EXTERNAL DIRECT SUM AND INTERNAL DIRECT SUM OF VECTOR SPACES

1. Direct Sum of Vector Spaces

Let V and W be vector spaces over a field F. On the cartesian product

$$V \times W = \{(v, w) : v \in V, w \in W\}$$

of V and W, we define the addition and the scalar multiplication of elements as follows. Let (v, w) and (v_1, w_1) and (v_2, w_2) be elements of $V \times W$ and $a \in F$. We define

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad a \cdot (v, w) = (av, aw).$$

Lemma 1.1. $(V \times W, +, \cdot)$ forms a vector space over F and is denoted by $V \oplus_e W$.

Proof. This is left to the reader as an exercise. □

Definition 1.1. The vector space $V \oplus_e W$ over F defined above is called the external direct sum of V and W.

Let Z be a vector space over F and X and Y be vector subspaces of Z. Suppose that X and Y satisfy the following properties:

(1) for each $z \in Z$, there exist $x \in X$ and $y \in Y$ such that $z = x + y$;
(2) $X \cap Y = \{0\}$.

In this case, we write $Z = X \oplus_i Y$ and say that Z is the internal direct sum of vector subspaces X and Y.

Theorem 1.1. Let X and Y be vector subspaces of a vector space Z over F such that Z is the internal direct sum of X and Y, i.e. $Z = X \oplus_i Y$. Then there is a linear isomorphism from Z onto $X \oplus_e Y$, i.e. $X \oplus_i Y$ is isomorphic to $X \oplus_e Y$.

Proof. Define $f : X \oplus_e Y \to Z$ by $f(x, y) = x + y$. Then f is a linear map. (Readers need to check). Since Z is the internal direct sum of X and Y, for any $z \in Z$, there exist $x \in X$ and $y \in Y$ such that $z = x + y$. Hence $f(x, y) = z$. This proves that f is surjective. To show that f is injective, we check that $\ker f = \{(0, 0)\}$. Let $(x, y) \in \ker f$. Then $f(x, y) = x + y = 0$. We see that $x = -y$ in Z. Therefore $x = -y \in X \cap Y = \{0\}$ (Z is the internal direct sum of X and Y.) We find $x = y = 0$. Hence $(x, y) = (0, 0)$. We conclude that $f : X \oplus_e Y \to Z$ is a linear isomorphism. □

Since $X \oplus_i Y$ is isomorphic to $X \oplus_e Y$, if $X \cap Y = \{0\}$, we do not distinguish $X \oplus_i Y$ and $X \oplus_e Y$ when $X \cap Y = \{0\}$ and X, Y are vector subspaces of Z. We use the notation $X \oplus Y$ for both of them when $X \cap Y = \{0\}$. We call $X \oplus Y$ the direct sum of X and Y for simplicity.