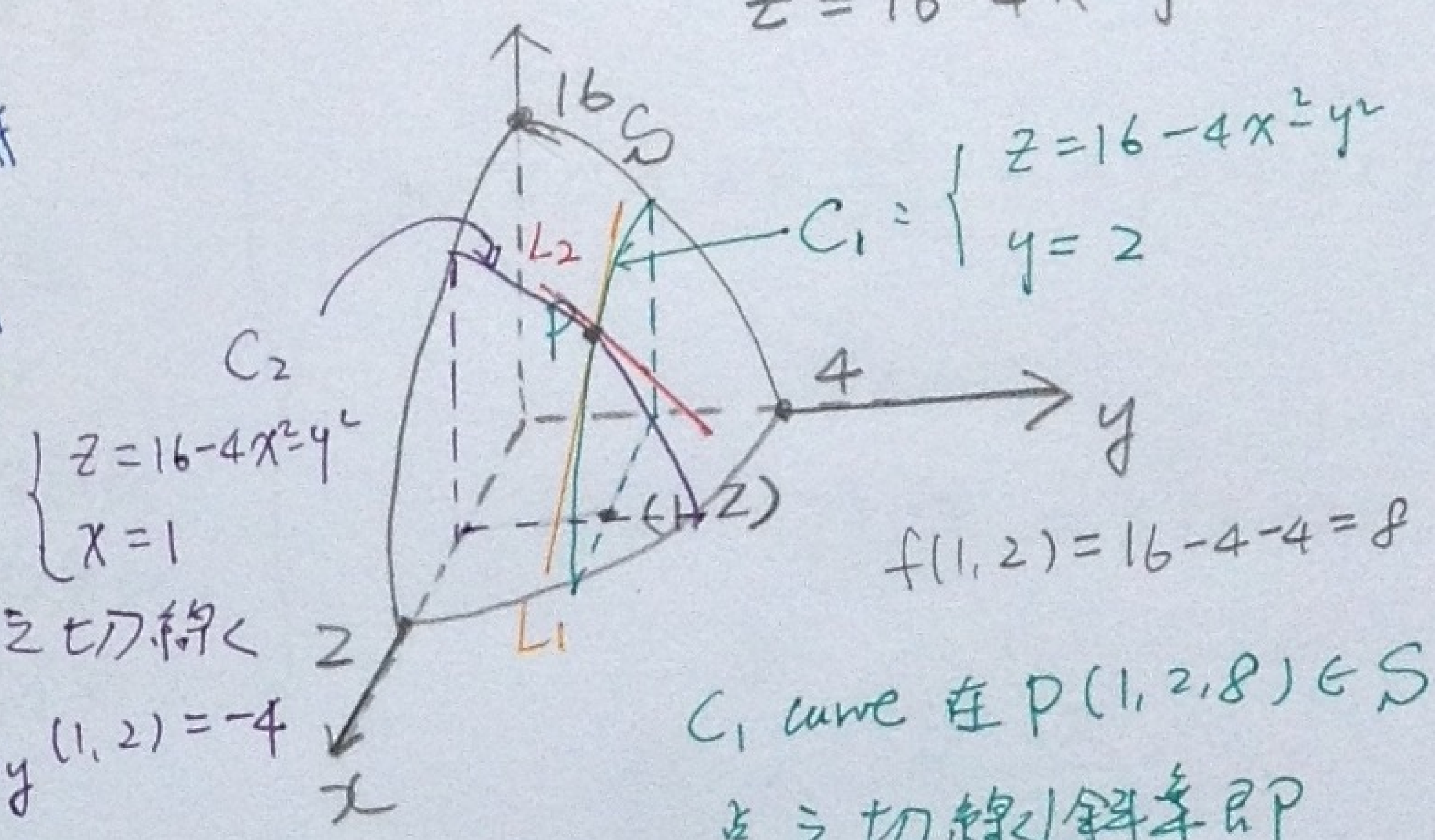
§11.4 P.d

#5. $f(x,y)=16-4x^2-y^2$, find $f_x(1,2)$ and $f_y(1,2)$ and interpret these numbers as slopes. sketch the graph

$z=16-4x^2-y^2$

sol: $f_x=-8x\big|_{(1,2)}=-8$ ※

$f_y=-2y\big|_{(1,2)}=-4$ ※



$f(1,2)=16-4-4=8$

C_2 curve 在 P点之切線 L_2 斜率即 $f_y(1,2)=-4$

C_1 curve 在 $P(1,2,8)\in S$ 点之切線 L_1 斜率即 $f_x(1,2)=-8$

7-28. Find the first p.d. of fct

#11. $f(x,y)=\dfrac{x-y}{x+y}$ sol: $f_x=\dfrac{(x+y)-(x-y)}{(x+y)^2}=\dfrac{2y}{(x+y)^2}$ ※

$f_y=\dfrac{-(x+y)-(x-y)}{(x+y)^2}=\dfrac{-2x}{(x+y)^2}$ ※

#17. $u=te^{\frac{w}{t}}$

sol: $u_t=e^{\frac{w}{t}}+t\,e^{\frac{w}{t}}\cdot\left(-\frac{w}{t^2}\right)=\left(1-\frac{w}{t}\right)e^{\frac{w}{t}}$ ※

$u_w=te^{\frac{w}{t}}\cdot\frac{1}{t}=e^{\frac{w}{t}}$ ※

#21. $w=\ln(x+2y+3z)$

sol: $w_x=\dfrac{1}{x+2y+3z}$, $w_y=\dfrac{2}{x+2y+3z}$, $w_z=\dfrac{3}{x+2y+3z}$

#27. $u=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$

sol: For each $i=1,2,\ldots,n$, $u_{x_i}=\frac{1}{2}(x_1^2+\cdots+x_n^2)^{-\frac{1}{2}}\cdot 2x_i$

$=\dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$ ※

#31. $f(x,y,z)=\dfrac{x}{y+z}$,

find $f_z(3,2,1)$ sol: $f=x(y+z)^{-1}\Rightarrow f_z=-x\dfrac{1}{(y+z)^2}$

$\Rightarrow f_z(3,2,1)=\dfrac{-3}{(2+1)^2}=-\dfrac{1}{3}$ ※

§11.4. P.d

#37 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$x^2+y^2+z^2=3xyz$$

<u>sol</u>: 兩邊对 x 微下去
心裡記得 x 是自變數
y 是常數
z 是 x 的函數
(z 是 x 和 y 的函數,
目前 y 固定不动)

$$\Rightarrow 2x+2z\frac{\partial z}{\partial x}=3yz+3xy\frac{\partial z}{\partial x}$$

↓ 化简

$$(2z-3xy)\frac{\partial z}{\partial x}=3yz-2x$$

$$\Downarrow$$

$$\frac{\partial z}{\partial x}=\frac{3yz-2x}{2z-3xy} \quad ①$$

※

同理 兩邊对 y 微下去, ……

$$2y+2z\frac{\partial z}{\partial y}=3xz+3xy\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y}=\frac{3xz-2y}{2z-3xy}$$

※

(亦可由对称性, 把①中 x 換 y, y 換 x 即答案)

#42. Find $\partial z/\partial x$ and $\partial z/\partial y$

(a) $z=f(x)g(y)$ <u>sol</u>: $\frac{\partial z}{\partial x}=f'(x)g(y)$, $\frac{\partial z}{\partial y}=f(x)g'(y)$ ※

(b) $z=f(xy)$

<u>sol</u>: $\frac{\partial z}{\partial x}=f'(xy)\frac{\partial(xy)}{\partial x}=yf'(xy)$ ※

$\frac{\partial z}{\partial y}=f'(xy)\frac{\partial(xy)}{\partial y}=xf'(xy)$ ※

(或 令 $xy=u$ $\quad (x,y)\xrightarrow{u} xy=u \xrightarrow{f} f(u)=f(xy)=z$

$\frac{\partial z}{\partial x}=f'(u)\big|_{u=xy}\cdot u_x=yf'(xy)$)

(c) $z=f(\frac{x}{y})$ <u>sol</u>: $\frac{\partial z}{\partial x}=f'(\frac{x}{y})\frac{\partial(\frac{x}{y})}{\partial x}=\frac{1}{y}f'(\frac{x}{y})$ ※

$\frac{\partial z}{\partial y}=f'(\frac{x}{y})\frac{\partial(\frac{x}{y})}{\partial y}=-\frac{x}{y^2}f'(\frac{x}{y})$ ※

(或同上 令 $\frac{x}{y}=u$
$(x,y)\xrightarrow{u}\frac{x}{y}=u\xrightarrow{f}f(u)=z$ . . . . .)

#45 Find all the second p.d. of $z = \frac{x}{x+y}$

sol: $z = x(x+y)^{-1}$

$\Rightarrow$ 1st p.d $\begin{cases} z_x = (x+y)^{-1} + x(-1)(x+y)^{-2} = \underline{y(x+y)^{-2}} \\ z_y = \underline{-x(x+y)^{-2}} \end{cases}$

$\Rightarrow$ 2nd p.d $\begin{cases} z_{xx} = y(-2)(x+y)^{-3} = \underline{\frac{-2y}{(x+y)^3}} \# \\ z_{xy} = (x+y)^{-2} + y(-2)(x+y)^{-3} = (x+y)^{-3}(x-y) = \underline{\frac{x-y}{(x+y)^3}} \# \\ \quad \parallel \\ z_{yx} \\ z_{yy} = -x(-2)(x+y)^{-3} = \underline{\frac{2x}{(x+y)^3}} \# \end{cases}$

#53 $f(x,y,z) = \cos(4x+3y+2z)$; find f_{xyz}, f_{yzz}

sol: $f_x = -4\sin(4x+3y+2z)$

$\Rightarrow f_{xy} = -12\cos(4x+3y+2z)$

$\Rightarrow \underline{f_{xyz} = 24\sin(4x+3y+2z)}$ ##

$f_y = -3\sin(4x+3y+2z)$

$\Rightarrow f_{yz} = -6\cos(4x+3y+2z)$

$\Rightarrow \underline{f_{yzz} = 12\sin(4x+3y+2z)}$ ##

#55. $u = e^{r\theta}\sin\theta$; $\frac{\partial^3 u}{\partial r^2 \partial\theta} = e^{r\theta}(r\theta^2\sin\theta + \theta^2\cos\theta + 2\theta\sin\theta)$

or $= \theta e^{r\theta}(r\theta\sin\theta + \theta\cos\theta + 2\sin\theta)$ #

sol: $\frac{\partial u}{\partial\theta} = re^{r\theta}\sin\theta + e^{r\theta}\cos\theta = \underline{e^{r\theta}(r\sin\theta + \cos\theta)}$

$\frac{\partial^2 u}{\partial r\partial\theta} = \theta e^{r\theta}(r\sin\theta + \cos\theta) + e^{r\theta}(\sin\theta)$

$= \underline{e^{r\theta}(r\theta\sin\theta + \theta\cos\theta + \sin\theta)}$

$\frac{\partial^3 u}{\partial r^2\partial\theta} = \theta e^{r\theta}(r\theta\sin\theta + \theta\cos\theta + \sin\theta) + e^{r\theta}(\theta\sin\theta)$

#65. 电阻並联公式 $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find $\frac{\partial R}{\partial R_1}$

sol：兩边对 R_1 微下去，心裡想 R_1 是自变数，R_2, R_2 視為常数

R 是 R_1 的函数（當然也是 R_2, R_3 的函数，但目前固定不动）

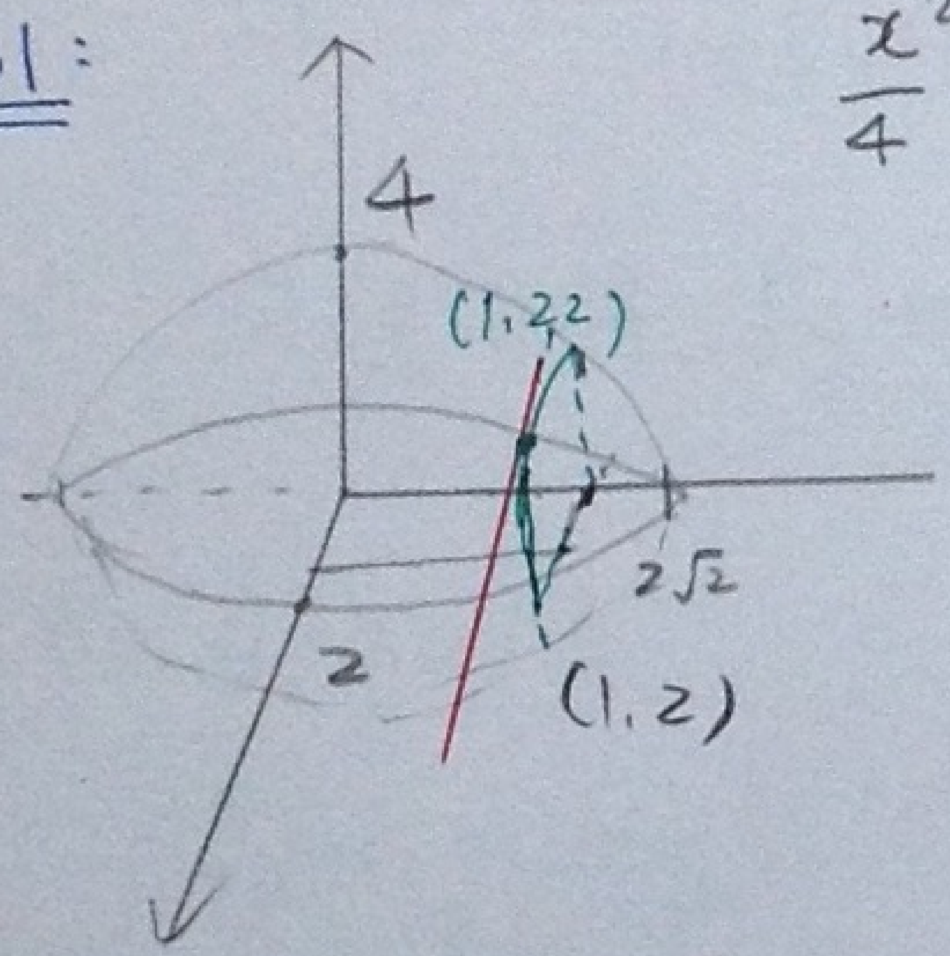
$$\Rightarrow \frac{\partial}{\partial R_1}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_1}\left(\frac{1}{R_1}\right)$$

$$\cancel{-}R^{-2}\boxed{\frac{\partial R}{\partial R_1}} = \cancel{-}R_1^{-2} \Rightarrow \underline{\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}} \ \#$$

#73. The ellipsoid $4x^2+2y^2+z^2=16$ intersects the plane $y=2$ in an ellipse. Find parametric eq for the tangent line to this ellipse at pt $(1,2,2)$

有点 $(1,2,2)$
缺方向向量 $\vec{v}$
‖
曲線 ellipse 之 tangent vector

sol：



$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{16} = 1$$

ellipse：$\begin{cases} 4x^2+z^2=8 \\ y=2 \end{cases} \Rightarrow \frac{x^2}{2} + \frac{z^2}{8} = 1$

parametric $\vec{r}(\theta) = \langle \sqrt{2}\cos\theta, 2, 2\sqrt{2}\sin\theta \rangle$

$\sqrt{2}\cos\theta = 1$, $2\sqrt{2}\sin\theta = 2$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\vec{r}'(\theta) = \langle -\sqrt{2}\sin\theta, 0, 2\sqrt{2}\cos\theta \rangle$$

$$\vec{r}'\left(\tfrac{\pi}{4}\right) = \langle -1, 0, 2 \rangle = \vec{v}$$

∴ parametric eq 為

$$\underline{x = 1 - t, \quad y = 2, \quad z = 2 + 2t} \ \#$$

#77. $f(x,y) = x(x^2+y^2)^{-\frac{3}{2}} e^{\sin(x^2 y)}$, find $f_x(1, \underset{\text{固定}}{0})$

sol：用定义做较易

令 $g(x) = f(x,0) = x(x^2)^{-\frac{3}{2}} e^0 = x|x|^{-3} = \begin{cases} x^{-2}, & x \ge 0 \\ -x^{-2}, & x < 0 \end{cases}$

求 $g'(1) = ?$ $\quad g'(x) = -2x^{-3} \Rightarrow g'(1) = -2$

$$\therefore \underline{f_x(1,0) = -2} \ \#$$

#79. $f(x,y)=\begin{cases}\dfrac{x^3y-xy^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$

(b) Find
(c) $f_x(x,y)=\begin{cases}\_\_\_\_, & (0,0)\\ 0, & (0,0)\end{cases}$, $f_y(x,y)=\begin{cases}\_\_\_\_, & (0,0)\\ 0, & (0,0)\end{cases}$

(d) Show that $f_{xy}(0,0)=-1\neq 1=f_{yx}(0,0)$

(e) 與 Clarrut's Thm 有相違背嗎？說明之！

Sol: $(x,y)\neq(0,0)$

(b)

$$f_x(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-xy^3)}{(x^2+y^2)^2}=\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$x=0,\ y=h$

同理可得 $f_y=\dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$　$x=h$, $y=0$

(原 f $x\leftrightarrow y$ 差一个负号
故 f_y 把 f_x 中 $x\leftrightarrow y$ 亦差一个负号)

小观察

(c) 在 (0,0) 点

$$f_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0}{h}=0$$

固定

同理可得 $f_y(0,0)=0$

(d)

$$\boxed{f_x}_y(0,0)=\lim_{h\to0}\frac{f_x(0,h)-f_x(0,0)}{h}=\lim_{h\to0}\frac{\frac{-h^5}{h^4}-0}{h}=-1$$

对y微　固定

$$\boxed{f_y}_x(0,0)=\lim_{h\to0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^5}{h^4}-0}{h}=1$$

对x微　固定

(e) $(x,y)\neq(0,0)$　$f_{xy}(x,y)=\dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$

$(x,y)\to(0,0)$ along y-axis, $f_{xy}(0,y)=-1\to-1$

$(x,y)\to(0,0)$ along x-axis, $f_{xy}(x,0)=1\to1$

$\therefore \lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ DNE 故 f_{xy} is Not conti at (0,0)，不可使用 Clarraut's Thm，故无違反 Thm. #