Qualified Examination: Partial Differentiation Equation

September, 2006 **Name:**____________________________

Do all problems. (E: easy, M: moderate, D:difficult)

1. (20 points) (M)

   Let $Lu = \sum_{k=1}^{3} a_k(x)\frac{\partial u}{\partial x_k}$, $x = (x_1, x_2, x_3) \in \Omega$, where Ω is an open set in R^3 and $a_k(x) \in C^\infty(\Omega)$. Given $f \in L^2(\Omega)$, we say that u is an L^2 weak solution of $Lu = f$ in Ω if $u \in L^2_{loc}(\Omega)$ and

   $$< u, L'\psi >=< f, \psi >, \quad \forall \psi \in C_c^\infty(\Omega),$$

   where $L'u = -\sum_{k=1}^{3} \frac{\partial(a_k u)}{\partial x_k}$.

   Suppose that there is a constant c such that

   $$< f, \phi > \leq c\|L'\phi\|_{L^2(\Omega)}, \quad \forall \phi \in C_c^\infty(\Omega).$$

   Please prove that there exists an L^2 weak solution of

   $$Lu = f.$$

   (Note: $< f, g >= \int_\Omega fg dx$, $\|f\|_{L^2} = (\int_\Omega f^2 dx)^{\frac{1}{2}}$.)

2. (20 points) (M)

   Use the Fourier transform method to solve the initial value problem

   $$\begin{aligned} u_t &= u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty. \end{aligned}$$

   And prove that u satisfies the following inequality

   $$\|u\|_p(t) \leq \frac{1}{(4\pi t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|f\|_q, \quad t > 0,$$

   for $1 \leq q \leq p \leq \infty$. (Note that the L^p, L^q norms are with respect to x.)

3. (20 points) (E)

   Solve the initial value problem

   $$u_t + u_x - 3u = t, \quad x \in R, t > 0.$$
   $$u(x,0) = x^2, \quad x \in R.$$

4. (20 points) (M)

(a) Find the Green's function for the quadrant

$$Q = \{(x, y) : x > 0, y > 0\}.$$

(b) Use your answer in (a) to solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \text{ for } (x, y) \in Q,$$

$$u(0, y) = g(y) \text{ for } y > 0,$$
$$u(x, 0) = h(x) \text{ for } x > 0.$$

5. (20 points) (M)
The three-dimensional wave equation is

$$u_{tt} - c^2 \Delta u = 0,$$

where $u = u(x, y, z, t)$ and Δ is the Laplacian operator. For waves with spherical symmetry, $u = u(\rho, t)$, where $\rho = \sqrt{x^2 + y^2 + z^2}$. Please derive the spherically symmetric wave equation in this special case and find its general solution.