

with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s} \right) f_s$$

where c is the speed of sound, about 332 m/s. (This is the **Doppler effect**.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2 . A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2 , and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

37–40 ■ Assume that all the given functions are differentiable.

37. If $z = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, (a) find $\partial z / \partial r$ and $\partial z / \partial \theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2$$

38. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = e^{-2s} \left[\left(\frac{\partial u}{\partial s} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right]$$

39. If $z = f(x - y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

40. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2 = \frac{\partial z}{\partial s} \frac{\partial z}{\partial t}$$

41–46 ■ Assume that all the given functions have continuous second-order partial derivatives.

41. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

[Hint: Let $u = x + at$, $v = x - at$.]

42. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right]$$

43. If $z = f(x, y)$, where $x = r^2 + s^2$, $y = 2rs$, find $\partial^2 z / \partial r \partial s$. (Compare with Example 7.)

44. If $z = f(x, y)$, where $x = r \cos \theta$, $y = r \sin \theta$, find (a) $\partial z / \partial r$, (b) $\partial z / \partial \theta$, and (c) $\partial^2 z / \partial r \partial \theta$.

45. If $z = f(x, y)$, where $x = r \cos \theta$, $y = r \sin \theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r}$$

46. Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$. (a) Show that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 \\ &\quad + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

(b) Find a similar formula for $\partial^2 z / \partial s \partial t$.

47. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x , y , and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x , F_y , and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$$

11.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x - and y -directions, that is, in the directions of the unit vectors \mathbf{i} and \mathbf{j} .

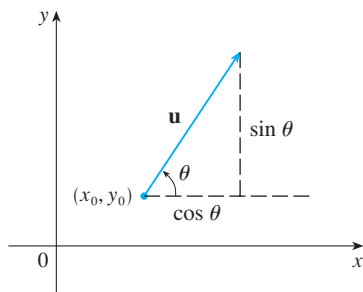


FIGURE 1

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos \theta, \sin \theta \rangle$



Visual 11.6A animates Figure 2 by rotating \mathbf{u} and therefore T .

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 1.) To do this we consider the surface S with equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S . The vertical plane that passes through P in the direction of \mathbf{u} intersects S in a curve C . (See Figure 2.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of \mathbf{u} .

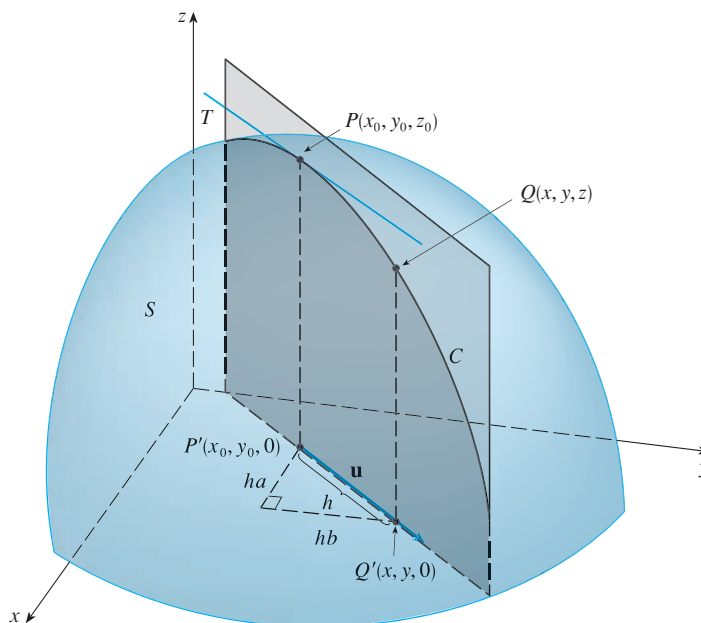


FIGURE 2

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q on the xy -plane, then the vector $\overrightarrow{P'Q'}$ is parallel to \mathbf{u} and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h . Therefore, $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \rightarrow 0$, we obtain the rate of change of z (with respect to distance) in the direction of \mathbf{u} , which is called the directional derivative of f in the direction of \mathbf{u} .

2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations (1), we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 THEOREM If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then by the definition of a derivative we have

$$\begin{aligned} \mathbf{4} \quad g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}}f(x_0, y_0) \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 11.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\mathbf{5} \quad g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \quad \blacksquare$$

If the unit vector \mathbf{u} makes an angle θ with the positive x -axis (as in Figure 1), then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula in Theorem 3 becomes

$$\mathbf{6} \quad D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

EXAMPLE 1 Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and \mathbf{u} is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} = (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2} \\ &= \frac{1}{2} [3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y] \end{aligned}$$

Therefore

$$D_{\mathbf{u}}f(1, 2) = \frac{1}{2} [3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2} \quad \blacksquare$$

■ The directional derivative $D_{\mathbf{u}}f(1, 2)$ in Example 1 represents the rate of change of z in the direction of \mathbf{u} . This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of \mathbf{u} shown in Figure 3.

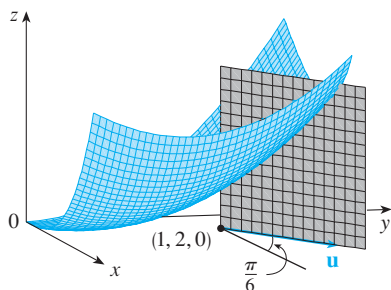


FIGURE 3

THE GRADIENT VECTOR

Notice from Theorem 3 that the directional derivative can be written as the dot product of two vectors:

$$\begin{aligned}
 \mathbf{7} \quad D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\
 &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\
 &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}
 \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f , which is read “del f ”).

8 DEFINITION If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

EXAMPLE 2 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and

$$\nabla f(0, 1) = \langle 2, 0 \rangle$$

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative as

$$\mathbf{9} \quad D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of \mathbf{u} as the scalar projection of the gradient vector onto \mathbf{u} .

EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\begin{aligned}
 \nabla f(x, y) &= 2xy^3 \mathbf{i} + (3x^2y^2 - 4) \mathbf{j} \\
 \nabla f(2, -1) &= -4 \mathbf{i} + 8 \mathbf{j}
 \end{aligned}$$

Note that \mathbf{v} is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of \mathbf{v} is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}} \mathbf{i} + \frac{5}{\sqrt{29}} \mathbf{j}$$

■ The gradient vector $\nabla f(2, -1)$ in Example 3 is shown in Figure 4 with initial point $(2, -1)$. Also shown is the vector \mathbf{v} that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f .

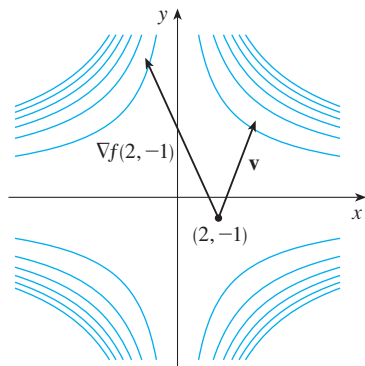


FIGURE 4

Therefore, by Equation 9, we have

$$\begin{aligned} D_{\mathbf{u}}f(2, -1) &= \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j} \right) \\ &= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}} \end{aligned}$$

FUNCTIONS OF THREE VARIABLES

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector \mathbf{u} .

10 DEFINITION The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

$$\mathbf{11} \quad D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable since the vector equation of the line through \mathbf{x}_0 in the direction of the vector \mathbf{u} is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 10.5.1) and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

$$\mathbf{12} \quad D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f , is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

$$\mathbf{13} \quad \nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

EXAMPLE 4 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle\end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore, Equation 14 gives

$$\begin{aligned}D_{\mathbf{u}}f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 3 \left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}}\end{aligned}$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

THEOREM 15 Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when \mathbf{u} has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

where θ is the angle between ∇f and \mathbf{u} . The maximum value of $\cos \theta$ is 1 and this occurs when $\theta = 0$. Therefore, the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when \mathbf{u} has the same direction as ∇f . \square



Visual 11.6B provides visual confirmation of Theorem 15.

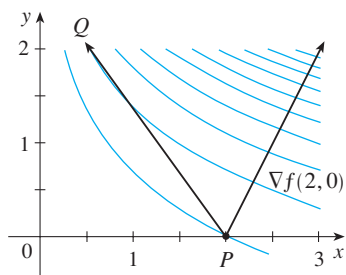


FIGURE 5

■ At $(2, 0)$ the function in Example 5 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. Notice from Figure 5 that this vector appears to be perpendicular to the level curve through $(2, 0)$. Figure 6 shows the graph of f and the gradient vector.

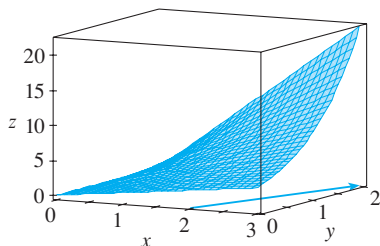


FIGURE 6

EXAMPLE 5

- (a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
 (b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}}f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle \\ &= 1(-\frac{3}{5}) + 2(\frac{4}{5}) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2} (-x \mathbf{i} - 2y \mathbf{j} - 3z \mathbf{k}) \end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \frac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \frac{5}{8} |-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \frac{5\sqrt{41}}{8}$$

Therefore, the maximum rate of increase of temperature is $5\sqrt{41}/8 \approx 4^\circ\text{C}/\text{m}$.

TANGENT PLANES TO LEVEL SURFACES

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S . Let C be any curve that lies on the surface S and passes through the point P . Recall from Section 10.7 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let t_0 be the parameter value corresponding to P ; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S , any point $(x(t), y(t), z(t))$ must satisfy the equation of S , that is,

$$16 \quad F(x(t), y(t), z(t)) = k$$

If x , y , and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$17 \quad \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$18 \quad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector at P , $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P .* (See Figure 7.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$** as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 10.5.7), we can write the equation of this tangent plane as

$$19 \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

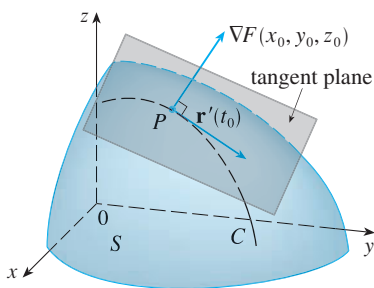


FIGURE 7

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 10.5.3, its symmetric equations are

$$20 \quad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F . Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$

$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 11.4.2. Thus our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 11.4.

EXAMPLE 7 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore, we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\frac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

■ Figure 8 shows the ellipsoid, tangent plane, and normal line in Example 7.

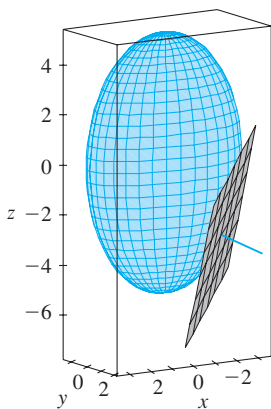


FIGURE 8

SIGNIFICANCE OF THE GRADIENT VECTOR

We now summarize the ways in which the gradient vector is significant. We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f . On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthog-

onal to the level surface S of f through P . (Refer to Figure 7.) These two properties are quite compatible intuitively because as we move away from P on the level surface S , the value of f does not change at all. So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f . Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P . Again this is intuitively plausible because the values of f remain constant as we move along the curve. (See Figure 9.)

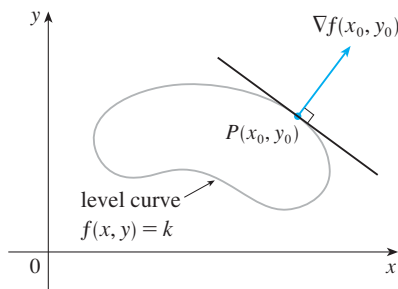


FIGURE 9

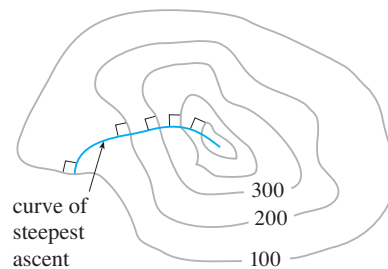


FIGURE 10

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y) , then a curve of steepest ascent can be drawn as in Figure 10 by making it perpendicular to all of the contour lines. This phenomenon can also be noticed in Figure 11 in Section 11.1, where Lonesome Creek follows a curve of steepest descent.

11.6 EXERCISES

1–2 ■ Find the directional derivative of f at the given point in the direction indicated by the angle θ .

1. $f(x, y) = \sqrt{5x - 4y}$, $(4, 1)$, $\theta = -\pi/6$

2. $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \pi/3$

3–6 ■

(a) Find the gradient of f .

(b) Evaluate the gradient at the point P .

(c) Find the rate of change of f at P in the direction of the vector \mathbf{u} .

3. $f(x, y) = 5xy^2 - 4x^3y$, $P(1, 2)$, $\mathbf{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

4. $f(x, y) = y \ln x$, $P(1, -3)$, $\mathbf{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$

5. $f(x, y, z) = xe^{2yz}$, $P(3, 0, 2)$, $\mathbf{u} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

6. $f(x, y, z) = \sqrt{x + yz}$, $P(1, 3, 1)$, $\mathbf{u} = \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle$

7–11 ■ Find the directional derivative of the function at the given point in the direction of the vector \mathbf{v} .

7. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

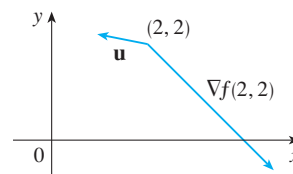
8. $f(x, y) = \ln(x^2 + y^2)$, $(2, 1)$, $\mathbf{v} = \langle -1, 2 \rangle$

9. $g(s, t) = s^2e^t$, $(2, 0)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

10. $f(x, y, z) = x/(y + z)$, $(4, 1, 1)$, $\mathbf{v} = \langle 1, 2, 3 \rangle$

11. $g(x, y, z) = (x + 2y + 3z)^{3/2}$, $(1, 1, 2)$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$

12. Use the figure to estimate $D_{\mathbf{u}}f(2, 2)$.



13. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

14. Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at $P(2, 1, 3)$ in the direction of the origin.

15–18 ■ Find the maximum rate of change of f at the given point and the direction in which it occurs.

15. $f(x, y) = y^2/x$, $(2, 4)$

16. $f(p, q) = qe^{-p} + pe^{-q}$, $(0, 0)$

17. $f(x, y, z) = \ln(xy^2z^3)$, $(1, -2, -3)$

18. $f(x, y, z) = \tan(x + 2y + 3z)$, $(-5, 1, 1)$

19. (a) Show that a differentiable function f decreases most rapidly at \mathbf{x} in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.

(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

20. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

21. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

22. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y , and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

23. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120° .

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

24. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in $^\circ\text{C}$ and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

(b) In which direction does the temperature increase fastest at P ?

(c) Find the maximum rate of increase at P .

25. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) In which direction does V change most rapidly at P ?

(c) What is the maximum rate of change at P ?

26. Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y , and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x -axis points east and the positive y -axis points north.

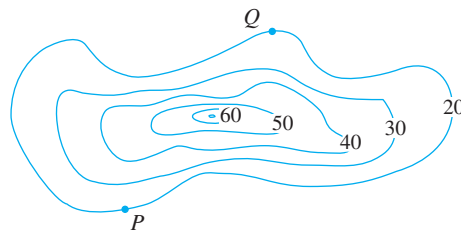
(a) If you walk due south, will you start to ascend or descend? At what rate?

(b) If you walk northwest, will you start to ascend or descend? At what rate?

(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

27. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector \overrightarrow{AB} is 3 and the directional derivative at A in the direction of \overrightarrow{AC} is 26. Find the directional derivative of f at A in the direction of the vector \overrightarrow{AD} .

28. For the given contour map draw the curves of steepest ascent starting at P and at Q .



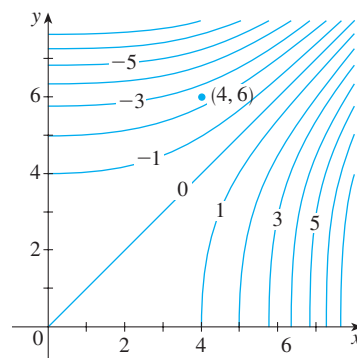
29. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and a, b are constants.

(a) $\nabla(au + bv) = a\nabla u + b\nabla v$

(b) $\nabla(uv) = u\nabla v + v\nabla u$

(c) $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\nabla u$

30. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.




31–34 ■ Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

31. $x^2 - 2y^2 + z^2 + yz = 2$, $(2, 1, -1)$

32. $x - z = 4 \arctan(yz)$, $(1 + \pi, 1, 1)$

33. $z + 1 = xe^y \cos z$, $(1, 0, 0)$

34. $yz = \ln(x + z)$, $(0, 0, 1)$

 **35–36** ■ Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

35. $xy + yz + zx = 3$, $(1, 1, 1)$

36. $xyz = 6$, $(1, 2, 3)$

37. If $f(x, y) = x^2 + 4y^2$, find the gradient vector $\nabla f(2, 1)$ and use it to find the tangent line to the level curve $f(x, y) = 8$ at the point $(2, 1)$. Sketch the level curve, the tangent line, and the gradient vector.

38. If $g(x, y) = x - y^2$, find the gradient vector $\nabla g(3, -1)$ and use it to find the tangent line to the level curve $g(x, y) = 2$ at the point $(3, -1)$. Sketch the level curve, the tangent line, and the gradient vector.

39. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

40. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

41. Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

42. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

43. Show that the sum of the x -, y -, and z -intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

44. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

45. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

46. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.



(b) Graph the cylinder, the plane, and the tangent line on the same screen.

47. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

48. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.



(b) Graph f near the origin and comment on how the graph confirms part (a).

49. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors \mathbf{u} and \mathbf{v} . Is it possible to find ∇f at this point? If so, how would you do it?

50. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[Hint: Use Definition 11.4.7 directly.]

11.7 MAXIMUM AND MINIMUM VALUES

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 5 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.