1. (15 points)
   (a) Give an example of a function that is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$.
   (b) Give an example of a function that is in $L^1((0,1))$ but not in $L^2((0,1))$.
   (c) Prove that any function $f \in L^1(I) \cap L^2(I)$ for any interval $I \subset \mathbb{R}$ must be in $L^p(I)$ for all p between 1 and 2.

2. (20 points) Suppose $f \in L^1(\mathbb{R})$.0 For each $x \in \mathbb{R}$, let $g(x) = \int_{\mathbb{R}} e^{-ixy^2} f(y)dy$.
   (a) Prove that the integral exists for every x.
   (b) Prove that g is a continuous function.
   (c) Prove that there is a dense subset S of $L^1(\mathbb{R})$ such that if $f \in S$, then $\lim_{|x|\to\infty} g(x) = 0$.
   (d) Prove that if $f \in L^1(\mathbb{R})$, then $\lim_{|x|\to\infty} g(x) = 0$

3. (10 points) Let $(X, \mathscr{A}, \mu)$ be a measure space. Let f be a positive integrable function on X. Prove that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any $A \in \mathscr{A}$, if $\mu(A) \leq \delta$ then $\int_A f d\mu \leq \varepsilon$.

4. (10 points) Prove that there exists an orthonormal basis
   $\mathscr{B} = \{f \in L^2([0,1]) \mid \int_0^1 \frac{|f(x)|\,dx}{x} < \infty, \text{and} \int_0^1 \frac{f(x)\,dx}{x} = 0\}$
   of $L^2([0,1])$.

   Hint: (i) Consider $\mathscr{S} = \{f \in L^2([0,1]) \mid \int_0^1 \frac{|f(x)|\,dx}{x} < \infty\}$, and let T be an operator defined on $\mathscr{S}$ by $Tf = \int_0^1 \frac{f(x)\,dx}{x}$ for each $f \in \mathscr{S}$.
   (ii) Consider, for each $n \in \mathbb{N}$, the characteristic function $g_n = \chi_{[1/n,1]}$ of $[\frac{1}{n}, 1]$.

5. (20 points) Let $(X, \mathscr{B}, \mu)$ be a finite measure space. Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions in $L^1(\mu)$, converging almost everywhere to an $L^1(\mu)$ function f. Suppose also that $\lim_{n \to \infty} \|f_n\|_1 = \|f\|_1$.

   (a) Prove that for every measurable set A, $\lim_{n \to \infty} \int_A |f_n| d\mu = \int_A |f| d\mu$.

   (b) Prove that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$.

6. (10 points) Let $C \in (0,1)$. For each $N \in \mathbb{N}$, show that there exists a δ_N, depending on C, with the following properties:

   (a) If $A_1, \ldots, A_N$ are measurable sets in $[0,1]$ each with measure C, then $m(A_i \cap A_j) \geq (1 - \delta_N) C^2$ for some $i \neq j$;

   (b) $\lim_{N \to \infty} \delta_N = 0$.

   Hint: Let $F = \sum_{n=1}^{N} \chi_{A_n}$, where χ_{A_n} denotes the characteristic function of the subset A_n. Find F^2.

7. (15 points) Let $(f_n)_{n \in \mathbb{N}}$ be an orthonormal sequence in $L^2([0,1])$. Let $S_n = \frac{1}{n} \sum_{j=1}^{n} f_j$.

   (a) Prove that $\|S_n\|_2^2 = \frac{1}{n}$.

   (b) Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence of positive integers such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty$. Prove that $\sum_{j=1}^{\infty} |S_{\lambda_j}|^2$ converges almost everywhere, and S_{λ_j} converges to 0 almost everywhere.