## 12.4 EXERCISES

**1.** Electric charge is distributed over the rectangle $1 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 2$ so that the charge density at $(x, y)$ is $\sigma(x, y) = 2xy + y^2$ (measured in coulombs per square meter). Find the total charge on the rectangle.

**2.** Electric charge is distributed over the disk $x^2 + y^2 \leqslant 4$ so that the charge density at $(x, y)$ is $\sigma(x, y) = x + y + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

**3–10** ▪ Find the mass and center of mass of the lamina that occupies the region $D$ and has the given density function $\rho$.

**3.** $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$; $\rho(x, y) = xy^2$

**4.** $D = \{(x, y) \mid 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$; $\rho(x, y) = cxy$

**5.** $D$ is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

**6.** $D$ is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(4, 0)$; $\rho(x, y) = x$

**7.** $D$ is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\rho(x, y) = y$

**8.** $D$ is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$; $\rho(x, y) = x$

**9.** $D$ is bounded by the parabola $x = y^2$ and the line $y = x - 2$; $\rho(x, y) = 3$

**10.** $D = \{(x, y) \mid 0 \leqslant y \leqslant \cos x,\ 0 \leqslant x \leqslant \pi/2\}$; $\rho(x, y) = x$

**11.** A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the $x$-axis.

**12.** Find the center of mass of the lamina in Exercise 11 if the density at any point is proportional to the square of its distance from the origin.

**13.** Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length $a$ if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

**14.** A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

**15.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 7.

**16.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 12.

**17.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 9.

**18.** Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the $x$-axis or the $y$-axis?

CAS **19–20** ▪ Use a computer algebra system to find the mass, center of mass, and moments of inertia of the lamina that occupies the region $D$ and has the given density function.

**19.** $D = \{(x, y) \mid 0 \leqslant y \leqslant \sin x,\ 0 \leqslant x \leqslant \pi\}$; $\rho(x, y) = xy$

**20.** $D$ is enclosed by the cardioid $r = 1 + \cos\theta$; $\rho(x, y) = \sqrt{x^2 + y^2}$

**21.** A lamina with constant density $\rho(x, y) = \rho$ occupies a square with vertices $(0, 0)$, $(a, 0)$, $(a, a)$, and $(0, a)$. Find the moments of inertia $I_x$ and $I_y$ and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

**22.** A lamina with constant density $\rho(x, y) = \rho$ occupies the region under the curve $y = \sin x$ from $x = 0$ to $x = \pi$. Find the moments of inertia $I_x$ and $I_y$ and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

## 12.5 TRIPLE INTEGRALS

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where $f$ is defined on a rectangular box:

$$\boxed{1} \qquad B = \left\{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\right\}$$

The first step is to divide $B$ into sub-boxes. We do this by dividing the interval $[a, b]$ into $l$ subintervals $[x_{i-1}, x_i]$ with lengths $\Delta x_i = x_i - x_{i-1}$, dividing $[c, d]$ into $m$ subintervals with lengths $\Delta y_j = y_j - y_{j-1}$, and dividing $[r, s]$ into $n$ subintervals with lengths $\Delta z_k = z_k - z_{k-1}$. The planes through the endpoints of these subintervals paral-

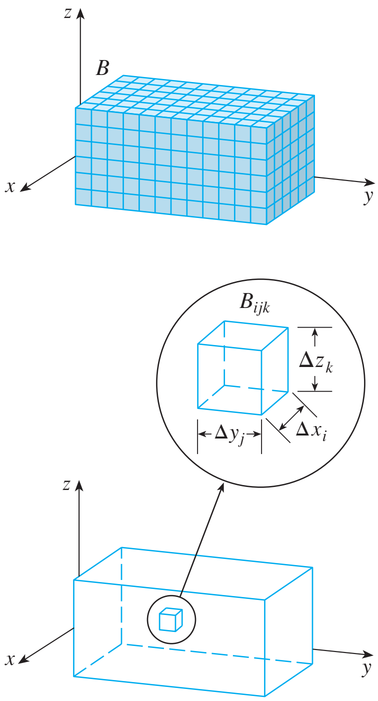



FIGURE 1

lel to the coordinate planes divide the box $B$ into $lmn$ sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. The sub-box $B_{ijk}$ has volume $\Delta V_{ijk} = \Delta x_i \, \Delta y_j \, \Delta z_k$.

Then we form the **triple Riemann sum**

**2**
$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V_{ijk}$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in $B_{ijk}$. By analogy with the definition of a double integral (12.1.5), we define the triple integral as the limit of the triple Riemann sums in (2) as the sub-boxes shrink.

**3 DEFINITION** The **triple integral** of $f$ over the box $B$ is

$$\iiint_B f(x, y, z) \, dV = \lim_{\max \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V_{ijk}$$

if this limit exists.

Again, the triple integral always exists if $f$ is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point $(x_i, y_j, z_k)$, and if we choose sub-boxes with the same dimensions. so that $\Delta V_{ijk} = \Delta V$, we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k) \, \Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

**4 FUBINI'S THEOREM FOR TRIPLE INTEGRALS** If $f$ is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to $x$ (keeping $y$ and $z$ fixed), then we integrate with respect to $y$ (keeping $z$ fixed), and finally we integrate with respect to $z$. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to $y$, then $z$, and then $x$, we have

$$\iiint_B f(x, y, z) \, dV = \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx$$

**V EXAMPLE 1** Evaluate the triple integral $\iiint_B xyz^2\, dV$, where $B$ is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

**SOLUTION** We could use any of the six possible orders of integration. If we choose to integrate with respect to $x$, then $y$, and then $z$, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz = \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\Big]_0^3 = \frac{27}{4}$$ ■

Now we define the **triple integral over a general bounded region $E$** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (12.2.2). We enclose $E$ in a box $B$ of the type given by Equation 1. Then we define a function $F$ so that it agrees with $f$ on $E$ but is 0 for points in $B$ that are outside $E$. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if $f$ is continuous and the boundary of $E$ is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 12.2).

We restrict our attention to continuous functions $f$ and to certain simple types of regions. A solid region $E$ is said to be of **type 1** if it lies between the graphs of two continuous functions of $x$ and $y$, that is,

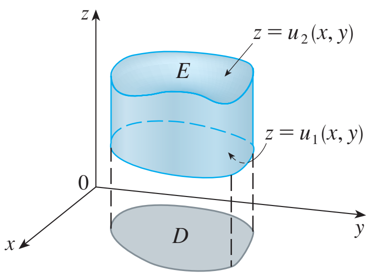



**FIGURE 2**
A type 1 solid region

**5** $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where $D$ is the projection of $E$ onto the $xy$-plane as shown in Figure 2. Notice that the upper boundary of the solid $E$ is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to (12.2.3), it can be shown that if $E$ is a type 1 region given by Equation 5, then

**6** $$\boxed{\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA}$$

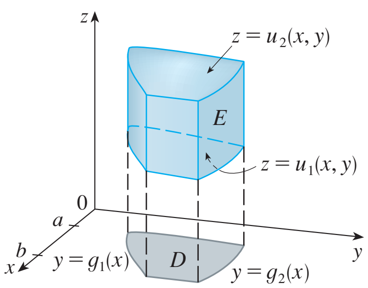



**FIGURE 3**
A type 1 solid region

The meaning of the inner integral on the right side of Equation 6 is that $x$ and $y$ are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to $z$.

In particular, if the projection $D$ of $E$ onto the $xy$-plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

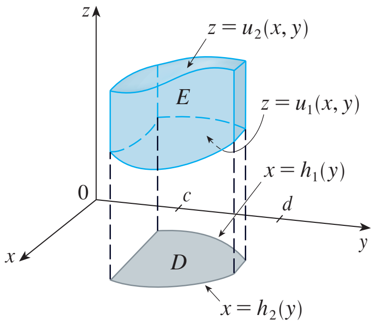



**FIGURE 4**
Another type 1 solid region

and Equation 6 becomes

**7**
$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

If, on the other hand, $D$ is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

**8**
$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

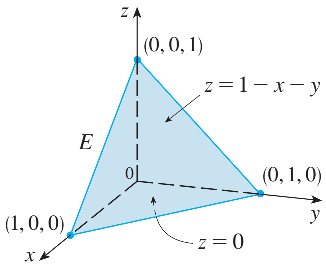



**FIGURE 5**

**EXAMPLE 2** Evaluate $\iiint_E z\, dV$, where $E$ is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

**SOLUTION** When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region $E$ (see Figure 5) and one of its projection $D$ on the $xy$-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the $xy$-plane. So the projection of $E$ is the triangular region shown in Figure 6, and we have

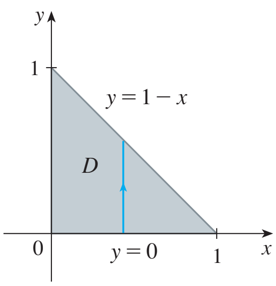



**FIGURE 6**

**9**
$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of $E$ as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$ ▪

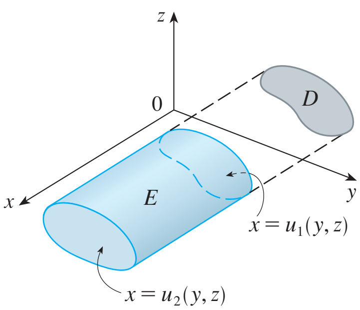



**FIGURE 7**
A type 2 region

A solid region $E$ is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, $D$ is the projection of $E$ onto the $yz$-plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

**10**
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx\right] dA$$

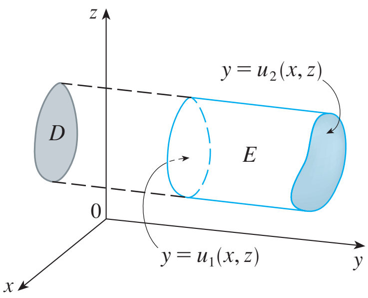



**FIGURE 8**
A type 3 region

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where $D$ is the projection of $E$ onto the $xz$-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8). For this type of region we have

**11**
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[ \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether $D$ is a type I or type II plane region (and corresponding to Equations 7 and 8).

**V EXAMPLE 3** Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where $E$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid $E$ is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection $D_1$ onto the $xy$-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

Visual 12.5 illustrates how solid regions (including the one in Figure 9) project onto coordinate planes.

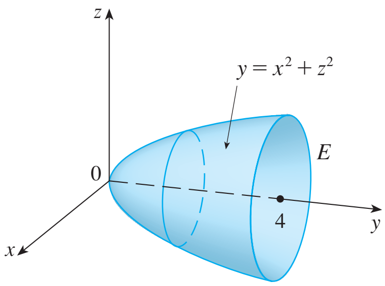



**FIGURE 9**
Region of integration

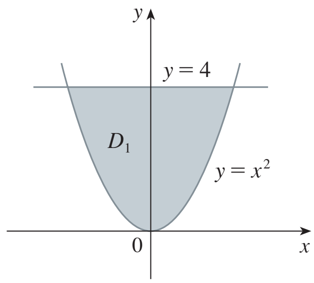



**FIGURE 10**
Projection on $xy$-plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of $E$ is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore, the description of $E$ as a type 1 region is

$$E = \{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y - x^2}}^{\sqrt{y - x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

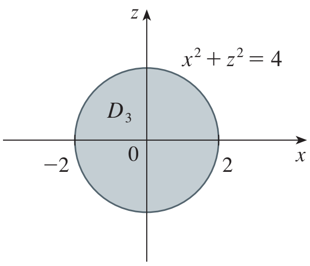



**FIGURE 11**
Projection on $xz$-plane

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider $E$ as a type 3 region. As such, its projection $D_3$ onto the $xz$-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 11.

Then the left boundary of $E$ is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_{D_3} \left[ \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right] dA$$

$$= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, dA$$

Although this integral could be written as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the $xz$-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr = 2\pi \left[ \frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}$$ ■

⊘ The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

## APPLICATIONS OF TRIPLE INTEGRALS

Recall that if $f(x) \geqslant 0$, then the single integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from $a$ to $b$, and if $f(x, y) \geqslant 0$, then the double integral $\iint_D f(x, y)\, dA$ represents the volume under the surface $z = f(x, y)$ and above $D$. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\, dV$, where $f(x, y, z) \geqslant 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that $E$ is just the *domain* of the function $f$; the graph of $f$ lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of $x$, $y$, $z$ and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in $E$. Then the triple integral does represent the volume of $E$:

**12**
$$\boxed{V(E) = \iiint_E dV}$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

and from Section 12.2 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

**EXAMPLE 4** Use a triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

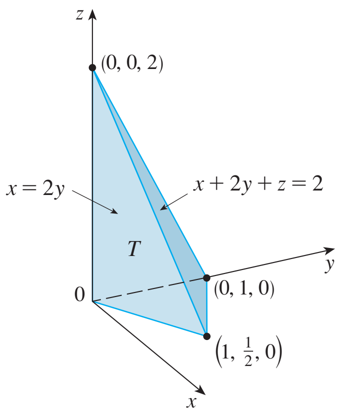



**FIGURE 12**

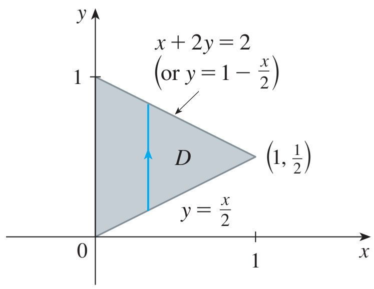



**FIGURE 13**

**SOLUTION** The tetrahedron $T$ and its projection $D$ on the $xy$-plane are shown in Figures 12 and 13. The lower boundary of $T$ is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$. Therefore, we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

by the same calculation as in Example 4 in Section 12.2.

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ■

All the applications of double integrals in Section 12.4 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region $E$ is $\rho(x, y, z)$, in units of mass per unit volume, at any given point $(x, y, z)$, then its **mass** is

**13** $$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

**14** $$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

**15** $$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of $E$. The **moments of inertia** about the three coordinate axes are

**16** $$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

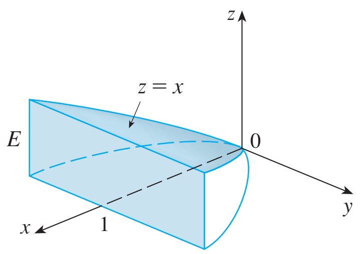
As in Section 12.4, the total **electric charge** on a solid object occupying a region $E$ and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

**EXAMPLE 5** Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

**SOLUTION** The solid $E$ and its projection onto the $xy$-plane are shown in Figure 14. The lower and upper surfaces of $E$ are the planes $z = 0$ and $z = x$, so we describe $E$ as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$



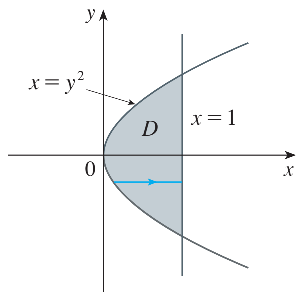




**FIGURE 14**

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$\begin{aligned} m &= \iiint_E \rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\, dz\, dx\, dy = \rho \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy \\ &= \rho \int_{-1}^{1} \left[ \frac{x^2}{2} \right]_{x=y^2}^{x=1} dy = \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\, dy \\ &= \rho \int_0^1 (1 - y^4)\, dy = \rho \left[ y - \frac{y^5}{5} \right]_0^1 = \frac{4\rho}{5} \end{aligned}$$

Because of the symmetry of $E$ and $\rho$ about the $xz$-plane, we can immediately say that $M_{xz} = 0$ and, therefore, $\bar{y} = 0$. The other moments are

$$\begin{aligned} M_{yz} &= \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy = \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy \\ &= \rho \int_{-1}^{1} \left[ \frac{x^3}{3} \right]_{x=y^2}^{x=1} dy = \frac{2\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3} \left[ y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7} \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy = \rho \int_{-1}^{1} \int_{y^2}^{1} \left[ \frac{z^2}{2} \right]_{z=0}^{z=x} dx\, dy \\ &= \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \frac{\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7} \end{aligned}$$

Therefore, the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left( \tfrac{5}{7}, 0, \tfrac{5}{14} \right)$$ ■

## 12.5 EXERCISES

**1.** Evaluate the integral in Example 1, integrating first with respect to $z$, then $x$, and then $y$.

**2.** Evaluate the integral $\iiint_E (xz - y^3)\, dV$, where

$$E = \{(x, y, z) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2, 0 \leqslant z \leqslant 1\}$$

using three different orders of integration.

**3–6** ▪ Evaluate the iterated integral.

**3.** $\displaystyle\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$

**4.** $\displaystyle\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$

**5.** $\displaystyle\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\, dx\, dz\, dy$

**6.** $\displaystyle\int_0^1 \int_0^z \int_0^y ze^{-y^2}\, dx\, dy\, dz$

**7–16** ■ Evaluate the triple integral.

**7.** $\iiint_E 2x\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant \sqrt{4 - y^2},\ 0 \leqslant z \leqslant y\}$

**8.** $\iiint_E yz \cos(x^5)\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x,\ x \leqslant z \leqslant 2x\}$

**9.** $\iiint_E 6xy\, dV$, where $E$ lies under the plane $z = 1 + x + y$ and above the region in the $xy$-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

**10.** $\iiint_E y\, dV$, where $E$ is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

**11.** $\iiint_E xy\, dV$, where $E$ is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$

**12.** $\iiint_E xz\, dV$, where $E$ is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$

**13.** $\iiint_E x^2 e^y\, dV$, where $E$ is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$

**14.** $\iiint_E (x + 2y)\, dV$, where $E$ is bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$, and $z = 0$

**15.** $\iiint_E x\, dV$, where $E$ is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

**16.** $\iiint_E z\, dV$, where $E$ is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

**17–20** ■ Use a triple integral to find the volume of the given solid.

**17.** The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

**18.** The solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$

**19.** The solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$

**20.** The solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 16$

**21.** (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

CAS (b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

**22.** (a) In the **Midpoint Rule for triple integrals** we use a triple Riemann sum to approximate a triple integral over a box $B$, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box $B_{ijk}$. Use the Midpoint Rule to estimate $\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV$, where $B$ is the cube defined by $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant 4$. Divide $B$ into eight cubes of equal size.

CAS (b) Use a computer algebra system to approximate the integral in part (a) correct to the nearest integer. Compare with the answer to part (a).

**23–24** ■ Use the Midpoint Rule for triple integrals (Exercise 22) to estimate the value of the integral. Divide $B$ into eight sub-boxes of equal size.

**23.** $\iiint_B \dfrac{1}{\ln(1 + x + y + z)}\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 8,\ 0 \leqslant z \leqslant 4\}$

**24.** $\iiint_B \sin(xy^2z^3)\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 1\}$

**25–26** ■ Sketch the solid whose volume is given by the iterated integral.

**25.** $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$

**26.** $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

**27–30** ■ Express the integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in six different ways, where $E$ is the solid bounded by the given surfaces.

**27.** $x^2 + z^2 = 4$, $y = 0$, $y = 6$

**28.** $z = 0$, $x = 0$, $y = 2$, $z = y - 2x$

**29.** $z = 0$, $z = y$, $x^2 = 1 - y$

**30.** $9x^2 + 4y^2 + z^2 = 1$

**31.** The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.