國立成功大學應用數學所 數値分析 博士班資格考
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1. Illustrate that $(a+b)c \neq ac+bc$ can happen in practice in a calculator. (10%)

2. Show that if $u(x)$ is a function that interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$ and $v(x)$ is a function that interpolates $f(x)$ at $x_1, x_2, \ldots, x_n$ then the function $w(x)$ given by
$$w(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}$$
interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$. (10%)

3. Is there a formula of the form
$$\int_0^1 f(x)dx \approx \alpha[f(x_0) + f(x_1)]$$
that correctly integrates all quadratic polynomials? (10%)

4. A sequence $\{p_n\}$ is said to be **superlinearly convergent** to p if
$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$
   (a) Show that if $p_n \to p$ of order α for $\alpha > 1$, then the sequence $\{p_n\}$ is certainly superlinearly convergent to p. (10%)
   (b) Show that $\{p_n = \frac{1}{n^n}\}$ is superlinearly convergent to 0, but does not converge to 0 of any order α for $\alpha > 1$. (10%)

5. Consider the initial value problem
$$\text{(I.V.P.)} \begin{cases} y' = f(t,y), & a \le t \le b, \\ y(a) = \alpha. \end{cases}$$
Show that the difference method
$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + a_1 f(t_i, w_i) + a_2 f\left(t_i + \beta, w_i + \delta f(t_i, w_i)\right), \end{aligned}$$
for each $i = 0, 1, \ldots, n-1$, cannot have local truncation error $O(h^3)$ for any choice of constants a_1, a_2, β and δ. (15%)

6. Show that every multi-step method defined by

$$w_{i+1} = w_i + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})]$$

with $\sum_{j=0}^{m} b_j = 1$ is stable, consistent and convergent. (10%)

7. Consider a 4-digit decimal system. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4+\varepsilon \\ 1 & 2+\varepsilon \end{bmatrix}$ where $\varepsilon = 10^{-2}$.

(a) Show that $\text{rank}(A) = 2$. (5%)

(b) Show that for a given $b \in \mathbb{R}^3$ the least square problem,

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2, \qquad (LS)$$

can not be usually solved by using the normal equation. (10%)

(c) Find $A^\dagger$, denotes the pseudo-inverse (generalized inverse) of A. Let $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, use $A^\dagger$ to construct a solution of problem (LS) so that the constructed solution has at least 3 significants. (10%)

Hint: $f\ell(1.0 + \varepsilon^2) = 1.0$, where $f\ell(\cdot)$ is the floating operator.