There are seven problems in the exam. Work out all seven of them.

[15%] 1. Let N be a finite normal subgroup of G, and H a subgroup of G. If $[G : H]$ is finite, and $[G : H]$ and $|N|$ are relatively prime, then $N < H$.

[15%] 2. We say that a group is indecomposable if G in nontrivial (i.e. G has more than one element), and $G \cong H \oplus K$ implies that H is trivial or K is trivial. Show that the additive group $\mathbb{Q}$ of rational numbers is indecomposable.

[15%] 3. Let G be a finite nilpotent group, i.e. G is the direct product of its Sylow subgroups, and let H be a minimal nontrivial normal subgroup of G. Show that H is contained in the center of G and that H has prime order.

[15%] 4. How many elements of of order 7 are there in a simple group of order 168?

[15%] 5. Let R be a ring and I an ideal in R. Let $[R : I] = \{r \in R \mid xr \in I \text{ for all } x \in R\}$. Show that $[R : I]$ is a two-sided ideal of R containing I.

[15%] 6. Let $R = \mathbb{Z}_6$ and $S = \{2, 4\} \subset R$. Then S is a multiplicative subset of R. Show that the ring of quotients of R by S, $S^{-1}R$, is isomorphic to $\mathbb{Z}_3$.

[10%] 7. Let F be an extension field of the field K. Show that

(a) If $[F : K]$ is prime, then there are no intermediate fields between F and K.

(b) If $u \in F$ has degree n over K, then n divides $[F : K]$.