49. $r = 1/\theta, \quad \theta = \pi$ **50.** $r = \sin 3\theta, \quad \theta = \pi/6$

51–54 ■ Find the points on the given curve where the tangent line is horizontal or vertical.

51. $r = 3\cos\theta$ **52.** $r = e^{\theta}$

53. $r = 1 + \cos\theta$ **54.** $r^2 = \sin 2\theta$

55. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

56. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

57–60 ■ Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

57. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

58. $r = \sin^2(4\theta) + \cos(4\theta)$

59. $r = 2 - 5\sin(\theta/6)$ **60.** $r = \cos(\theta/2) + \cos(\theta/3)$

61. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

62. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

63. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?
(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

64. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

65. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

66. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

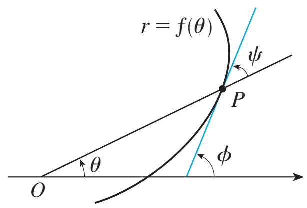
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

67. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



68. (a) Use Exercise 67 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.
(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.
(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

9.4 AREAS AND LENGTHS IN POLAR COORDINATES

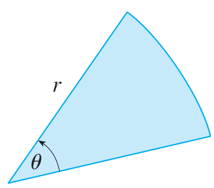


FIGURE 1

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

1 $$A = \tfrac{1}{2}r^2\theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

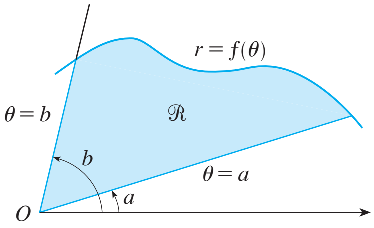


FIGURE 2

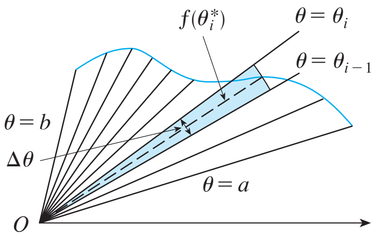


FIGURE 3

Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$. (See also Exercise 67 in Section 6.2.)

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leqslant 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints θ_0, $\theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

and so an approximation to the total area A of $\mathcal{R}$ is

2 $$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathcal{R}$ is

3 $$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

Formula 3 is often written as

4 $$A = \int_a^b \tfrac{1}{2}r^2\,d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

V EXAMPLE 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION The curve $r = \cos 2\theta$ was sketched in Example 8 in Section 9.3. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore, Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\,d\theta = \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$ ■

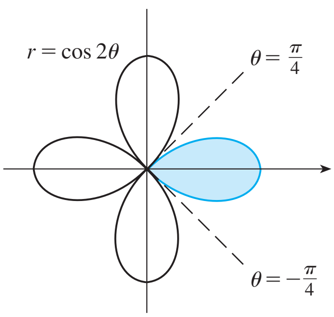


FIGURE 4

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

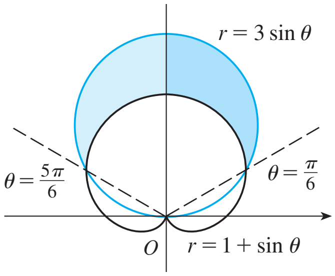


FIGURE 5

SOLUTION The cardioid (see Example 7 in Section 9.3) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$, which gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\,d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2\,d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$\begin{aligned} A &= 2\left[\tfrac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \tfrac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\,d\theta\right] \\ &= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\,d\theta \\ &= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\,d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right] \\ &= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi \end{aligned}$$

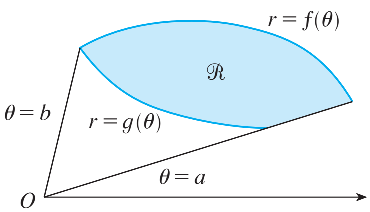


FIGURE 6

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let $\mathscr{R}$ be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ and $0 < b - a \leqslant 2\pi$. The area A of $\mathscr{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \tfrac{1}{2}[g(\theta)]^2\,d\theta = \tfrac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right)d\theta$$

CAUTION The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$ and found only two such points, $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3\sin\theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin\theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

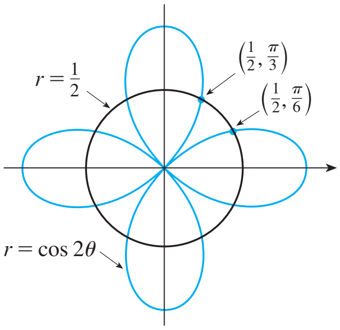


FIGURE 7

EXAMPLE 3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

SOLUTION If we solve the equations $r = \cos 2\theta$ and $r = \frac{1}{2}$, we get $\cos 2\theta = \frac{1}{2}$ and, therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. Thus the values of θ between 0 and 2π that satisfy both equations are $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$. We have found four points of intersection: $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$, $\left(\frac{1}{2}, 7\pi/6\right)$, and $\left(\frac{1}{2}, 11\pi/6\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\frac{1}{2}, \pi/3\right)$, $\left(\frac{1}{2}, 2\pi/3\right)$, $\left(\frac{1}{2}, 4\pi/3\right)$, and $\left(\frac{1}{2}, 5\pi/3\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\frac{1}{2}$. ■

ARC LENGTH

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, we can use Formula 9.2.5 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

5 $$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

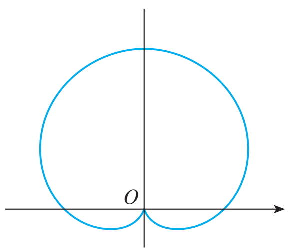


FIGURE 8
$r = 1 + \sin\theta$

V EXAMPLE 4 Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 7 in Section 9.3.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so

Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\; d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\; d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$. ■

9.4 EXERCISES

1–4 ▪ Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \sqrt{\theta}$, $0 \leqslant \theta \leqslant \pi/4$

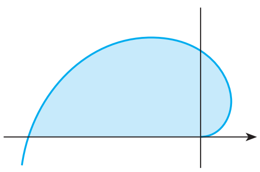
2. $r = e^{\theta/2}$, $\pi \leqslant \theta \leqslant 2\pi$

3. $r = \sin\theta$, $\pi/3 \leqslant \theta \leqslant 2\pi/3$

4. $r = \sqrt{\sin\theta}$, $0 \leqslant \theta \leqslant \pi$

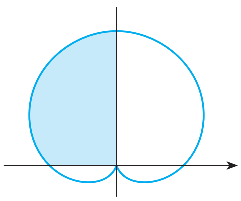
5–8 ▪ Find the area of the shaded region.

5.



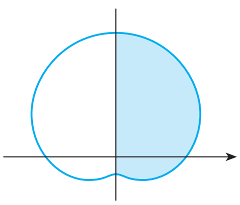
$r = \theta$

6.



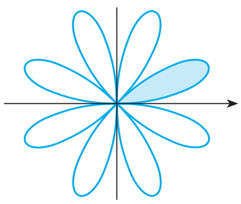
$r = 1 + \sin\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sin 4\theta$

9–12 ▪ Sketch the curve and find the area that it encloses.

9. $r^2 = 4\cos 2\theta$

10. $r = 3(1 + \cos\theta)$

11. $r = 2\cos 3\theta$

12. $r = 2 + \cos 2\theta$

13–14 ▪ Graph the curve and find the area that it encloses.

13. $r = 1 + 2\sin 6\theta$

14. $r = 2\sin\theta + 3\sin 9\theta$

15–18 ▪ Find the area of the region enclosed by one loop of the curve.

15. $r = \sin 2\theta$

16. $r = 4\sin 3\theta$

17. $r = 1 + 2\sin\theta$ (inner loop)

18. $r = 2\cos\theta - \sec\theta$

19–22 ▪ Find the area of the region that lies inside the first curve and outside the second curve.

19. $r = 4\sin\theta$, $r = 2$

20. $r = 1 - \sin\theta$, $r = 1$

21. $r = 3\cos\theta$, $r = 1 + \cos\theta$

22. $r = 2 + \sin\theta$, $r = 3\sin\theta$

23–26 ▪ Find the area of the region that lies inside both curves.

23. $r = \sin\theta$, $r = \cos\theta$

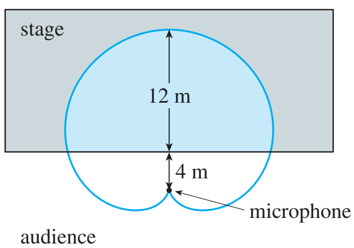
24. $r = \sin 2\theta$, $r = \sin\theta$

25. $r = \sin 2\theta$, $r = \cos 2\theta$

26. $r^2 = 2\sin 2\theta$, $r = 1$

27. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

28. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is



given by the cardioid $r = 8 + 8 \sin \theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.

29–32 ■ Find all points of intersection of the given curves.

29. $r = \cos \theta, \quad r = 1 - \cos \theta$

30. $r = \cos 3\theta, \quad r = \sin 3\theta$

31. $r = \sin \theta, \quad r = \sin 2\theta$

32. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

33–36 ■ Find the exact length of the polar curve.

33. $r = 3 \sin \theta, \quad 0 \leqslant \theta \leqslant \pi/3$

34. $r = e^{2\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$

35. $r = \theta^2, \quad 0 \leqslant \theta \leqslant 2\pi$

36. $r = \theta, \quad 0 \leqslant \theta \leqslant 2\pi$

37–38 ■ Use a calculator to find the length of the curve correct to four decimal places.

37. $r = 3 \sin 2\theta$

38. $r = 4 \sin 3\theta$

9.5 CONIC SECTIONS IN POLAR COORDINATES

In your previous study of conic sections, parabolas were defined in terms of a focus and directrix whereas ellipses and hyperbolas were defined in terms of two foci. After reviewing those definitions and equations, we present a more unified treatment of all three types of conic sections in terms of a focus and directrix. Furthermore, if we place the focus at the origin, then a conic section has a simple polar equation. In Chapter 10 we will use the polar equation of an ellipse to derive Kepler's laws of planetary motion.

CONICS IN CARTESIAN COORDINATES

Here we provide a brief reminder of what you need to know about conic sections. A more thorough review can be found on the website www.stewartcalculus.com.

Recall that a **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 1. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

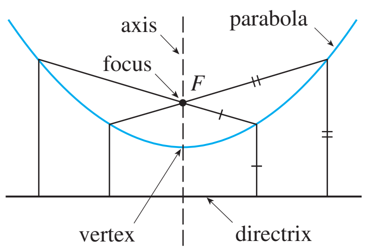
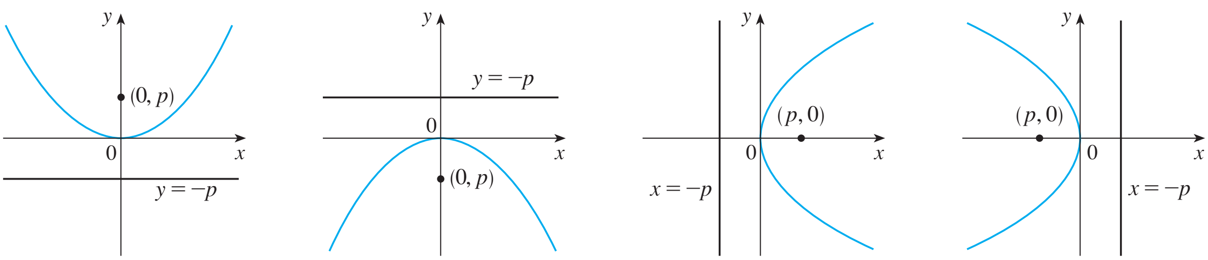


FIGURE 1

A parabola has a very simple equation if its vertex is placed at the origin and its directrix is parallel to the x-axis or y-axis. If the focus is on the y-axis at the point $(0, p)$, then the directrix has the equation $y = -p$ and an equation of the parabola is $x^2 = 4py$. [See parts (a) and (b) of Figure 2.] If the focus is on the x-axis at $(p, 0)$, then the directrix is $x = -p$ and an equation is $y^2 = 4px$ as in parts (c) and (d).



(a) $x^2 = 4py,\ p > 0$ (b) $x^2 = 4py,\ p < 0$ (c) $y^2 = 4px,\ p > 0$ (d) $y^2 = 4px,\ p < 0$

FIGURE 2