Ph.D. Qualifying Examination (2005.9.23)
Algebra

Answer all the problems and show all your works.

1. (15%) Show that no group of order 48 is simple.

2. (10%) Let H be a subgroup of a finite group G with $[G : H] = p$, where p is the smallest prime dividing the order of G. Prove that H is normal in G. (Hint: Consider the action of G on the coset of H.)

3. (10%) Let R be a commutative Noetherian ring with identity. Show that $R[x]$ is also Noetherian.

4. (15%) Let $A \subset R$ be two integral domains containing identity such that R is integral over A. Let P and Q be prime ideals in R with $P \subseteq Q$. Show that $P = Q$ if $P \cap A = Q \cap A$.

5. (10%) Find all prime ideals in the ring $\mathbb{C}[x,y]/(xy-1)$, where $\mathbb{C}$ is the field of all complex numbers.

6. (15%) Let R and S be two rings. Let M be a right R-module, N a right S-module and P a R-S-bimodule with R acting on the left and S acting on the right. Show that there is an isomorphism of abelian groups from $\mathrm{Hom}_S(M \otimes_R P, N)$ to $\mathrm{Hom}_R(M, \mathrm{Hom}_S(P,N))$.

7. (10%) Let $\mathbb{Z}_4$ is a cyclic group of order 4. We consider $\mathbb{Z}_4$ to be a $\mathbb{Z}$-module.

   (i) (5%) Find a projective $\mathbb{Z}$-module P and a surjective $\mathbb{Z}$-homomorphism from P to $\mathbb{Z}_4$.

   (ii) (5%) Find an injective $\mathbb{Z}$-module J and an injective $\mathbb{Z}$-homomorphism from $\mathbb{Z}_4$ to J.

8. (15%) Let E be a splitting field over $\mathbb{Q}$ of the equation $f(x) = x^4 - 5$, where $\mathbb{Q}$ is the field of all rational numbers.

   (i) (10%) Determine the Galois group of E over $\mathbb{Q}$.

   (ii) (5%) Find all the intermediate fields K between E and $\mathbb{Q}$ satisfying $[E : K] = 2$.