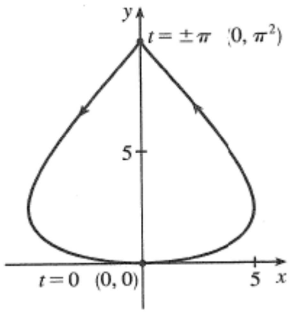
9.1

3. Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

$x = 5\sin t, y = t^2, -\pi \le t \le \pi$

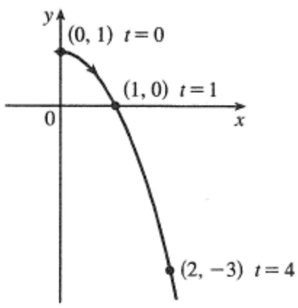


7. $x = \sqrt{t}, y = 1 - t$

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

(b) $x = \sqrt{t}, y = 1 - t \Rightarrow y = 1 - x^2, t \ge 0, x \ge 0$

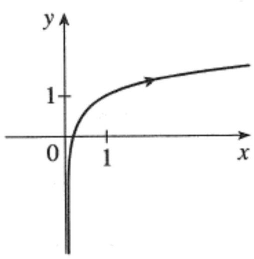


13. $x = e^{2t}, y = t + 1$

(a) Eliminate the parameter to find a Cartesian equation of the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

(a) $x = e^{2t} \Rightarrow t = \dfrac{1}{2}\ln x \Rightarrow y = t + 1 = \dfrac{1}{2}\ln x + 1$ ， $x > 0$

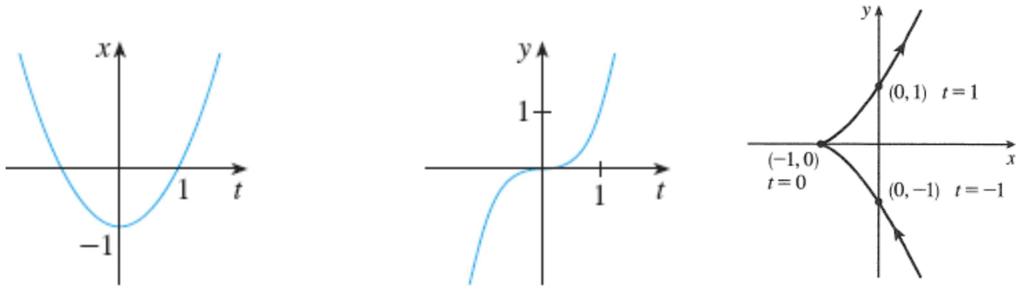


17. $x = 5\sin t, y = 2\cos t, -\pi \le t \le 5\pi$. Describe the motion of a particle with position (x, y) as t varies in the given interval.

由此參數式可知 $\sin t=\dfrac{x}{5},\cos t=\dfrac{y}{2}\Rightarrow \sin^2 t+\cos^2 t=\dfrac{x^2}{25}+\dfrac{y^2}{4}=1$

為中心在 $(0,0)$，長軸長10、短軸長4的橢圓，當 $t=-\pi$ 時，起點為 $(0,-2)$，逆時針旋繞3圈。

19. Use the graphs of $x=f(t)$ and $y=g(t)$ to sketch the parametric curve $x=f(t)$, $y=g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



27. Find parametric equations for the path of a particle that moves along the circle $x^2+(y-1)^2=4$ in the manner described.

(a) Once around clockwise, starting at $(2,1)$.

(b) Three times around counterclockwise, starting at $(2,1)$.

(c) Half way around counterclockwise, starting at $(0,3)$.

(a) $x^2+(y-1)^2=4\Rightarrow x=2\cos\theta, y=1+2\sin\theta$，為逆時針旋繞，起點為 $(2,1)$

順時針 $\theta=-t\Rightarrow x=2\cos t, y=1-2\sin t, 0\le t\le 2\pi$

(b) 繞轉三次，即周期減為三分之一，所以 $\theta=3t\Rightarrow x=2\cos 3t, y=1+2\sin 3t, 0\le t\le 2\pi$

(c) 轉半圈，且起點為 $(0,3)$，故 $\theta=t+\dfrac{\pi}{2}\Rightarrow x=2\cos(t+\dfrac{\pi}{2}), y=1+2\sin(t+\dfrac{\pi}{2}), 0\le t\le \pi$

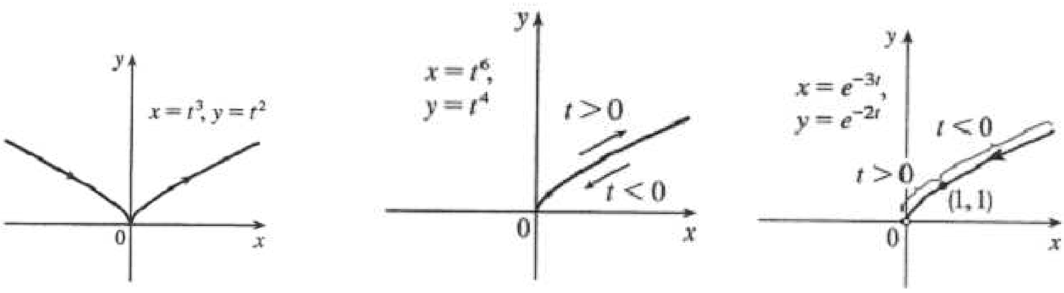
$\Rightarrow x=-2\sin\dfrac{\pi}{2}, y=1+2\cos t, 0\le t\le \pi$

31. Compare the curves represented by the parametric equations. How do they differ?

(a) $x=t^3, y=t^2$ (b) $x=t^6, y=t^4$ (c) $x=e^{-3t}, y=e^{-2t}$

(a) $\Rightarrow y=x^{2/3}, x\in R, y\in R^+\cup\{0\}$ (b) $\Rightarrow y=x^{2/3}, x\in R^+\cup\{0\}, y\in R^+\cup\{0\}$

(c) $\Rightarrow y=x^{2/3}, x\in R^+, y\in R^+$



 廖彥法 整理

9.2

3. $x = t^4 + 1, y = t^3 + t; t = -1$. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$\frac{dx}{dt} = 4t^3, \frac{dy}{dt} = 3t^2 + 1 \Rightarrow \frac{dy}{dx} = \frac{3t^2+1}{4t^3} \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = \frac{3+1}{-4} = -1$$

切點為 $(2,-2)$，所以切線為 $y + 2 = -(x-2) \Rightarrow y = -x$

7. Find an equation of the tangent to the curve $x = e^t, y = (t-1)^2$ at the point $(1,1)$ by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

(a) $\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2(t-1) \Rightarrow \frac{dy}{dx} = \frac{2t-2}{e^t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{-2}{1} = -2$

切點為 $(1,1)$，所以切線為 $y - 1 = -2(x-1) \Rightarrow y = -2x + 3$

(b) $x = e^t \Rightarrow t = \ln x \Rightarrow y = (\ln x - 1)^2 \Rightarrow \frac{dy}{dx} = 2(\ln x - 1)\cdot\frac{1}{x}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{-2}{1} = -2$，切點為 $(1,1)$，所以切線為 $y - 1 = -2(x-1) \Rightarrow y = -2x + 3$

9. $x = 4 + t^2, y = t^2 + t^3$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d(dy/dx)/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t}$$

當 $\frac{d^2y}{dx^2} > 0$ 時，曲線凹向上，則 $t > 0$。

15. $x = 2\cos\theta, y = \sin 2\theta$. Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

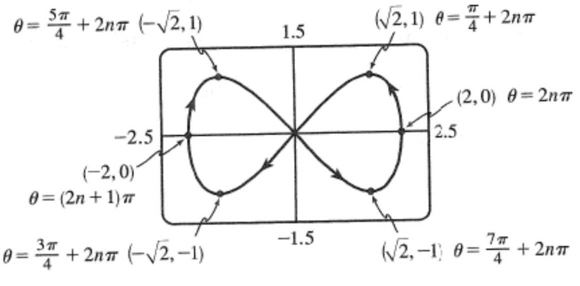
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{-2\sin\theta} = \frac{-\cos 2\theta}{\sin\theta}$$

當 $\frac{dy}{dx} = 0$ 時有水平切線，則 $\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

所有的切點為 $(\pm\sqrt{2}, \pm 1)$

當 $\frac{dx}{dy} = 0$ 時有垂直切線，則 $\sin\theta = 0 \Rightarrow \theta = 0, \pi$

所有的切點為 $(\pm 2, 0)$



25. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with equations $x = -7t$, $y = 12t - 5$?

$$x = -7t \text{ , } y = 12t - 5 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{12}{7}$$

$$x = t^3 + 4t \text{ , } y = 6t^2 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t}{3t^2 + 4}$$

$$\frac{12t}{3t^2 + 4} = -\frac{12}{7} \Rightarrow t = -1, \frac{-4}{3} \Rightarrow (x, y) = (-5,6), (\frac{-208}{27}, \frac{32}{3})$$

27. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.

可用第一象限的區域面積計算，則 $A = 4\int ydx = \int_0^{\pi/2} y(\theta)dx(\theta) = 4ab\int_0^{\pi/2} \sin^2\theta d\theta = \pi ab$

29. Find the area bounded by the curve $x = \cos t$, $y = e^t$, $0 \le t \le \frac{\pi}{2}$, and the lines $y = 1$ and $x = 0$.

$$A = \int_0^1 (y-1)dx = \int_{\pi/2}^0 (e^t - 1)(-\sin t)dt = \left[\frac{1}{2}e^t(\sin t - \cos t) + \cos t\right]_0^{\pi/2} = \frac{1}{2}(e^{\pi/2} - 1)$$

33-36. Set up, but do not evaluate, an integral that represents the length of the curve.

33. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$

$$\Rightarrow \frac{dx}{dt} = 1 - 2t, \frac{dy}{dt} = 2t^{1/2} \Rightarrow (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1 + 4t^2$$

$$L = \int_1^2 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int_1^2 \sqrt{1 + 4t^2}\, dt$$

35. $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$

$$\Rightarrow \frac{dx}{dt} = 1 - \sin t, \frac{dy}{dt} = 1 - \cos t \Rightarrow (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 3 - 2\sin t - 2\cos t$$

$$L = \int_0^{2\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int_0^{2\pi} \sqrt{3 - 2\sin t - 2\cos t}\, dt$$

39. $x = \frac{t}{1+t}$, $y = \ln(1+t)$, $0 \le t \le 2$. Find the length of the curve.

$$\Rightarrow \frac{dx}{dt} = \frac{1}{(1+t)^2}, \frac{dy}{dt} = \frac{1}{1+t} \Rightarrow (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = \frac{t^2 + 2t + 2}{(1+t)^4}$$

$$L = \int_0^2 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int_0^2 \frac{\sqrt{t^2 + 2t + 2}}{(1+t)^2}\, dt$$ ，

令 $u = 1 + t \Rightarrow du = dt$ ，則 $\int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2} dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2} du$

令 $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$ ，則 $\int_1^3 \frac{\sqrt{u^2+1}}{u^2} du = \int_{\tan^{-1}1}^{\tan^{-1}3} \frac{\sec^3\theta}{\tan^2\theta} d\theta = \int_{\tan^{-1}1}^{\tan^{-1}3} (\sec\theta + \frac{\cos\theta}{\sin^2\theta}) d\theta$

$= \left[\ln|\sec\theta + \tan\theta| - \frac{1}{\sin\theta} \right]_{\tan^{-1}1}^{\tan^{-1}3} = \ln(\sqrt{10}+3) - \frac{\sqrt{10}}{3} - \ln(\sqrt{2}+1) + \sqrt{2}$

$= -\frac{\sqrt{10}}{3} + \sqrt{2} + \ln(\frac{\sqrt{10}+3}{\sqrt{2}+1})$

47. Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

$x = \sin^2 t$, $y = \cos^2 t$, $0 \le t \le 3\pi$

$$L = \int_0^{3\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \sqrt{2}\int_0^{3\pi} |\sin 2t| dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t dt = -3\sqrt{2}\cos 2t\Big|_0^{\pi/2} = 6\sqrt{2}$$

50. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

$$L = 4\int_0^{\pi/2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta = 12a\int_0^{\pi/2} \sin\theta\cos\theta d\theta = 6a\int_0^{\pi/2} \sin 2\theta d\theta = -3a\cos 2t\Big|_0^{\pi/2} = 6a$$

9.3

5. (a) $(1,1)$ (b) $(2\sqrt{3}, -2)$

The Cartesian coordinates of a point are given.

(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \le \theta < 2\pi$.

(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \le \theta < 2\pi$.

(a) (i) $r\cos\theta = r\sin\theta = 1 \Rightarrow r = \sqrt{2}, \theta = \frac{\pi}{4}$

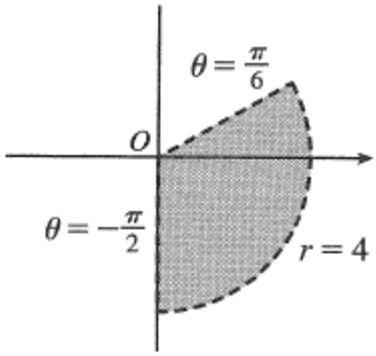
(ii) $r\cos\theta = r\sin\theta = 1 \Rightarrow r = -\sqrt{2}, \theta = \frac{5\pi}{4}$

(b) (i) $r\cos\theta = 2\sqrt{3}, r\sin\theta = -2 \Rightarrow r = 4, \theta = \frac{11\pi}{6}$

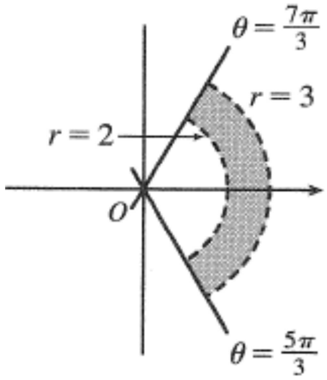
(ii) $r\cos\theta = 2\sqrt{3}, r\sin\theta = -2 \Rightarrow r = -4, \theta = \frac{5\pi}{6}$

7-12. Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

9. $0 \le r < 4, -\dfrac{\pi}{2} \le \theta < \dfrac{\pi}{6}$



11. $2 < r < 3, \dfrac{5\pi}{3} \le \theta \le \dfrac{7\pi}{3}$



13. $r = 3\sin\theta$. Identify the curve by finding a Cartesian equation for the curve.

$$r = 3\sin\theta \Rightarrow x = 3\sin\theta\cos\theta, y = 3\sin^2\theta \Rightarrow x = \frac{3}{2}\sin 2\theta, y = \frac{3}{2} - \frac{3}{2}\cos 2\theta$$

$\Rightarrow x^2 + (y - \dfrac{3}{2})^2 = (\dfrac{3}{2})^2$ 為一圓心 $(0, \dfrac{3}{2})$ 半徑 $\dfrac{3}{2}$ 的圓。

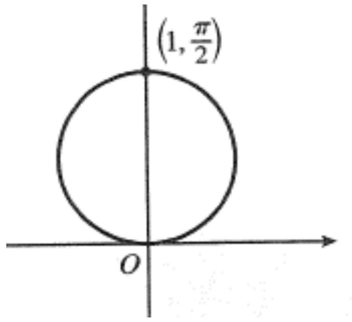
19. $x^2 + y^2 = 2cx$. Find a polar equation for the curve represented by the given Cartesian equation.

$x = r\cos\theta, y = r\sin\theta$

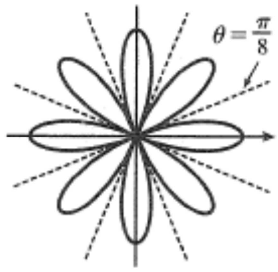
$\Rightarrow r^2 = 2cr\cos\theta \Rightarrow r(r - 2c\cos\theta) = 0 \Rightarrow r = 0, r = 2c\cos\theta$

23-40. Sketch the curve with the given polar equation.

25. $r = \sin\theta$



33. $r = 2\cos 4\theta$



47. $r = 2\sin\theta, \theta = \pi/6$. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$r = 2\sin\theta \Rightarrow x = 2\sin\theta\cos\theta = \sin 2\theta, y = 2\sin^2\theta = 1-\cos 2\theta$$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin 2\theta}{2\cos 2\theta} = \tan 2\theta \Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \sqrt{3}$$

53. $r = 1+\cos\theta$. Find the points on the given curve where the tangent line is horizontal or vertical.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$r = 1+\cos\theta \Rightarrow x = (1+\cos\theta)\cos\theta, y = (1+\cos\theta)\sin\theta$$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\cos^2\theta - 1}{-\sin\theta - 2\sin\theta\cos\theta}$$

水平切線，斜率為零，即 $\cos\theta + 2\cos^2\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2}, -1 \Rightarrow \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

此切點為 $(\frac{3}{2}, \frac{\pi}{3}), (0, \pi), (\frac{3}{2}, \frac{5\pi}{3})$

垂直切線，斜率不存在，即 $-\sin\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow \sin\theta = 0, \cos\theta = -\frac{1}{2} \Rightarrow \theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$，此切點為 $(2,0), (\frac{1}{2}, \frac{2\pi}{3}), (\frac{1}{2}, \frac{4\pi}{3})$

55. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

$$r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx$$

$$\Rightarrow (x^2 - bx + \frac{b^2}{4}) + (y^2 - ay + \frac{a^2}{4}) = \frac{a^2+b^2}{4} \Rightarrow (x - \frac{b}{2})^2 + (y - \frac{a}{2})^2 = (\frac{\sqrt{a^2+b^2}}{2})^2$$

即圓心 $(\frac{b}{2}, \frac{a}{2})$，半徑 $\frac{\sqrt{a^2+b^2}}{2}$ 的圓。

9.4

3. $r = \sin\theta, \pi/3 \le \theta \le 2\pi/3$. Find the area of the region that is bounded by the given curve and lies in the specified sector.

$$A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_{\pi/3}^{2\pi/3} \sin^2\theta d\theta = \frac{1}{4}(\theta - \frac{1}{2}\sin 2\theta)_{\pi/3}^{2\pi/3} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

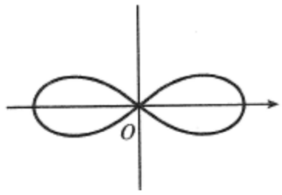
7. $r = 4 + 3\sin\theta$. Find the area of the shaded region.

$$A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (4 + 3\sin\theta)^2 d\theta = (\frac{41}{4}\theta - \frac{9}{8}\sin 2\theta)_{-\pi/2}^{\pi/2} = \frac{41}{4}\pi$$

9-12. Sketch the curve and find the area that it encloses.

9. $r^2 = 4\cos 2\theta$

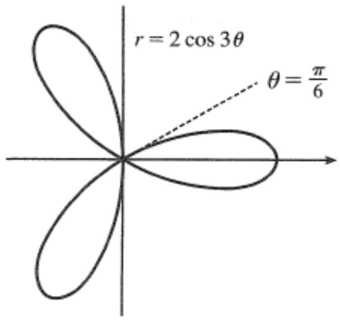
由圖可看出，此圖形上下左右對稱，故僅需積分$0 \le \theta \le \frac{\pi}{4}$，再乘以 4。



$$A = 4 \times \frac{1}{2}\int_0^{\pi/4} 4\cos 2\theta d\theta = 4\sin 2\theta\Big|_0^{\pi/4} = 4$$

11. $r = 2\cos 3\theta$

由圖可看出，此圖形上下對稱，且每葉相等，故僅需積分$0 \le \theta \le \frac{\pi}{6}$，再乘以 6。

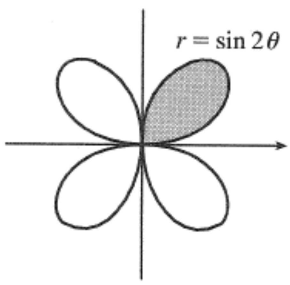


$$A = 6 \times \frac{1}{2}\int_0^{\pi/6} 4\cos^2 3\theta d\theta = 6(\theta + \frac{1}{6}\sin 6\theta)\Big|_0^{\pi/6} = \pi$$

15-18. Find the area of the region enclosed by one loop of the curve.

15. $r = \sin 2\theta$

由圖可看出，此圖形環繞成一圈，$0 \le \theta \le \frac{\pi}{2}$ 。

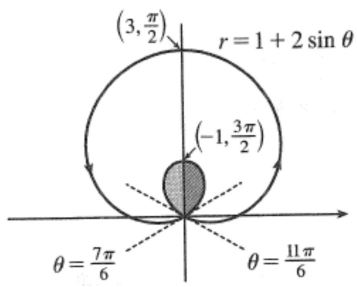


$$A = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta d\theta = \frac{1}{4}(\theta - \frac{1}{4}\sin 4\theta)\Big|_0^{\pi/2} = \frac{\pi}{8}$$

17. $r = 1 + 2\sin\theta$ (inner loop)

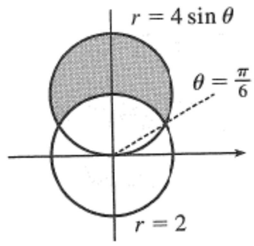
由圖可看出，此圖形環繞成兩圈，內圈$\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$ 。

$$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2 d\theta = \frac{1}{2}(3\theta - 4\cos\theta - \sin 2\theta)\Big|_{7\pi/6}^{11\pi/6} = \pi - \frac{3\sqrt{3}}{2}$$



19-22. Find the area of the region that lies inside the first curve and outside the second curve.

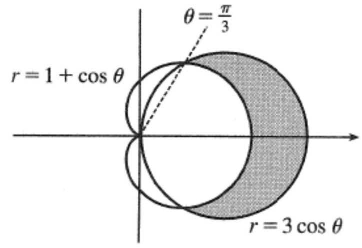
19. $r = 4\sin\theta, r = 2$



$r = 4\sin\theta, r = 2$ 兩函數相交於 $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6}[(4\sin\theta)^2 - 2^2]d\theta = (2\theta - 2\sin 2\theta)\Big|_{\pi/6}^{5\pi/6} = \frac{4}{3}\pi + 2\sqrt{3}$$

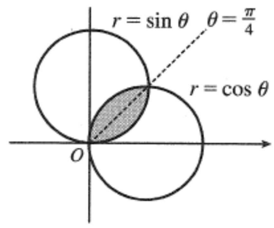
21. $r = 3\cos\theta, r = 1 + \cos\theta$



$r = 3\cos\theta, r = 1+\cos\theta$ 兩函數相交於 $\theta = \pm\dfrac{\pi}{3}$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}[(3\cos\theta)^2 - (1+\cos\theta)^2]d\theta = \left[\frac{3}{2}\theta + \sin 2\theta - \sin\theta\right]_{-\pi/3}^{\pi/3} = \pi$$

23. $r = \sin\theta, r = \cos\theta$. Find the area of the region the lies inside both curves.

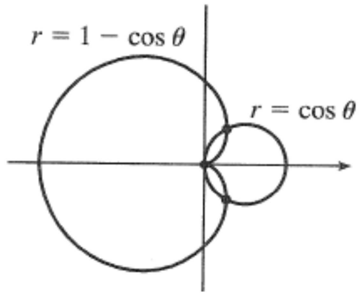


由圖形可知此區域對稱於 $\theta = \dfrac{\pi}{4}$

所以 $A = 2 \times \dfrac{1}{2}\displaystyle\int_0^{\pi/4}\sin^2\theta d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$

29-32. Find all points of intersection of the given curves.

29. $r = \cos\theta, r = 1 - \cos\theta$



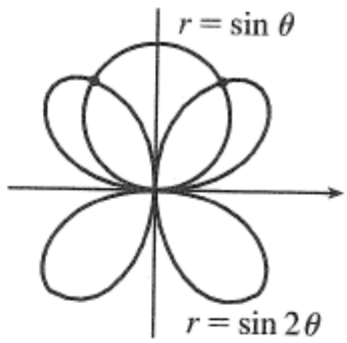
$r = \cos\theta, r = 1-\cos\theta \Rightarrow \cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

所以交點為 $(\dfrac{1}{2}, \dfrac{\pi}{3})$ 、 $(\dfrac{1}{2}, \dfrac{5\pi}{3})$

31. $r = \sin\theta, r = \sin 2\theta$

$r = \sin\theta, r = \sin 2\theta \Rightarrow \sin\theta = \sin 2\theta \Rightarrow \sin\theta(1 - 2\cos\theta) = 0 \Rightarrow \theta = 0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

所以交點為 $(0,0)$ 、 $(0,\pi)$ 、 $(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{3})$ 、 $(-\dfrac{\sqrt{3}}{2}, \dfrac{5\pi}{3})$



33-36. Find the exact length of the polar curve.

33. $r = 3\sin\theta$, $0 \le \theta \le \dfrac{\pi}{3}$

$$L = \int\sqrt{r^2 + (dr/d\theta)^2}d\theta = \int_0^{\pi/3} 3d\theta = 3\theta\Big|_0^{\pi/3} = \pi$$

35. $r = \theta^2, 0 \le \theta \le 2\pi$

$$L = \int\sqrt{r^2 + (dr/d\theta)^2}d\theta = \int_0^{2\pi}\theta\sqrt{\theta^2 + 4}d\theta$$

令 $u = \theta^2 + 4 \Rightarrow du = 2\theta d\theta$

則 $\displaystyle\int_0^{2\pi}\theta\sqrt{\theta^2+4}d\theta = \frac{1}{2}\int_4^{4\pi^2+4}u^{1/2}du = \frac{1}{3}u^{3/2}\Big|_4^{4\pi^2+4} = \frac{8}{3}[(\pi^2+1)^{3/2} - 1]$

廖彥法 整理