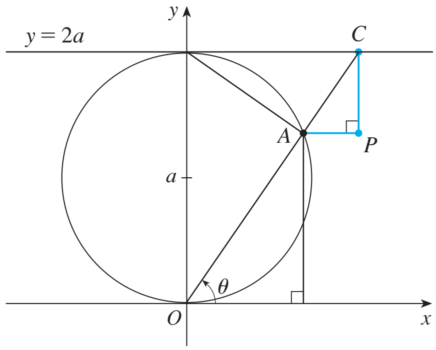
**36.** A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point $P$ in the figure. Show that parametric equations for this curve can be written as

$$x = 2a \cot \theta \qquad y = 2a \sin^2\theta$$

Sketch the curve.




**37.** Suppose that the position of one particle at time $t$ is given by

$$x_1 = 3 \sin t \qquad y_1 = 2 \cos t \qquad 0 \leqslant t \leqslant 2\pi$$

and the position of a second particle is given by

$$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(a) Graph the paths of both particles. How many points of intersection are there?

(b) Are any of these points of intersection *collision points*? In other words, are the particles ever at the same place at the same time? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

**38.** If a projectile is fired with an initial velocity of $v_0$ meters per second at an angle $\alpha$ above the horizontal and air resistance is assumed to be negligible, then its position after $t$ seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where $g$ is the acceleration due to gravity (9.8 m/s$^2$).

(a) If a gun is fired with $\alpha = 30°$ and $v_0 = 500$ m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?

(b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle $\alpha$ to see where it hits the ground. Summarize your findings.

(c) Show that the path is parabolic by eliminating the parameter.

**39.** Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as $c$ increases? Illustrate by graphing several members of the family.

**40.** The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when $c$ increases?

**41.** The curves with equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when $a$, $b$, and $n$ vary. (Take $n$ to be a positive integer.)

**42.** Investigate the family of curves defined by the parametric equations

$$x = \sin t\,(c - \sin t) \qquad y = \cos t\,(c - \sin t)$$

How does the shape change as $c$ changes? In particular, you should identify the transitional values of $c$ for which the basic shape of the curve changes.

## 9.2 CALCULUS WITH PARAMETRIC CURVES

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, areas, and arc length.

### TANGENTS

Suppose $f$ and $g$ are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$ where $y$ is also a differentiable function of $x$. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for $dy/dx$:

▪ If we think of the curve as being traced out by a moving particle, then $dy/dt$ and $dx/dt$ are the vertical and horizontal velocities of the particle and Formula 1 says that the slope of the tangent is the ratio of these velocities.

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0} \tag{1}$$

Equation 1 (which you can remember by thinking of canceling the $dt$'s) enables us to find the slope $dy/dx$ of the tangent to a parametric curve without having to eliminate the parameter $t$. We see from (1) that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

As we know from Chapter 4, it is also useful to consider $d^2y/dx^2$. This can be found by replacing $y$ by $dy/dx$ in Equation 1:

⊘ Note that $\dfrac{d^2y}{dx^2} \neq \dfrac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

**EXAMPLE 1** A curve $C$ is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.
(a) Show that $C$ has two tangents at the point $(3, 0)$ and find their equations.
(b) Find the points on $C$ where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

**SOLUTION**
(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore, the point $(3, 0)$ on $C$ arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that $C$ crosses itself at $(3, 0)$. Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at $(3, 0)$ are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

(b) $C$ has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on $C$ are $(1, -2)$ and $(1, 2)$. $C$ has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on $C$ is $(0, 0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

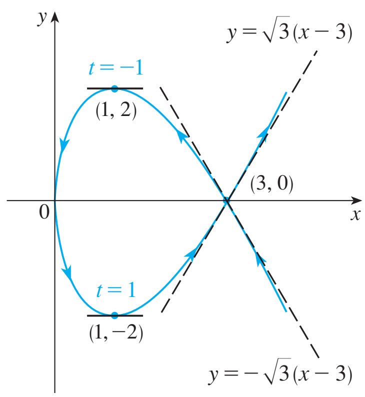



**FIGURE 1**

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch $C$ in Figure 1. ■

## V EXAMPLE 2

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$. (See Example 7 in Section 9.1.)

(b) At what points is the tangent horizontal? When is it vertical?

**SOLUTION**

(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and
$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$

Therefore, the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \qquad \text{or} \qquad \sqrt{3}\,x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.

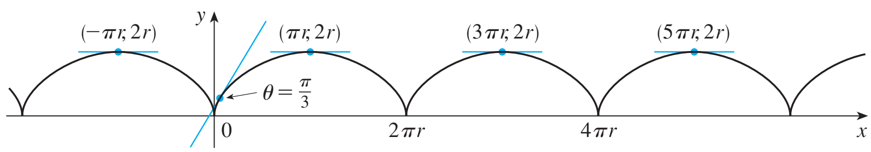



**FIGURE 2**

(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n-1)\pi$, $n$ an integer. The corresponding point on the cycloid is $\big((2n-1)\pi r, 2r\big)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta\to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta\to 2n\pi^+} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$. ■

## AREAS

We know that the area under a curve $y = F(x)$ from $a$ to $b$ is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is given by parametric equations $x = f(t)$, $y = g(t)$ and is traversed once as $t$ increases from $\alpha$ to $\beta$, then we can adapt the earlier formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

$$\left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt \quad \text{if } \big(f(\beta), g(\beta)\big) \text{ is the leftmost endpoint}\right]$$

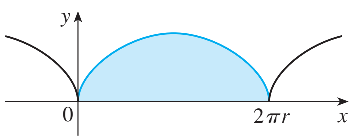


**FIGURE 3**

■ The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 7 in Section 9.1). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

**V EXAMPLE 3** Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$. (See Figure 3.)

**SOLUTION** One arch of the cycloid is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = r^2\left(\tfrac{3}{2} \cdot 2\pi\right) = 3\pi r^2 \end{aligned}$$ ■

## ARC LENGTH

We already know how to find the length $L$ of a curve $C$ given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 7.4.3 says that if $F'$ is continuous, then

**2** $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Suppose that $C$ can also be described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that $C$ is traversed once, from left to right, as $t$ increases from $\alpha$ to $\beta$ and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

Since $dx/dt > 0$, we have

**3** $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Even if $C$ can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into $n$ subintervals of equal width $\Delta t$. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subinter-

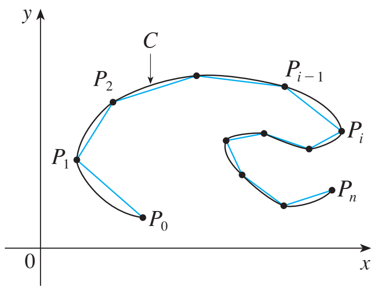



**FIGURE 4**

vals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on $C$ and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates $C$ (see Figure 4).

As in Section 7.4, we define the length $L$ of $C$ to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to $f$ on the interval $[t_{i-1}, t_i]$, gives a number $t_i^*$ in $(t_{i-1}, t_i)$ such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\,\Delta t$$

Similarly, when applied to $g$, the Mean Value Theorem gives a number $t_i^{**}$ in $(t_{i-1}, t_i)$ such that

$$\Delta y_i = g'(t_i^{**})\,\Delta t$$

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\,\Delta t]^2 + [g'(t_i^{**})\,\Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t \end{aligned}$$

and so

**4** $$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if $f'$ and $g'$ are continuous, it can be shown that the limit in (4) is the same as if $t_i^*$ and $t_i^{**}$ were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as (3).

**5 THEOREM** If a curve $C$ is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $f'$ and $g'$ are continuous on $[\alpha, \beta]$ and $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$ of Section 7.4.

**EXAMPLE 4** If we use the representation of the unit circle given in Example 2 in Section 9.1,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 5 gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation given in Example 3 in Section 9.1,

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in Theorem 5 gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} 2\, dt = 4\pi$$

Notice that the integral gives twice the arc length of the circle because as $t$ increases from 0 to $2\pi$, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve $C$ from a parametric representation, we have to be careful to ensure that $C$ is traversed only once as $t$ increases from $\alpha$ to $\beta$. ■

**V EXAMPLE 5** Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

**SOLUTION** From Example 3 we see that one arch is described by the parameter interval $0 \leqslant \theta \leqslant 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta = r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

■ The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 5). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

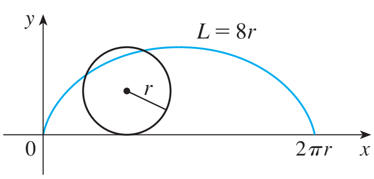



**FIGURE 5**

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\left[-2\cos(\theta/2)\right]_0^{2\pi}$$

$$= 2r[2 + 2] = 8r$$ ■

## 9.2 EXERCISES

**1–2** ■ Find $dy/dx$.

**1.** $x = t - t^3, \quad y = 2 - 5t$

**2.** $x = te^t, \quad y = t + e^t$

**3–6** ■ Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

**3.** $x = t^4 + 1, \quad y = t^3 + t; \quad t = -1$

**4.** $x = 2t^2 + 1, \quad y = \frac{1}{3}t^3 - t; \quad t = 3$

**5.** $x = e^{\sqrt{t}}, \quad y = t - \ln t^2; \quad t = 1$

**6.** $x = \cos\theta + \sin 2\theta, \quad y = \sin\theta + \cos 2\theta; \quad \theta = 0$

**7.** Find an equation of the tangent to the curve $x = e^t$, $y = (t - 1)^2$ at the point $(1, 1)$ by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

**8.** Find equations of the tangents to the curve $x = \sin t$, $y = \sin(t + \sin t)$ at the origin. Then graph the curve and the tangents.

**9–12** ■ Find $dy/dx$ and $d^2y/dx^2$. For which values of $t$ is the curve concave upward?

**9.** $x = 4 + t^2, \quad y = t^2 + t^3$

**10.** $x = t^3 - 12t, \quad y = t^2 - 1$

**11.** $x = t - e^t, \quad y = t + e^{-t}$

**12.** $x = t + \ln t, \quad y = t - \ln t$

**13–16** ■ Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

**13.** $x = 10 - t^2, \quad y = t^3 - 12t$

**14.** $x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$

**15.** $x = 2\cos\theta, \quad y = \sin 2\theta$

**16.** $x = \cos 3\theta, \quad y = 2\sin\theta$

**17.** Use a graph to estimate the coordinates of the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$. Then use calculus to find the exact coordinates.

**18.** Try to estimate the coordinates of the highest point and the leftmost point on the curve $x = te^t$, $y = te^{-t}$. Then find the exact coordinates. What are the asymptotes of this curve?

**19–20** ■ Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

**19.** $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

**20.** $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

**21.** Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

**22.** At what point does the curve $x = 1 - 2\cos^2 t$, $y = (\tan t)(1 - 2\cos^2 t)$ cross itself? Find the equations of both tangents at that point.

**23.** (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of $\theta$. (See Exercise 34 in Section 9.1.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

**24.** (a) Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ in terms of $\theta$.
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or $-1$?

**25.** At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with equations $x = -7t$, $y = 12t - 5$?

**26.** Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

**27.** Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

**28.** Find the area bounded by the curve $x = t - 1/t$, $y = t + 1/t$ and the line $y = 2.5$.

**29.** Find the area bounded by the curve $x = \cos t$, $y = e^t$, $0 \leqslant t \leqslant \pi/2$, and the lines $y = 1$ and $x = 0$.

**30.** Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

**31.** Find the area under one arch of the trochoid of Exercise 34 in Section 9.1 for the case $d < r$.

**32.** Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the $x$-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

**33–36** ■ Set up, but do not evaluate, an integral that represents the length of the curve.

**33.** $x = t - t^2, \quad y = \frac{4}{3}t^{3/2}, \quad 1 \leqslant t \leqslant 2$

**34.** $x = 1 + e^t, \quad y = t^2, \quad -3 \leqslant t \leqslant 3$

**35.** $x = t + \cos t, \quad y = t - \sin t, \quad 0 \leqslant t \leqslant 2\pi$

**36.** $x = \ln t, \quad y = \sqrt{t+1}, \quad 1 \leqslant t \leqslant 5$

**37–40** ▪ Find the length of the curve.

**37.** $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

**38.** $x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta),$
$0 \leqslant \theta \leqslant \pi$

**39.** $x = \dfrac{t}{1+t}, \quad y = \ln(1+t), \quad 0 \leqslant t \leqslant 2$

**40.** $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \leqslant t \leqslant 3$

**41–43** ▪ Graph the curve and find its length.

**41.** $x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \leqslant t \leqslant \pi$

**42.** $x = \cos t + \ln\left(\tan\frac{1}{2}t\right), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

**43.** $x = e^t - t, \quad y = 4e^{t/2}, \quad -8 \leqslant t \leqslant 3$

**44.** Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

**45.** Use Simpson's Rule with $n = 6$ to estimate the length of the curve $x = t - e^t$, $y = t + e^t$, $-6 \leqslant t \leqslant 6$.

**46.** In Exercise 36 in Section 9.1 you were asked to derive the parametric equations $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

**47–48** ▪ Find the distance traveled by a particle with position $(x, y)$ as $t$ varies in the given time interval. Compare with the length of the curve.

**47.** $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

**48.** $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

**49.** Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where $e$ is the eccentricity of the ellipse $\left(e = c/a, \text{ where } c = \sqrt{a^2 - b^2}\right)$.

**50.** Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

CAS **51.** (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?

(b) Use your CAS to find the approximate length of this curve.

CAS **52.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\,du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2)\,du$$

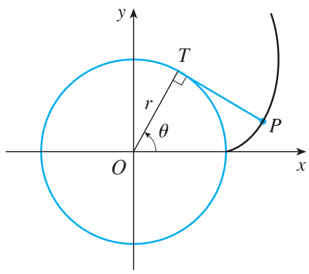
where $C$ and $S$ are the Fresnel functions that were introduced in Chapter 5.

(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?

(b) Find the length of Cornu's spiral from the origin to the point with parameter value $t$.

**53.** A string is wound around a circle and then unwound while being held taut. The curve traced by the point $P$ at the end of the string is called the **involute** of the circle. If the circle has radius $r$ and center $O$ and the initial position of $P$ is $(r, 0)$, and if the parameter $\theta$ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$




**54.** A cow is tied to a silo with radius $r$ by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.