FINAL EXAM OF CALCULUS

Date: 2000, June 19, 13:10–14:55

An answer without reasoning will not be accepted.

1. [8%] Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x,y,z) = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$.

2. [8%] Show that there is no vector field $\mathbf{G}$ such that curl $\mathbf{G} = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}$.

3. [10%] Compute the outward flux of $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ through the ellipsoid $4x^2+9y^2+6z^2 = 36$.

4. [10%] Compute the surface integral

$$\iint_S (x^2+y^2)\,d\sigma,$$

where S is the hemisphere $z = \sqrt{1-(x^2+y^2)}$.

5. [14%] Let Ω be a Jordan region (on the xy-plane) with a piece-wise smooth boundary C, and let f and g be continuously differentiable functions on an open set containing Ω.

(i) Use the vector form of the Green's theorem to prove *Green first identity*:

$$\iint_\Omega f\nabla^2 g\,d\sigma = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_\Omega \nabla f\cdot\nabla g\,d\sigma$$

where $\mathbf{n}$ is the outer unit normal vector.

(ii) Use above Green's first identity to prove *Green's second identity*:

$$\iint_\Omega (f\nabla^2 g - g\nabla^2 f)\,d\sigma = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds.$$

6. [12%] Show that the vector field $\mathbf{v}(x,y,z) = (2xz+\sin y)\mathbf{i}+x\cos y\mathbf{j}+x^2\mathbf{k}$ is a gradient. Then evaluate the line integral of $\mathbf{v}$ over the curve $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$ for $u \in [0,\pi]$.

7. [10%] Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $x^2+4y^2+z^2 = 36$.

8. [6%] Compute the sum of the series

$$1 - \ln\pi + \frac{(\ln\pi)^2}{2!} - \frac{(\ln\pi)^3}{3!} + \cdots$$

9. [8%] Compute the interval of convergence of the Taylor series in x of $\ln(1-x)$.

10. [8%] Find the tangential component of the acceleration vector of $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

11. [8%] A function $f(x,y)$ is called *homogeneous of degree n* if f has continuous second-order partial derivatives and $f(tx,ty) = t^n f(x,y)$ for all t. Use the chain rule to show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y).$$