Functional Analysis

96.10.1

1. Show that a normed linear space X is complete if every absolutely convergent series is convergent in X. (14%)

2. Prove that if X is a normed linear space and $x \in X$, then $\|x\| = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \leq 1\}$. (12%)

3. Prove that an orthonormal sequence $\{\varphi_n\}$ is complete in $L^2(a,b)$ if $\sum_{n=1}^{\infty} \left(\int_a^x \varphi_n(t)\,dt \right)^2 = x - a$ for all $x \in (a,b)$. (16%)

4. Let X and Y be Banach spaces and let $A \in \mathcal{B}(X,Y)$. Prove that there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in X$ if and only if $\ker A = \{0\}$ and ran A is closed. (16%)

5. Let T be a compact linear operator from a Banach space X onto itself. Show that if T^{-1} is a bounded operator, then X is finite-dimensional. (12%)

6. Let A be a real symmetric $n \times n$ matrix. Consider A as an operator in $\mathbb{R}^n$ given by $x \to Ax$. Prove that $\|A\| = \max_j |\lambda_j|$, where λ_j are the eigenvalues of A. (15%)

7. Let X and Y be normed linear spaces and $T \in \mathcal{B}(X,Y)$. Prove that if $\{x_n\}$ is a sequence in X that is weakly convergent to x_0, then $\{Tx_n\}$ is weakly convergent to Tx_0. (15%)