15.8 Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates

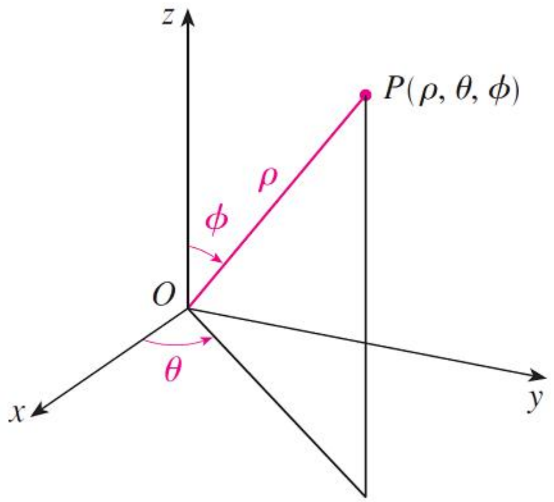
Another useful coordinate system in three dimensions is the *spherical coordinate system*.

It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point *P* in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to *P*, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive *z*-axis and the line segment *OP*.



The spherical coordinates of a point

Figure 1

Spherical Coordinates

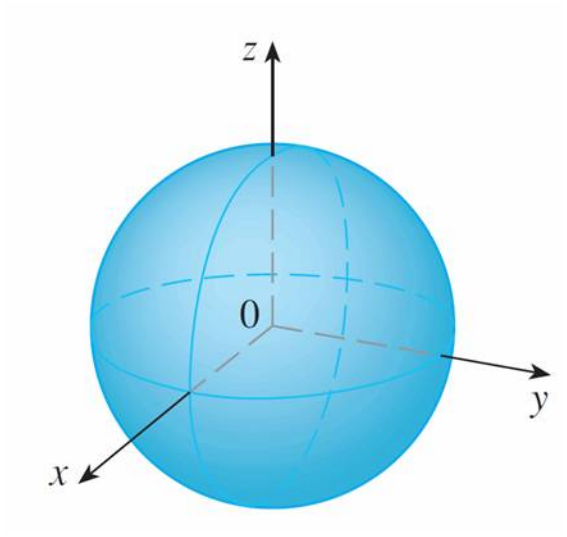
Note that

$$\rho \geq 0 \qquad\qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates

For example, the sphere with center the origin and radius *c* has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name “spherical” coordinates.

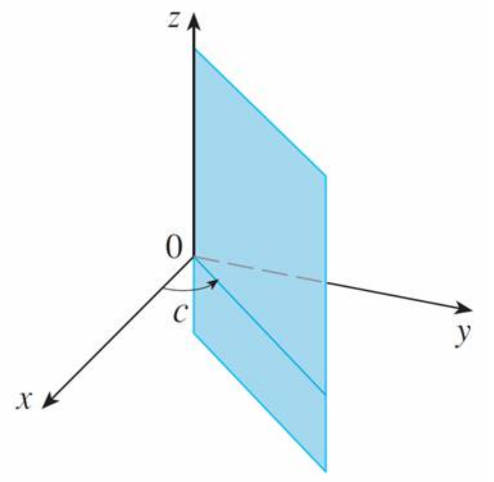


$\rho = c$, a sphere

Figure 2

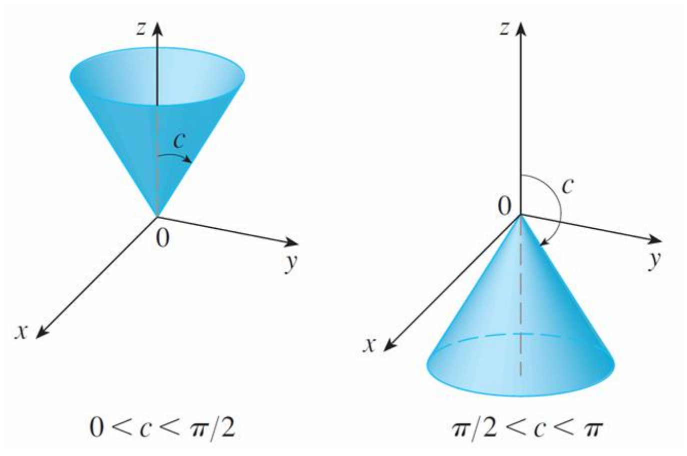
Spherical Coordinates

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the *z*-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\phi = c$, a half-cone

Figure 4

Spherical Coordinates

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles *OPQ* and *OPP′* we have

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

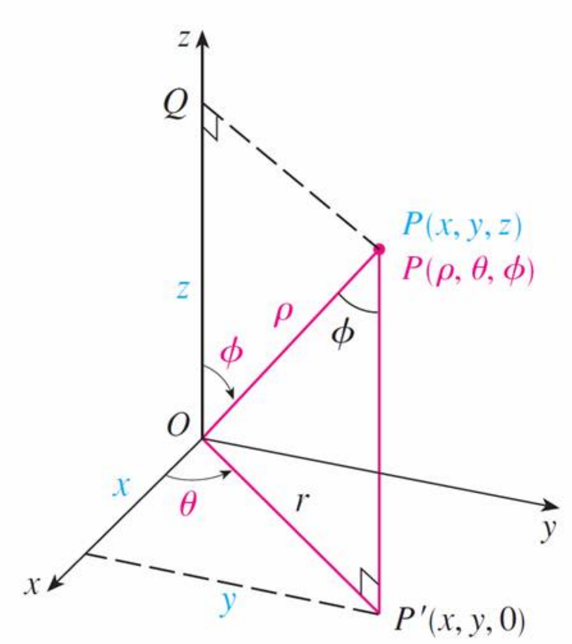


Figure 5

Spherical Coordinates

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

1

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

Also, the distance formula shows that

2

$$\rho^2 = x^2 + y^2 + z^2$$

We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:

We plot the point in Figure 6.

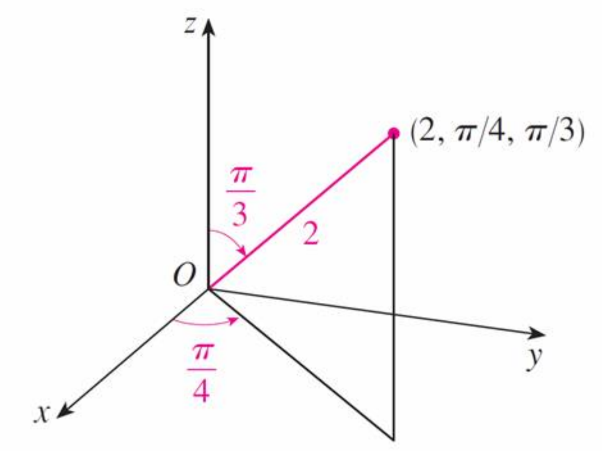


Figure 6

Example 1 – *Solution*

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From Equations 1 we have

$$= 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point (2, π/4, π/3) is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.

Evaluating Triple Integrals with Spherical Coordinates

Evaluating Triple Integrals with Spherical Coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

Evaluating Triple Integrals with Spherical Coordinates

Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \, \Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin \phi_k \, \Delta\theta$ (arc of a circle with radius $\rho_i \sin \phi_k$, angle $\Delta\theta$).

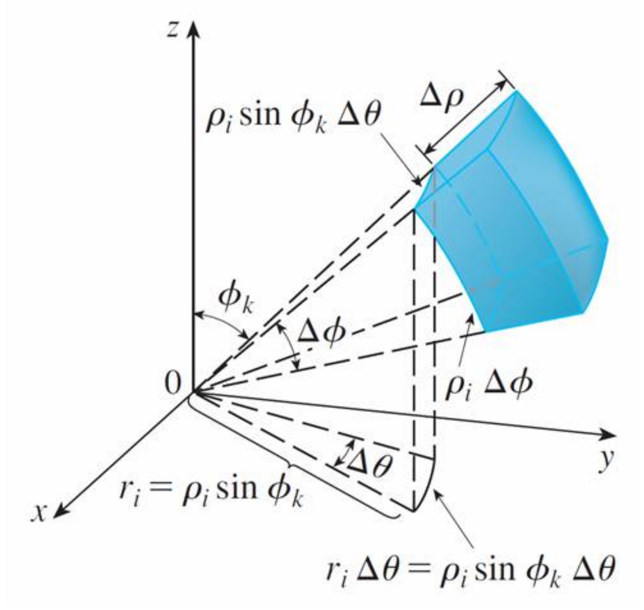


Figure 7

Evaluating Triple Integrals with Spherical Coordinates

So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \, \Delta\phi)(\rho_i \sin \phi_k \, \Delta\theta) = \rho_i^2 \sin \phi_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

In fact, it can be shown, with the aid of the Mean Value Theorem, that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin \tilde{\phi}_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}.

Evaluating Triple Integrals with Spherical Coordinates

Let $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ be the rectangular coordinates of this point. Then

$$\iiint_E f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$$

$$= \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \sin \tilde{\phi}_k \cos \tilde{\theta}_j,\ \tilde{\rho}_i \sin \tilde{\phi}_k \sin \tilde{\theta}_j,\ \tilde{\rho}_i \cos \tilde{\phi}_k)\ \tilde{\rho}_i^2 \sin \tilde{\phi}_k\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

Evaluating Triple Integrals with Spherical Coordinates

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Evaluating Triple Integrals with Spherical Coordinates

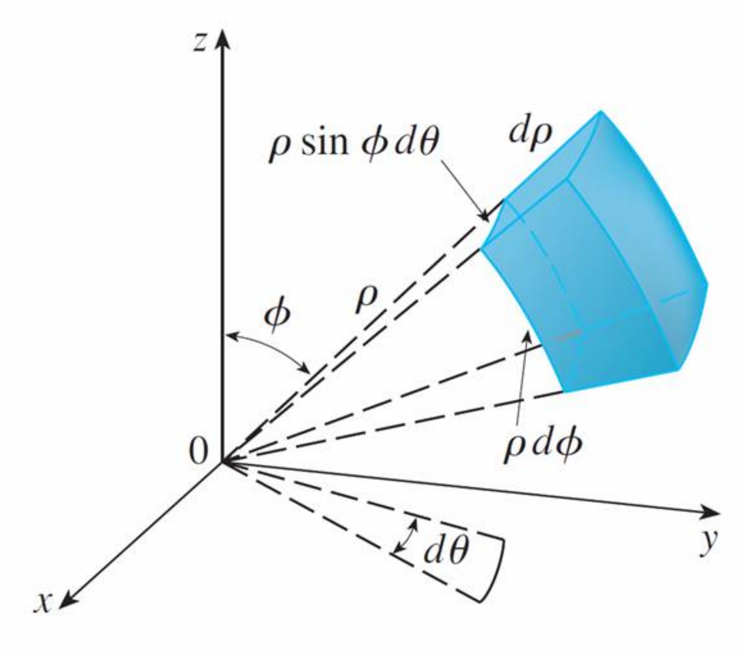
Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

using the appropriate limits of integration, and replacing *dv* by $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$.

Evaluating Triple Integrals with Spherical Coordinates

This is illustrated in Figure 8.



Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

Figure 8

Evaluating Triple Integrals with Spherical Coordinates

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$$

In this case the formula is the same as in (3) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 4

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

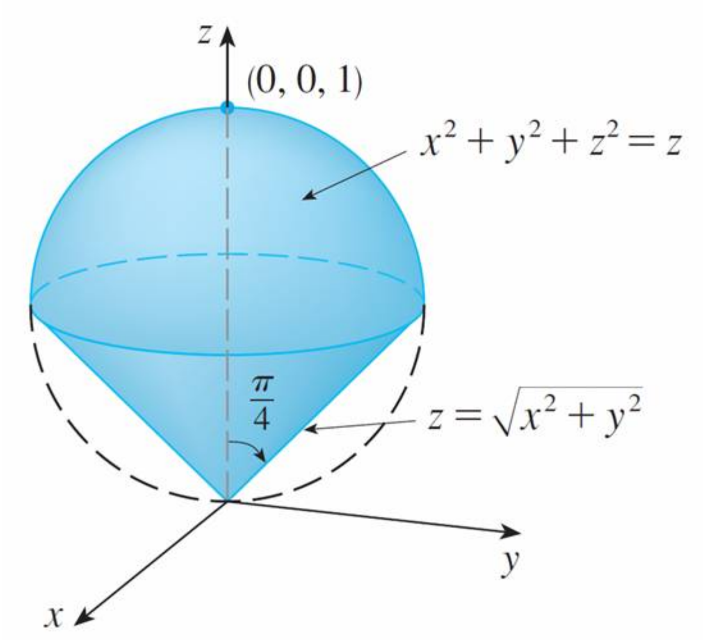


Figure 9

Example 4 – *Solution*

Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \cos \phi$$

The equation of the cone can be written as

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2\phi \, \cos^2\theta + \rho^2 \sin^2\phi \, \sin^2\theta}$$

$$= \rho \sin \phi$$

Example 4 – *Solution*

This gives sin ϕ = cos ϕ, or $\phi = \pi/4$. Therefore the description of the solid *E* in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\}$$

Example 4 – *Solution*

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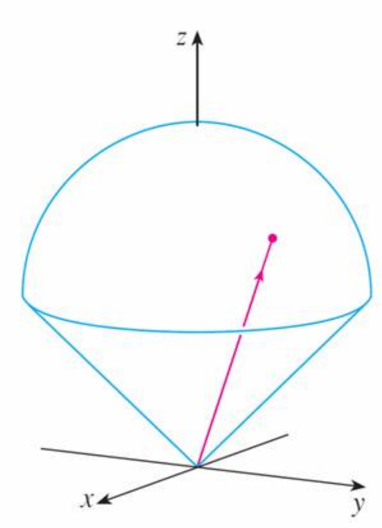
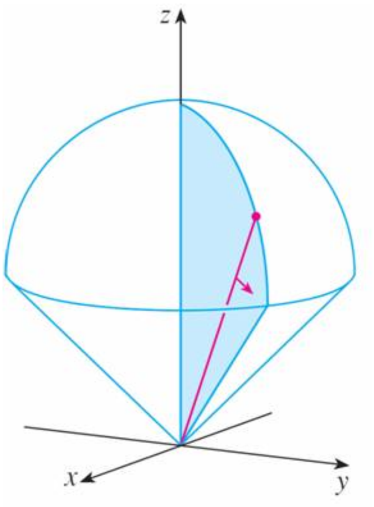
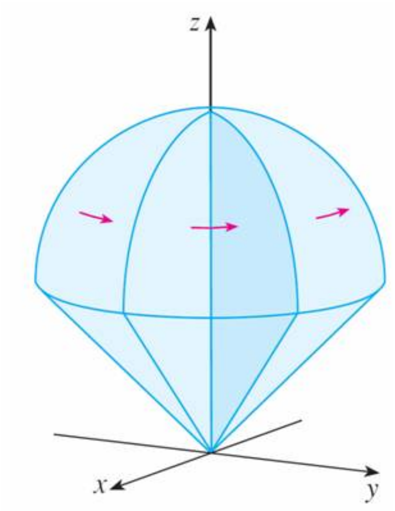
Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ.



ρ varies from 0 to cos ϕ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π.

Figure 11

Example 4 – *Solution*

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The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$