8.1

9-28. Determine whether the sequence converges or diverges. If it converges, find the limit.

9. $a_n = \dfrac{3+5n^2}{n+n^2}$

$$\Rightarrow a_n = \frac{(3/n^2)+5}{(1/n)+1} \Rightarrow \lim_{n\to\infty} a_n = \frac{5}{1} = 5 \quad \text{收斂}$$

21. $a_n = \dfrac{\cos^2 n}{2^n}$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos^2 n}{2^n} \Rightarrow 0 \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{2^n} = 0$$ ，依夾擠定理，此數列收斂

23. $a_n = (1+\dfrac{2}{n})^n$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+\frac{2}{n})^n = \exp\left[\lim_{n\to\infty} n\ln(1+\frac{2}{n})\right] = \exp\left[\lim_{n\to\infty} \frac{\ln(1+\frac{2}{n})}{\frac{1}{n}}\right]$$

$$\xrightarrow{L'Hospital} \exp\left[\lim_{n\to\infty} \frac{2}{1+2/n}\right] = e^2$$

32. (a) If $\{a_n\}$ is convergent, show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = \dfrac{1}{1+a_n}$ for $n \ge 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

(a) 令 $\lim_{n\to\infty} a_n = L$ ，則存在一個 $\varepsilon > 0$ 及一個正整數 N，使得 $|a_n - L| < \varepsilon, n > N$

則 $|a_{n+1} - L| < \varepsilon, n+1 > N$ ，即 $n > N-1 = M$ ，

故依定義， $\lim_{n\to\infty} a_{n+1} = L$ ，所以 $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$

(b) $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{1+a_n} \Rightarrow L = \frac{1}{1+L} \Rightarrow L = \frac{-1\pm\sqrt{5}}{2}$$ (負不合)

34. $a_n = \dfrac{2n-3}{3n+4}$. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

$$f(n) = a_n = \frac{2n-3}{3n+4} \Rightarrow f'(n) = \frac{17}{(3n+4)^2} > 0$$ ，故 $\{a_n\}$ 為一嚴格遞增數列。

廖彥法 整理

且 $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-3}{3n+4} = \frac{2}{3}$ ，為其數列上界。

39. Use induction to show that the sequence defined by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and $a_n < 3$ for all *n*. Deduce that $\{a_n\}$ is convergent and find its limit.

證明數列遞增，當 $n=1$ 時，$a_2 = 3 - \frac{1}{a_1} = 2 > a_1$ ，原式成立

設 $n = k$ 時，$a_{k+1} > a_k$ 成立，

則 $n = k+1$ 時，$a_{k+2} = 3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k} = a_{k+1}$ ，原式成立

故依數學歸納法，此數列遞增。

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L \Rightarrow L = 3 - \frac{1}{L} \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$ 且 $\lim_{n\to\infty} a_n > a_1 = 1$ ，則 $L = \frac{3+\sqrt{5}}{2} < 3$

8.2

3-8. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

3. $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$

此為無窮等比級數，首項 5，公比 $r = -\frac{2}{3}$ ，其和收斂，$S = \frac{5}{1-(-\frac{2}{3})} = 3$

7. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

此為無窮等比級數，首項 $\frac{1}{3}$ ，公比 $r = \frac{\pi}{3} > 1$ ，其和發散。

9-18. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

9. $\sum_{n=1}^{\infty} \frac{1}{2n}$

$\Rightarrow \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n} = \frac{1}{2}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots) > \frac{1}{2}(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\ldots)$

$= \frac{1}{2}(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots)$ 發散，故 $\sum_{n=1}^{\infty}\frac{1}{2n}$ 發散

13. $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}$

$$\Rightarrow \sum_{n=1}^{\infty}\left[(\frac{1}{3})^n+(\frac{2}{3})^n\right]=\frac{1/3}{1-1/3}+\frac{2/3}{1-2/3}=\frac{5}{2}$$

17. $\sum_{n=1}^{\infty}\tan^{-1}n$

$a_n=\tan^{-1}n\Rightarrow \lim_{n\to\infty}a_n=\frac{\pi}{2}\neq 0$ ，故此數列發散。

19-22. Determine whether the geometric series is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

19. $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

$$\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)=\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+...\right)=1+\frac{1}{2}=\frac{3}{2}$$ 收斂

21. $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+3)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right)=\lim_{n\to\infty}\left(1-\frac{1}{4}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{6}+...\right)=1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}$$ 收斂

25. $3.\overline{417}$. Express the number as a ratio of integers.

$$S=3.\overline{417}\Rightarrow 1000S=3417.\overline{417}$$

$$\Rightarrow (1000-1)S=3417-3\Rightarrow S=\frac{3414}{999}=\frac{1138}{333}$$

27. $\sum_{n=1}^{\infty}\frac{x^n}{3^n}$. Find the value of *x* for which the series converges. Find the sum of the series for those value of *x*.

$\sum_{n=1}^{\infty}\frac{x^n}{3^n}$ 為無窮等比級數，首項 $\frac{x}{3}$ ，公比 $r=\frac{x}{3}$ ，若其和收斂，則 $|\frac{x}{3}|<1\Rightarrow -3<x<3$

且其和為 $\frac{x/3}{1-x/3}=\frac{x}{3-x}$

31. If the *n*th partial sum of a series $\sum_{n=1}^{\infty}a_n$ is $s_n=\frac{n-1}{n+1}$ find a_n and $\sum_{n=1}^{\infty}a_n$.

$$a_n=s_n-s_{n-1}=\frac{n-1}{n+1}-\frac{n-2}{n}=\frac{2}{n(n+1)}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{n-1}{n+1} = 1$$

35. What is the value of c if $\sum_{n=2}^{\infty}(1+c)^{-n} = 2$

$\sum_{n=2}^{\infty}(1+c)^{-n}$ 為無窮等比級數，首項 $(1+c)^{-2}$ ，公比 $r=\frac{1}{1+c}$ ，若其和收斂，則 $|\frac{1}{1+c}|<1 \Rightarrow c>0, c<-2$

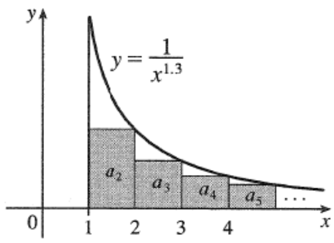
其和為 $\frac{(1+c)^{-2}}{1-\frac{1}{1+c}} = \frac{1}{c(1+c)} = 2 \Rightarrow c = \frac{-1\pm\sqrt{3}}{2}$ (負不合)

8.3

1. Draw a picture to show that $\sum_{n=2}^{\infty}\frac{1}{n^{1.3}} < \int_1^{\infty}\frac{1}{x^{1.3}}dx$. What can you conclude about the series?

由右圖可看出 $\sum_{n=2}^{\infty}\frac{1}{n^{1.3}} < \int_1^{\infty}\frac{1}{x^{1.3}}dx$ ，且 $\int_1^{\infty}\frac{1}{x^{1.3}}dx, p=1.3>1 \Rightarrow$ 收斂

故依比較審斂法，知 $\sum_{n=2}^{\infty}\frac{1}{n^{1.3}}$ 收斂



5. It is important to distinguish between $\sum_{n=1}^{\infty} n^b$ and $\sum_{n=1}^{\infty} b^n$. What name is given to the first series? To the second? For what values of b does the first series converge? For what values of b does the second series converge?

$\sum_{n=1}^{\infty} n^b$ 為p-級數(p-series)， $p=-b$ ； $\sum_{n=1}^{\infty} b^n$ 為等比級數，公比為 b

$\sum_{n=1}^{\infty} n^b$ ， $p=-b>1$ 時，級數收斂； $\sum_{n=1}^{\infty} b^n$ ，公比為 b ，$|b|<1$ 時，級數收斂。

9. Use the Comparison Test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n+1}$$

$\sum_{n=1}^{\infty}\frac{1}{n^2+n+1} < \sum_{n=1}^{\infty}\frac{1}{n^2}, p=2>1$ ，收斂

11-26. Determine whether the series is convergent or divergent.

13. $\sum_{n=1}^{\infty} ne^{-n}$

利用積分審歛法，$\int_1^{\infty} xe^{-x}dx = \lim_{t\to\infty}\int_1^{t} xe^{-x}dx = \lim_{t\to\infty}[-xe^{-x} - e^{-x}]_1^t = \frac{2}{e}$，故$\sum_{n=1}^{\infty} ne^{-n}$ 收斂。

15. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

利用積分審歛法，$\int_2^{\infty}\frac{1}{x\ln x} = \lim_{t\to\infty}\int_2^{t}\frac{1}{x\ln x} = \lim_{t\to\infty}[\ln(\ln x)]_2^t = \infty$，故$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ 發散。

21. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$

設 $a_n = \frac{1}{\sqrt{n^2+1}}, b_n = \frac{1}{n}$，則 $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = 1 > 0$

且$\sum_{n=1}^{\infty} b_n$ 發散，故依極限比較審歛法知$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$ 發散。

25. $\sum_{n=1}^{\infty}\sin(\frac{1}{n})$

設 $a_n = \sin(\frac{1}{n}), b_n = \frac{1}{n}$，則 $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{1/n} = 1 > 0$

且$\sum_{n=1}^{\infty} b_n$ 發散，故依極限比較審歛法知$\sum_{n=1}^{\infty}\sin(\frac{1}{n})$ 發散。

27. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$. Find the values of p for which the series is convergent.

設 $f(x) = \frac{1}{x(\ln x)^p}$，則 $f(x)$ 為連續且遞減函數，

故使用積分審歛法，$\int_2^{\infty}\frac{1}{x(\ln x)^p}dx = \lim_{t\to\infty}\int_2^{t}\frac{1}{x(\ln x)^p}dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t$

$= \lim_{t\to\infty}\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}$

欲收斂，則 $\lim_{t\to\infty}(\ln t)^{1-p}$ 收斂，即 $1-p<0 \Rightarrow p>1$.