

10.7 VECTOR FUNCTIONS AND SPACE CURVES

In general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions \mathbf{r} whose values are three-dimensional vectors. This means that for every number t in the domain of \mathbf{r} there is a unique vector in V_3 denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f , g , and h are real-valued functions called the **component functions** of \mathbf{r} and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

EXAMPLE 1 If

$$\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$$

then the component functions are

$$f(t) = t^3 \qquad g(t) = \ln(3 - t) \qquad h(t) = \sqrt{t}$$

By our usual convention, the domain of \mathbf{r} consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions t^3 , $\ln(3 - t)$, and \sqrt{t} are all defined when $3 - t > 0$ and $t \geq 0$. Therefore, the domain of \mathbf{r} is the interval $[0, 3)$. ■

The **limit** of a vector function \mathbf{r} is defined by taking the limits of its component functions as follows.

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

provided the limits of the component functions exist.

■ If $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$, this definition is equivalent to saying that the length and direction of the vector $\mathbf{r}(t)$ approach the length and direction of the vector \mathbf{L} .

Equivalently, we could have used an ε - δ definition (see Exercise 68). Limits of vector functions obey the same rules as limits of real-valued functions (see Exercise 67).

EXAMPLE 2 Find $\lim_{t \rightarrow 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$.

SOLUTION According to Definition 1, the limit of \mathbf{r} is the vector whose components are the limits of the component functions of \mathbf{r} :

$$\begin{aligned} \lim_{t \rightarrow 0} \mathbf{r}(t) &= \left[\lim_{t \rightarrow 0} (1 + t^3) \right] \mathbf{i} + \left[\lim_{t \rightarrow 0} te^{-t} \right] \mathbf{j} + \left[\lim_{t \rightarrow 0} \frac{\sin t}{t} \right] \mathbf{k} \\ &= \mathbf{i} + \mathbf{k} \quad (\text{by Equation 1.4.5}) \end{aligned}$$

■

A vector function \mathbf{r} is **continuous at a** if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that \mathbf{r} is continuous at a if and only if its component functions f , g , and h are continuous at a .

There is a close connection between continuous vector functions and space curves. Suppose that f , g , and h are continuous real-valued functions on an interval I . Then the set C of all points (x, y, z) in space, where

$$\mathbf{2} \quad x = f(t) \quad y = g(t) \quad z = h(t)$$

and t varies throughout the interval I , is called a **space curve**. The equations in (2) are called **parametric equations of C** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C . Thus any continuous vector function \mathbf{r} defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

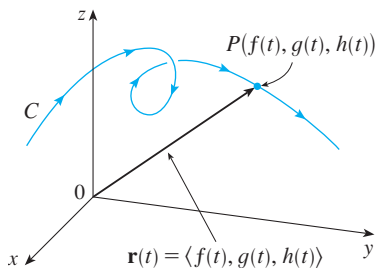


FIGURE 1

C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.



Visual 10.7A shows several curves being traced out by position vectors, including those in Figures 1 and 2.

EXAMPLE 3 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

SOLUTION The corresponding parametric equations are

$$x = 1 + t \quad y = 2 + 5t \quad z = -1 + 6t$$

which we recognize from Equations 10.5.2 as parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\langle 1, 5, 6 \rangle$. Alternatively, we could observe that the function can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5, 6 \rangle$, and this is the vector equation of a line as given by Equation 10.5.1. ■

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ (see Example 1 in Section 9.1) could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\mathbf{i} + (t + 1)\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

EXAMPLE 4 Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

SOLUTION The parametric equations for this curve are

$$x = \cos t \quad y = \sin t \quad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy -plane. (See Example 2 in Section 9.1.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**. ■

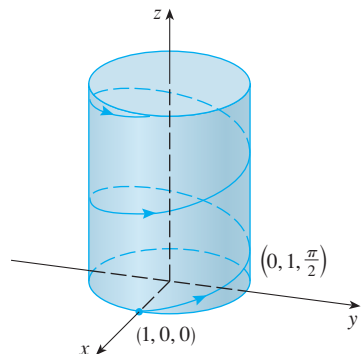


FIGURE 2



FIGURE 3

■ Figure 4 shows the line segment PQ in Example 5.

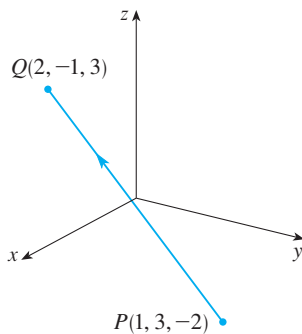


FIGURE 4

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helices that are intertwined as in Figure 3.

In Examples 3 and 4 we were given vector equations of curves and asked for a geometric description or sketch. In the next two examples we are given a geometric description of a curve and are asked to find parametric equations for the curve.

EXAMPLE 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

SOLUTION In Section 10.5 we found a vector equation for the line segment that joins the tip of the vector \mathbf{r}_0 to the tip of the vector \mathbf{r}_1 :

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

(See Equation 10.5.4.) Here we take $\mathbf{r}_0 = \langle 1, 3, -2 \rangle$ and $\mathbf{r}_1 = \langle 2, -1, 3 \rangle$ to obtain a vector equation of the line segment from P to Q :

$$\mathbf{r}(t) = (1 - t)\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle \quad 0 \leq t \leq 1$$

or
$$\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle \quad 0 \leq t \leq 1$$

The corresponding parametric equations are

$$x = 1 + t \quad y = 3 - 4t \quad z = -2 + 5t \quad 0 \leq t \leq 1$$

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

SOLUTION Figure 5 shows how the plane and the cylinder intersect, and Figure 6 shows the curve of intersection C , which is an ellipse.

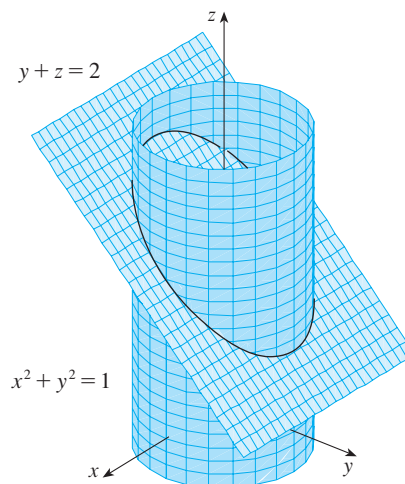


FIGURE 5

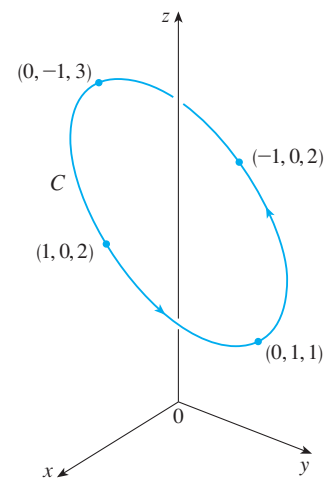


FIGURE 6

The projection of C onto the xy -plane is the circle $x^2 + y^2 = 1$, $z = 0$. So we know from Example 2 in Section 9.1 that we can write

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

So we can write parametric equations for C as

$$x = \cos t \quad y = \sin t \quad z = 2 - \sin t \quad 0 \leq t \leq 2\pi$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + (2 - \sin t) \mathbf{k} \quad 0 \leq t \leq 2\pi$$

This equation is called a *parametrization* of the curve C . The arrows in Figure 6 indicate the direction in which C is traced as the parameter t increases. ■

USING COMPUTERS TO DRAW SPACE CURVES

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology. For instance, Figure 7 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t) \cos t \quad y = (4 + \sin 20t) \sin t \quad z = \cos 20t$$

It's called a **toroidal spiral** because it lies on a torus. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t) \cos t \quad y = (2 + \cos 1.5t) \sin t \quad z = \sin 1.5t$$

is graphed in Figure 8. It wouldn't be easy to plot either of these curves by hand.

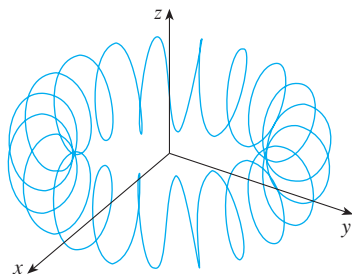


FIGURE 7 A toroidal spiral

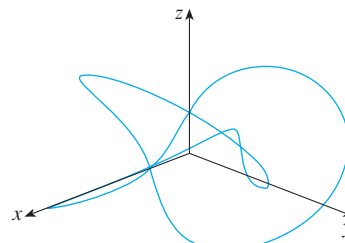


FIGURE 8 A trefoil knot

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 8.) The next example shows how to cope with this problem.

EXAMPLE 7 Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

SOLUTION We start by using the computer to plot the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$ for $-2 \leq t \leq 2$. The result is shown in Figure 9(a), but it's hard to see the true nature of the curve from that graph alone. Most three-



In Visual 10.7B you can rotate the box in Figure 9 to see the curve from any viewpoint.

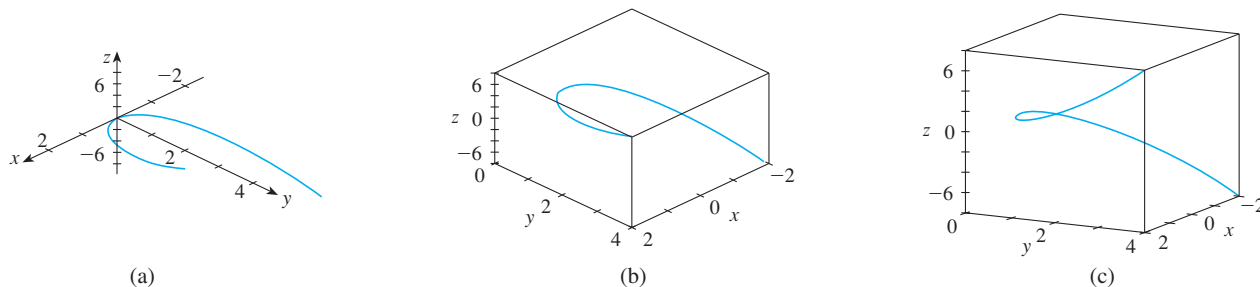


FIGURE 9
Views of the twisted cubic

dimensional computer graphing programs allow the user to enclose a curve or surface in a box instead of displaying the coordinate axes. When we look at the same curve in a box in Figure 9(b), we have a much clearer picture of the curve. We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs. We get an even better idea of the curve when we view it from different vantage points. Part (c) shows the result of rotating the box to give another viewpoint.

DERIVATIVES

The **derivative** \mathbf{r}' of a vector function \mathbf{r} is defined in much the same way as for real-valued functions:

3

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 10. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then \overrightarrow{PQ} represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \rightarrow 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by \mathbf{r} at the point P , provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$



Visual 10.7C shows an animation of Figure 10.

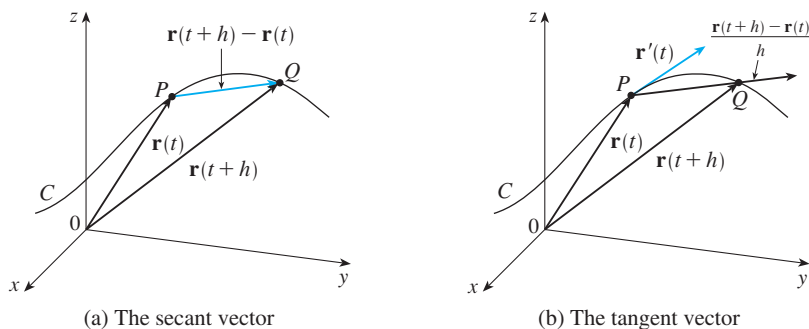


FIGURE 10

(a) The secant vector

(b) The tangent vector

The following theorem gives us a convenient method for computing the derivative of a vector function \mathbf{r} : just differentiate each component of \mathbf{r} .

4 THEOREM If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

PROOF

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\mathbf{r}(t + \Delta t) - \mathbf{r}(t)] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\ &= \lim_{\Delta t \rightarrow 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \left\langle \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \langle f'(t), g'(t), h'(t) \rangle \quad \square \end{aligned}$$

EXAMPLE 8

- (a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$.
 (b) Find the unit tangent vector at the point where $t = 0$.

SOLUTION

- (a) According to Theorem 4, we differentiate each component of \mathbf{r} :

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos 2t\mathbf{k}$$

- (b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k} \quad \blacksquare$$

EXAMPLE 9 For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

SOLUTION We have

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$$

The curve is a plane curve and elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives $y = 2 - x^2$, $x \geq 0$. In Figure 11 we draw the position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point $(1, 1)$. ■

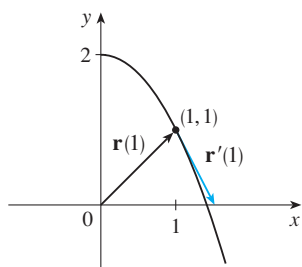


FIGURE 11

EXAMPLE 10 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2 \cos t \quad y = \sin t \quad z = t$$

at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, so

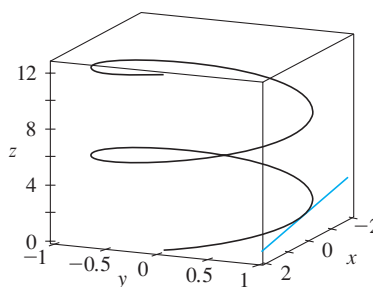
$$\mathbf{r}'(t) = \langle -2 \sin t, \cos t, 1 \rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1 \rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1 \rangle$, so by Equations 10.5.2 its parametric equations are

$$x = -2t \quad y = 1 \quad z = \frac{\pi}{2} + t$$

■ The helix and the tangent line in Example 10 are shown in Figure 12.

FIGURE 12



■ In Section 10.9 we will see how $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ can be interpreted as the velocity and acceleration vectors of a particle moving through space with position vector $\mathbf{r}(t)$ at time t .

Just as for real-valued functions, the **second derivative** of a vector function \mathbf{r} is the derivative of \mathbf{r}' , that is, $\mathbf{r}'' = (\mathbf{r}')'$. For instance, the second derivative of the function in Example 10 is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle$$

A curve given by a vector function $\mathbf{r}(t)$ on an interval I is called **smooth** if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ (except possibly at any endpoints of I). For instance, the helix in Example 10 is smooth because $\mathbf{r}'(t)$ is never $\mathbf{0}$.

EXAMPLE 11 Determine whether the semicubical parabola $\mathbf{r}(t) = \langle 1 + t^3, t^2 \rangle$ is smooth.

SOLUTION Since

$$\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$$

we have $\mathbf{r}'(0) = \langle 0, 0 \rangle = \mathbf{0}$ and, therefore, the curve is not smooth. The point that corresponds to $t = 0$ is $(1, 0)$, and we see from the graph in Figure 13 that there is a sharp corner, called a **cusp**, at $(1, 0)$. Any curve with this type of behavior—an abrupt change in direction—is not smooth.

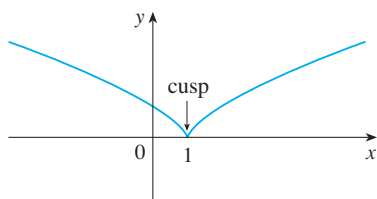


FIGURE 13

The curve $\mathbf{r}(t) = \langle 1 + t^3, t^2 \rangle$ is not smooth.

A curve, such as the semicubical parabola, that is made up of a finite number of smooth pieces is called **piecewise smooth**.