By assumption, U_i are all irreducible. Now we want to show that $U_i \cap U_j \neq \emptyset$. Suppose $U_i \cap U_j$ is an empty set. Since U_i and U_j are affine, so is $U_i \coprod U_j$. Since both U_i and U_j are nonempty, $U_i \coprod U_j$ is also nonempty. Since every affine open subset of X is irreducible, $U_i \coprod U_j$ is irreducible but this is impossible. Hence $U_i \cap U_j$ is nonempty for any i, j.

□