35. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). The level curves of T are called *isothermals* because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2)$$

36. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

37–40 ▪ Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

37. $f(x, y) = e^x \cos y$

38. $f(x, y) = (1 - 3x^2 + y^2)e^{1-x^2-y^2}$

39. $f(x, y) = xy^2 - x^3$ (monkey saddle)

40. $f(x, y) = xy^3 - yx^3$ (dog saddle)

41–46 ▪ Match the function (a) with its graph (labeled A–F on page 600) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

41. $z = \sin(xy)$

42. $z = e^x \cos y$

43. $z = \sin(x - y)$

44. $z = \sin x - \sin y$

45. $z = (1 - x^2)(1 - y^2)$

46. $z = \dfrac{x - y}{1 + x^2 + y^2}$

47–50 ▪ Describe the level surfaces of the function.

47. $f(x, y, z) = x + 3y + 5z$

48. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

49. $f(x, y, z) = x^2 - y^2 + z^2$

50. $f(x, y, z) = x^2 - y^2$

51–52 ▪ Describe how the graph of g is obtained from the graph of f.

51. (a) $g(x, y) = f(x, y) + 2$ (b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$ (d) $g(x, y) = 2 - f(x, y)$

52. (a) $g(x, y) = f(x - 2, y)$ (b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

53. Use a computer to investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c?

54. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2} \qquad f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2} \qquad f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$$

and $$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$$

obtained from the graph of g?

11.2 LIMITS AND CONTINUITY

The limit of a function of two or more variables is similar to the limit of a function of a single variable. We use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f. In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \ \text{as} \ (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

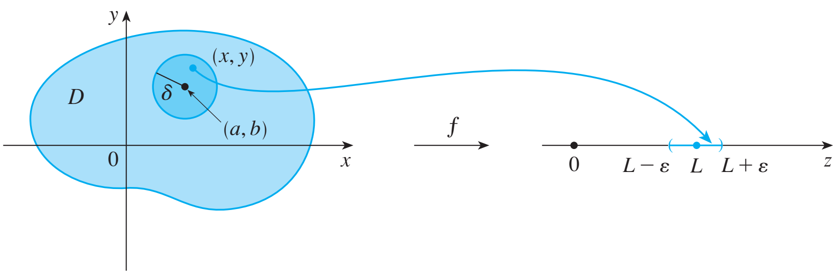

FIGURE 1

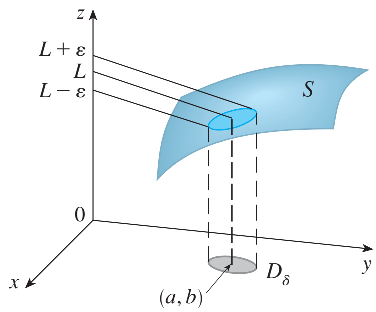

FIGURE 2

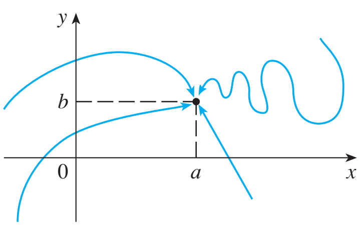

FIGURE 3

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. We recall from Chapter 1 that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

> If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

V EXAMPLE 1 Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

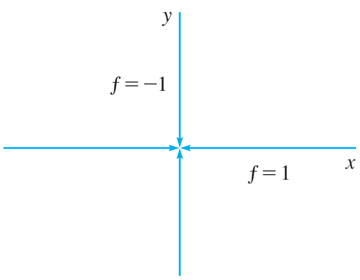

FIGURE 4

(See Figure 4.) Since f has two different limits along two different lines, the given limit does not exist. ▪

EXAMPLE 2 If $f(x, y) = xy/(x^2 + y^2)$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\qquad f(x, y) \to \frac{1}{2} \qquad \text{as} \qquad (x, y) \to (0, 0)$ along $y = x$

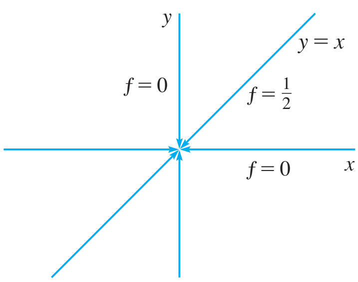

FIGURE 5

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ▪

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

In Visual 11.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

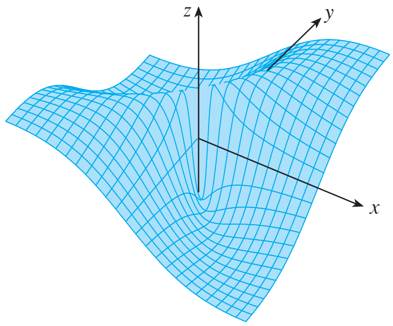

FIGURE 6

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\displaystyle\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any nonvertical line through the origin. Then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $\qquad f(x, y) \to 0 \qquad$ as $\qquad (x, y) \to (0, 0)$ along $y = mx$

Thus f has the same limiting value along every nonvertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $\qquad f(x, y) \to \frac{1}{2} \qquad$ as $\qquad (x, y) \to (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist. ▪

▪ Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

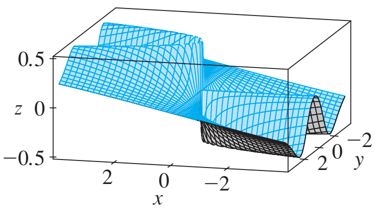

FIGURE 7

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 1.4 can be extended to functions of two variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

2 $$\lim_{(x, y)\to(a, b)} x = a \qquad \lim_{(x, y)\to(a, b)} y = b \qquad \lim_{(x, y)\to(a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is,

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2 + y^2) \leqslant 1$ and therefore

3
$$\frac{3x^2|y|}{x^2 + y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leqslant 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

■ Another way to do Example 4 is to use the Squeeze Theorem instead of Definition 1. From (2) it follows that

$$\lim_{(x, y) \to (0, 0)} 3|y| = 0$$

and so the first inequality in (3) shows that the given limit is 0.

Hence, by Definition 1,

$$\lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0$$ ■

CONTINUITY

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 DEFINITION A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

V EXAMPLE 5 Evaluate $\lim_{(x, y) \to (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y) \to (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$ ■

EXAMPLE 6 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at (0, 0) because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at (0, 0) but g is still discontinuous there because $\lim_{(x, y) \to (0, 0)} g(x, y)$ does not exist (see Example 1). ■

▪ Figure 8 shows the graph of the continuous function in Example 8.

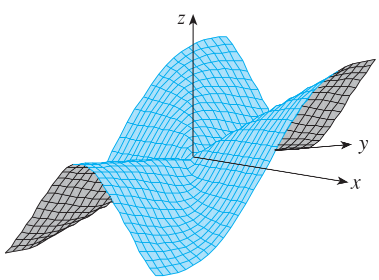

FIGURE 8

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore, f is continuous at (0, 0), and so it is continuous on $\mathbb{R}^2$. ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

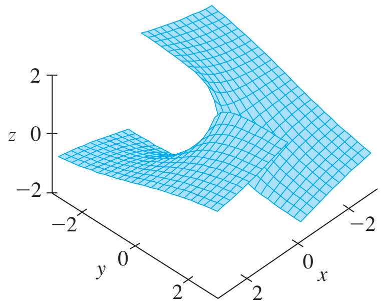

FIGURE 9

The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y-axis. ■

FUNCTIONS OF THREE OR MORE VARIABLES

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. Because the distance between two points (x, y, z) and (a, b, c) is $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y, z) \text{ is in the domain of } f \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$$

$$\text{then} \quad |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 11.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (5) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

11.2 EXERCISES

1. Suppose that $\lim_{(x, y)\to(3, 1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?

2. Explain why each function is continuous or discontinuous.
(a) The outdoor temperature as a function of longitude, latitude, and time
(b) Elevation (height above sea level) as a function of longitude, latitude, and time
(c) The cost of a taxi ride as a function of distance traveled and time

3–16 ■ Find the limit, if it exists, or show that the limit does not exist.

3. $\lim_{(x, y)\to(5, -2)} (x^5 + 4x^3y - 5xy^2)$

4. $\lim_{(x, y)\to(6, 3)} xy\cos(x - 2y)$

5. $\lim_{(x, y)\to(0, 0)} \dfrac{y^4}{x^4 + 3y^4}$

6. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 + \sin^2 y}{2x^2 + y^2}$

7. $\lim_{(x, y)\to(0, 0)} \dfrac{xy\cos y}{3x^2 + y^2}$

8. $\lim_{(x, y)\to(0, 0)} \dfrac{6x^3y}{2x^4 + y^4}$

9. $\lim_{(x, y)\to(0, 0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$

10. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2}$

11. $\lim_{(x, y)\to(0, 0)} \dfrac{2x^2y}{x^4 + y^2}$

12. $\lim_{(x, y)\to(0, 0)} \dfrac{xy^4}{x^2 + y^8}$

13. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

14. $\lim_{(x, y)\to(0, 0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

15. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

16. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$

17–18 ■ Use a computer graph of the function to explain why the limit does not exist.

17. $\lim_{(x, y)\to(0, 0)} \dfrac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$

18. $\lim_{(x, y)\to(0, 0)} \dfrac{xy^3}{x^2 + y^6}$

19–20 ■ Find $h(x, y) = g(f(x, y))$ and the set on which h is continuous.

19. $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$

20. $g(t) = \dfrac{\sqrt{t} - 1}{\sqrt{t} + 1}, \quad f(x, y) = x^2 - y$

21–28 ■ Determine the set of points at which the function is continuous.

21. $F(x, y) = \dfrac{\sin(xy)}{e^x - y^2}$

22. $F(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$

23. $G(x, y) = \ln(x^2 + y^2 - 4)$

24. $F(x, y) = e^{x^2y} + \sqrt{x + y^2}$

25. $f(x, y, z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$

26. $f(x, y, z) = \sqrt{x + y + z}$

27. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

28. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

29–30 ■ Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

29. $\lim_{(x, y)\to(0, 0)} \dfrac{x^3 + y^3}{x^2 + y^2}$

30. $\lim_{(x, y)\to(0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

31. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint*: Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

32. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.