39. If f is a constant function, $f(x, y) = k$, and $R = [a, b] \times [c, d]$, show that $\iint_R k\, dA = k(b - a)(d - c)$.

40. If $R = [0, 1] \times [0, 1]$, show that $0 \leqslant \iint_R \sin(x + y)\, dA \leqslant 1$.

CAS **41.** Use your CAS to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

42. (a) In what way are the theorems of Fubini and Clairaut similar?

(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\, dt\, ds$$

for $a < x < b$, $c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

12.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

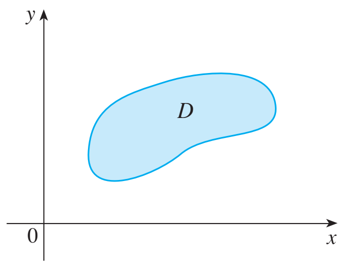


FIGURE 1

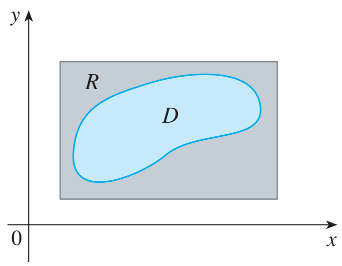


FIGURE 2

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

$$\mathbf{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

If the double integral of F exists over R, then we define the **double integral of f over D** by

$$\boxed{\mathbf{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 12.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$ we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

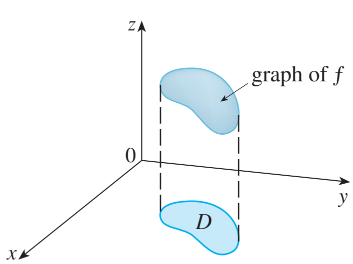


FIGURE 3

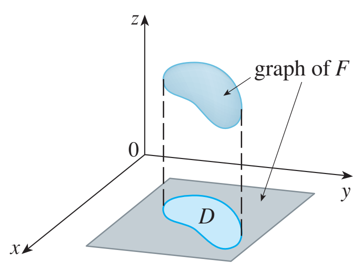


FIGURE 4

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved" (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists. In particular, this is the case for the following types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

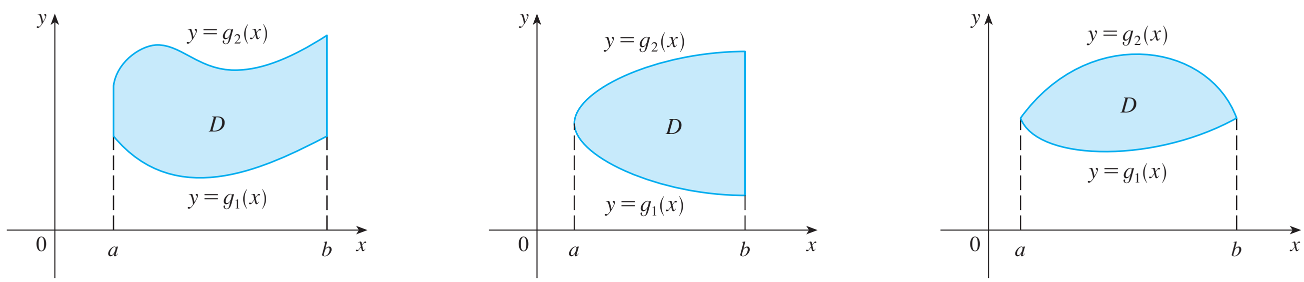


FIGURE 5 Some type I regions

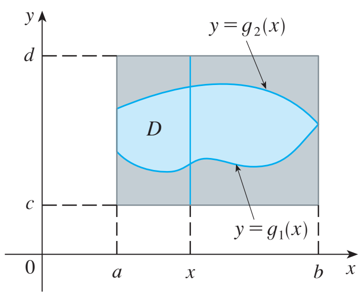


FIGURE 6

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

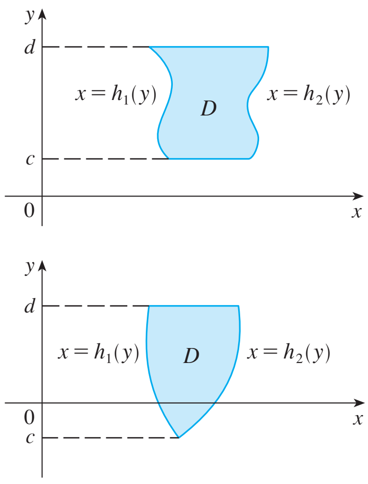


FIGURE 7
Some type II regions

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

Using the same methods that were used in establishing (3), we can show that

5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

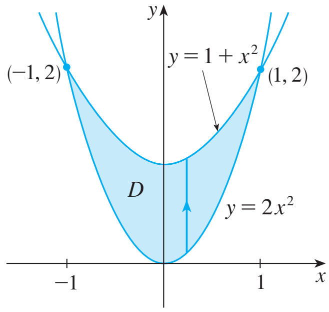


FIGURE 8

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx = \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$ ■

NOTE When we set up a double integral as in Example 1, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.

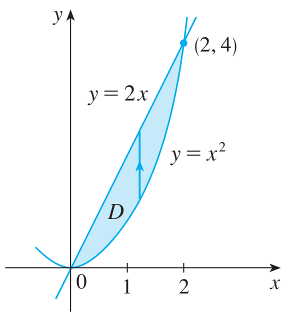


FIGURE 9
D as a type I region

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 2x\}$$

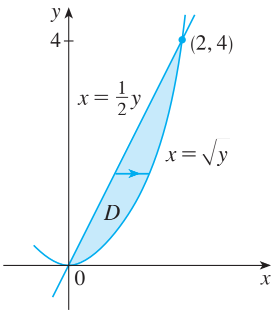


FIGURE 10
D as a type II region

▪ Figure 11 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.

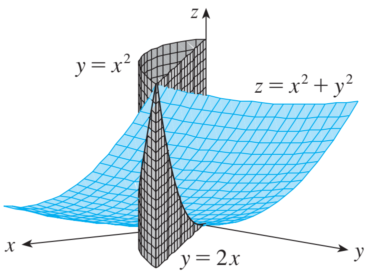


FIGURE 11

Therefore, the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \bigg]_0^2 = \frac{216}{35}$$

SOLUTION 2 From Figure 10 we see that D can also be written as a type II region:

$$D = \left\{ (x, y) \mid 0 \leqslant y \leqslant 4,\ \tfrac{1}{2}y \leqslant x \leqslant \sqrt{y} \right\}$$

Therefore, another expression for V is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy$$

$$= \tfrac{2}{15}y^{5/2} + \tfrac{2}{7}y^{7/2} - \tfrac{13}{96}y^4 \Big]_0^4 = \tfrac{216}{35}$$ ■

V EXAMPLE 3 Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore, we prefer to express D as a type II region:

$$D = \left\{ (x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1 \right\}$$

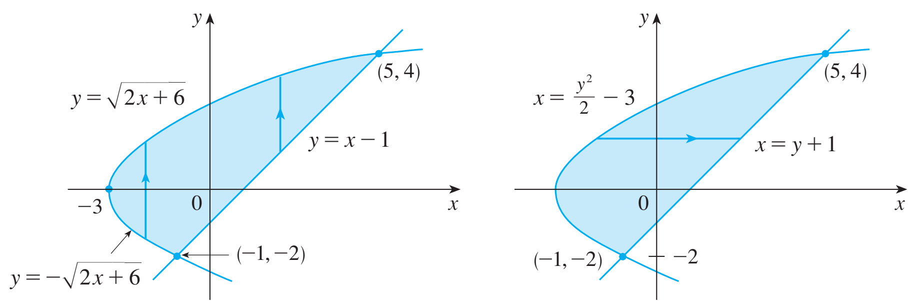


FIGURE 12 (a) D as a type I region (b) D as a type II region

Then (5) gives

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}\,y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2-3\right)^2\right] dy$$

$$= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\,\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method. ■

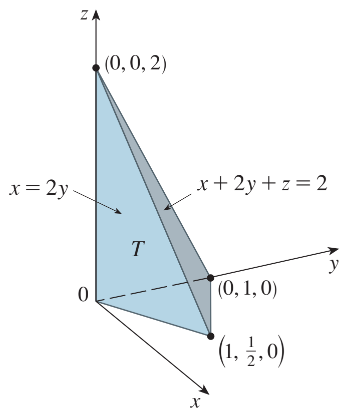


FIGURE 13

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

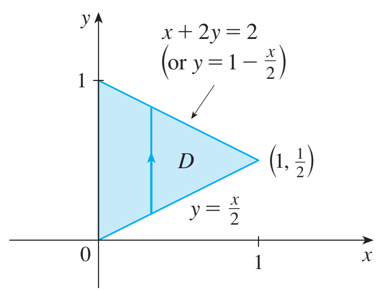


FIGURE 14

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$V = \iint_D (2 - x - 2y)\,dA = \int_0^1\int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx$$

$$= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$= \int_0^1 (x^2 - 2x + 1)\,dx = \frac{x^3}{3} - x^2 + x\Big]_0^1 = \frac{1}{3}$$ ■

V EXAMPLE 5 Evaluate the iterated integral $\int_0^1\int_x^1 \sin(y^2)\,dy\,dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\,dy$. But it's impossible to do so in finite terms since

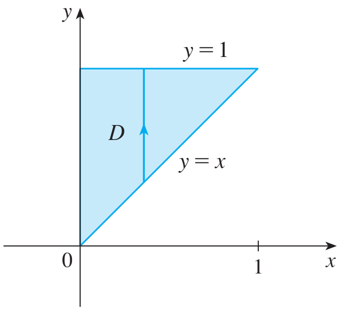


FIGURE 15
D as a type I region

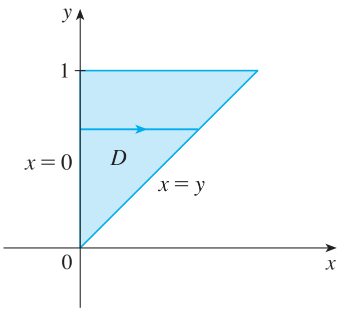


FIGURE 16
D as a type II region

$\int \sin(y^2)\,dy$ is not an elementary function. (See the end of Section 6.4.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

where
$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

This enables us to use (5) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x\sin(y^2)\Big]_{x=0}^{x=y} dy \\ &= \int_0^1 y\sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1 \\ &= \tfrac{1}{2}(1 - \cos 1)\end{aligned}$$

PROPERTIES OF DOUBLE INTEGRALS

We assume that all of the following integrals exist. The first three properties of double integrals over a region D follow immediately from Definition 2 and Properties 12, 13, and 14 in Section 12.1.

6 $$\iint_D [f(x, y) + g(x, y)]\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA$$

7 $$\iint_D cf(x, y)\,dA = c\iint_D f(x, y)\,dA$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\,dA \geqslant \iint_D g(x, y)\,dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

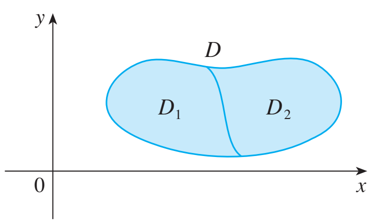
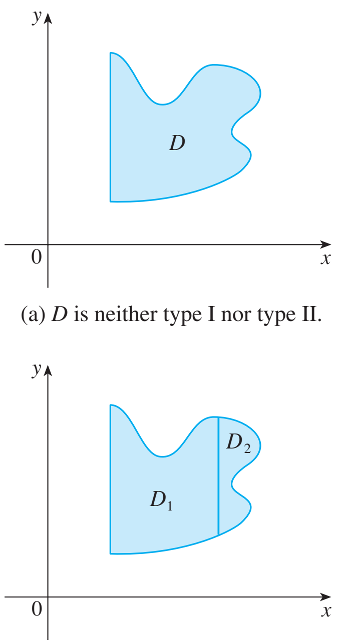


FIGURE 17

9 $$\boxed{\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA}$$



(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

FIGURE 18

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 43 and 44.)

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10
$$\iint_D 1 \, dA = A(D)$$

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1 \, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 47.)

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y) \, dA \leqslant MA(D)$$

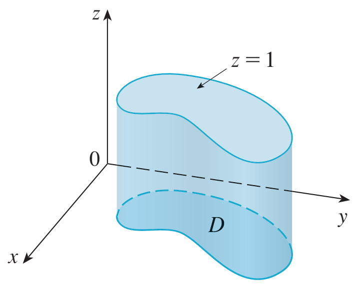


FIGURE 19
Cylinder with base D and height 1

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y} \, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y} \, dA \leqslant 4\pi e$$ ■

12.2 EXERCISES

1–6 ■ Evaluate the iterated integral.

1. $\displaystyle\int_0^1 \int_0^{x^2} (x + 2y) \, dy \, dx$

2. $\displaystyle\int_1^2 \int_y^2 xy \, dx \, dy$

3. $\displaystyle\int_0^1 \int_y^{e^y} \sqrt{x} \, dx \, dy$

4. $\displaystyle\int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx$

5. $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta} \, dr \, d\theta$

6. $\displaystyle\int_0^1 \int_0^v \sqrt{1 - v^2} \, du \, dv$

7–16 ■ Evaluate the double integral.

7. $\displaystyle\iint_D x^3 y^2 \, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2, \ -x \leqslant y \leqslant x\}$

8. $\displaystyle\iint_D \frac{4y}{x^3 + 2} \, dA, \quad D = \{(x, y) \mid 1 \leqslant x \leqslant 2, \ 0 \leqslant y \leqslant 2x\}$

9. $\displaystyle\iint_D \frac{2y}{x^2 + 1} \, dA, \quad D = \left\{(x, y) \mid 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant \sqrt{x}\right\}$

10. $\displaystyle\iint_D e^{y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

11. $\displaystyle\iint_D x\cos y\, dA$, D is bounded by $y = 0,\ y = x^2,\ x = 1$

12. $\displaystyle\iint_D (x + y)\, dA$, D is bounded by $y = \sqrt{x}$ and $y = x^2$

13. $\displaystyle\iint_D y^3\, dA$,
D is the triangular region with vertices (0, 2), (1, 1), (3, 2)

14. $\displaystyle\iint_D xy^2\, dA$, D is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

15. $\displaystyle\iint_D (2x - y)\, dA$,
D is bounded by the circle with center the origin and radius 2

16. $\displaystyle\iint_D 2xy\, dA$, D is the triangular region with vertices (0, 0), (1, 2), and (0, 3)

17–26 ▪ Find the volume of the given solid.

17. Under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$

18. Under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$

19. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

20. Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, $z = 0$

21. Bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$

22. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

23. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

24. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

25. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

26. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

27–28 ▪ Find the volume of the solid by subtracting two volumes.

27. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

28. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

CAS **29–30** ▪ Use a computer algebra system to find the exact volume of the solid.

29. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

30. Enclosed by $z = x^2 + y^2$ and $z = 2y$

31–36 ▪ Sketch the region of integration and change the order of integration.

31. $\displaystyle\int_0^4\int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

32. $\displaystyle\int_0^1\int_{4x}^4 f(x, y)\, dy\, dx$

33. $\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\, dx\, dy$

34. $\displaystyle\int_0^3\int_0^{\sqrt{9-y}} f(x, y)\, dx\, dy$

35. $\displaystyle\int_1^2\int_0^{\ln x} f(x, y)\, dy\, dx$

36. $\displaystyle\int_0^1\int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

37–42 ▪ Evaluate the integral by reversing the order of integration.

37. $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\, dx\, dy$

38. $\displaystyle\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\, dx\, dy$

39. $\displaystyle\int_0^3\int_{y^2}^9 y\cos(x^2)\, dx\, dy$

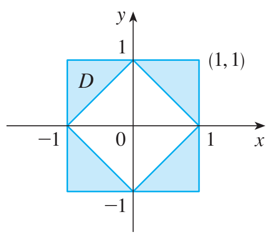
40. $\displaystyle\int_0^1\int_{x^2}^1 x^3\sin(y^3)\, dy\, dx$

41. $\displaystyle\int_0^1\int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1 + \cos^2 x}\, dx\, dy$

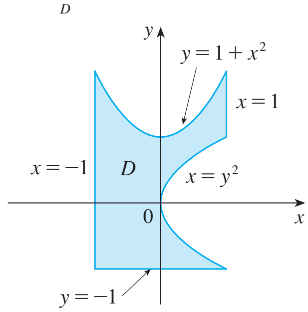
42. $\displaystyle\int_0^8\int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

43–44 ▪ Express D as a union of regions of type I or type II and evaluate the integral.

43. $\displaystyle\iint_D x^2\, dA$



44. $\displaystyle\iint_D xy\, dA$



45–46 ▪ Use Property 11 to estimate the value of the integral.

45. $\displaystyle\iint_D \sqrt{x^3 + y^3}\, dA, \quad D = [0, 1] \times [0, 1]$

46. $\displaystyle\iint_D e^{x^2+y^2}\,dA$,
D is the disk with center the origin and radius $\frac{1}{2}$

47. Prove Property 11.

48. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\,dA = \int_0^1 \int_0^{2y} f(x, y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x, y)\,dx\,dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

49. Evaluate $\iint_D (x^2 \tan x + y^3 + 4)\,dA$, where $D = \{(x, y) \mid x^2 + y^2 \leqslant 2\}$. [*Hint:* Exploit the fact that D is symmetric with respect to both axes.]

50. Use symmetry to evaluate $\iint_D (2 - 3x + 4y)\,dA$, where D is the region bounded by the square with vertices $(\pm 5, 0)$ and $(0, \pm 5)$.

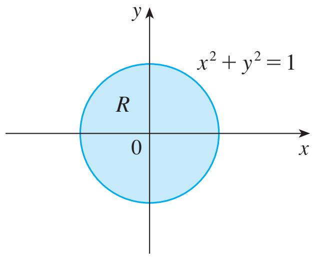
51. Compute $\iint_D \sqrt{1 - x^2 - y^2}\,dA$, where D is the disk $x^2 + y^2 \leqslant 1$, by first identifying the integral as the volume of a solid.

CAS **52.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

12.3 DOUBLE INTEGRALS IN POLAR COORDINATES

■ Polar coordinates were introduced in Section 9.3.

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\,dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated but R is easily described using polar coordinates.



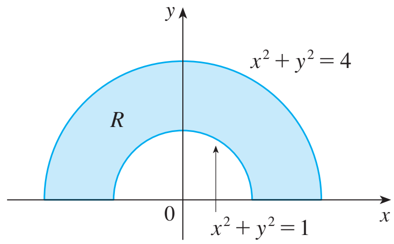


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

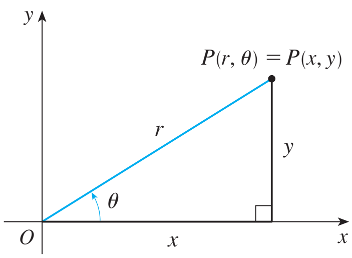


FIGURE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\,dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ with lengths $\Delta r_i = r_i - r_{i-1}$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ with lengths $\Delta\theta_j = \theta_j - \theta_{j-1}$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles shown in Figure 4.