Calculus (II)

Lecture Note 2023 Spring

Calculus (Metric Version) 9th Ed., James Stewart

Contents

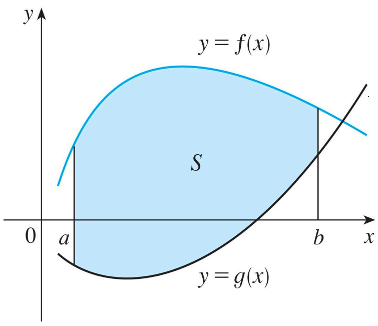
Chapter 5

Applications of Integration

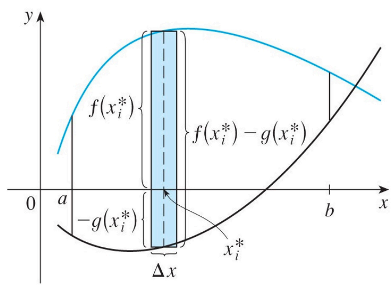
5.1 Areas Between Curves

In the present section, we try to evaluate the integrals to find areas of regions that lie between the graphs of two functions.

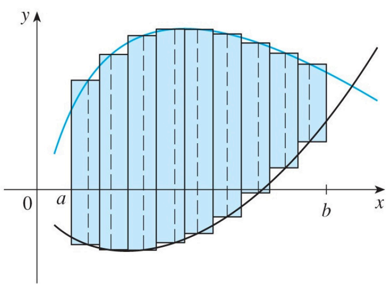
Let f and g be two continuous functions satisfying $f(x) \geq g(x)$ for every $x \in [a, b]$. Let S be the region between the two curves $y = f(x)$ and $y = g(x)$, and the vertical lines $x = a$ and $x = b$. We use the approximating rectangles method to evaluate the area of S.



$S = \{(x, y) \mid a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)\}$



(a) Typical rectangle



(b) Approximating rectangles

Let P be a partition of $[a, b]$. The Riemann sum

$$\sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \triangle x_i$$

is an approximation to the area of S. We define the area A of the region S as the limiting value of the sum of the area of these approximating rectangles

$$A = \lim_{n\to\infty} \sum_{n=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \triangle x_i.$$

Theorem 5.1.1. *The area A of the region bounded by the cruve $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are integrable and $f(x) \geq g(x)$ for all $x \in [a, b]$, is*

$$A = \int_a^b \left[f(x) - g(x)\right]\, dx$$

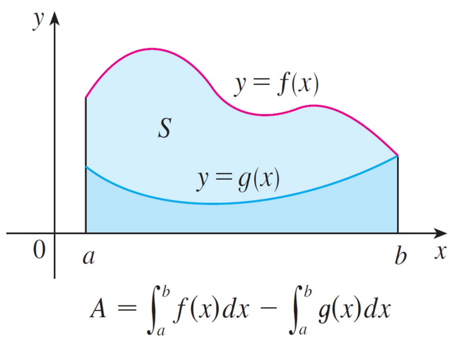
Note. (1) If $g(x) \equiv 0$, S is the region under the graph of f. The area of S is

$$A = \int_a^b \left[f(x) - 0\right]\, dx = \int_a^b f(x)\, dx$$

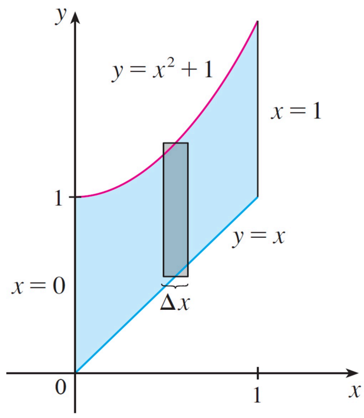
is the same as the area we discussed before.

(2) If $f(x) \geq g(x) \geq 0$ for all $x \in [a, b]$

$$\begin{aligned} A &= \left[\text{area under } y = f(x)\right] - \left[\text{area under } y = g(x)\right] \\ &= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx \\ &= \int_a^b \left[f(x) - g(x)\right]\, dx. \end{aligned}$$



Example 5.1.2. Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



Proof.

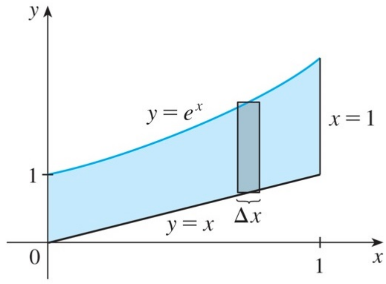
$$\begin{aligned} A &= \int_0^1 \left[(x^2 + 1 - x)\right]\, dx = \int_0^1 (x^2 - x + 1)\, dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x\Big]_0^1 = \frac{5}{6} \end{aligned}$$

□

Example 5.1.3. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$ and bounded on the sides by $x = 0$ and $x = 1$.

Proof.

$$\begin{aligned} A &= \int_0^1 [e^x - x]\,dx \\ &= e^x - \frac{1}{2}x^2\Big|_0^1 = e - \frac{3}{2}. \end{aligned}$$



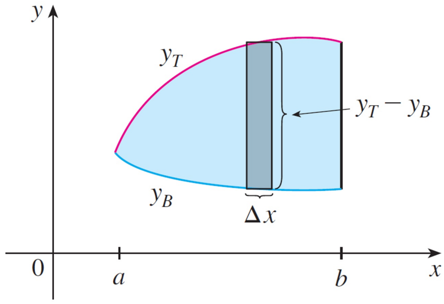
□

Remark. When we set up an integral for an area, it is helpful to sketch the region to identify the top curve y_T, the bottom curve y_B and a typical approximating rectangle.

Then the area of a typical rectangle is $(y_T - y_B)\triangle x$ and the equation

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}(y_T - y_B)\triangle x = \int_a^b (y_T - y_B)\,dx$$

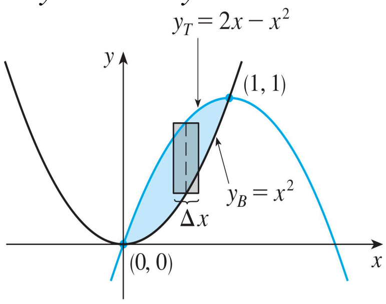
summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.



Example 5.1.4. Find the area of the region enclosed by the parabola $y = x^2$ and $y = 2x - x^2$.

Proof. The points of intersection of $y = x^2$ and $y = 2x - x^2$ are given by solving the equation $x^2 = x - x^2$. They are $x = 0$ and $x = 1$. The graph $y = 2x - x^2$ is above the graph of $y = x^2$ for all $x \in [0, 1]$. The area of the region is

$$A = \int_0^1 \left[(2x - x^2) - x^2\right]\,dx = x^2 + x - \frac{2}{3}x^3\Big|_0^1 = \frac{1}{3}.$$



□

To find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values.

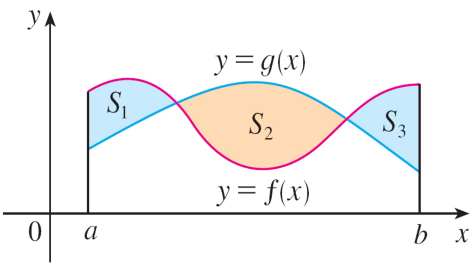
We splits the region S into several subregions $S_1, S_2, \cdots S_n$ with areas $A_1, A_2, \cdots A_n$. Then the area of S is

$$A = A_1 + A_2 + \cdots + A_n.$$

Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } f(x) \leq g(x), \end{cases}$$

we have the following results.

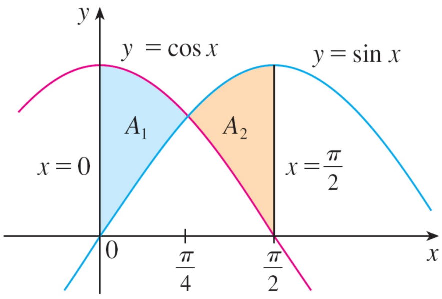


Theorem 5.1.5. *The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is*

$$A = \int_a^b \left|f(x) - g(x)\right|\,dx.$$

Example 5.1.6. Find the area of the region bounded by the cruves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

Proof. The points of intersection of two curves in $[0, \frac{\pi}{2}]$ is $\frac{\pi}{4}$. Also, $\cos x \geq \sin x$ when $0 \leq x \leq \frac{\pi}{4}$ and $\sin x \geq \cos x$ when $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$. The area of the region is

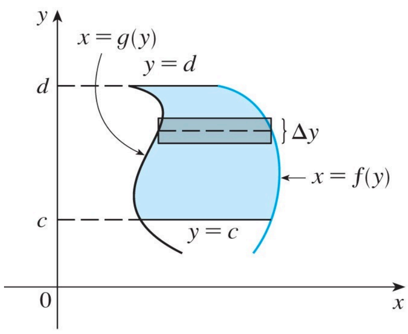


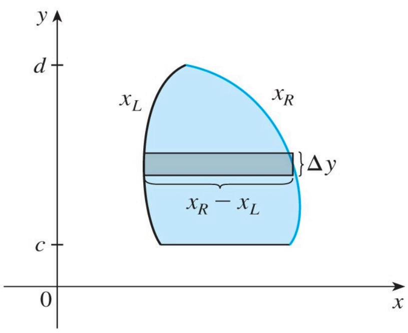
$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{2}} \left|\cos x - \sin x\right| dx \\
&= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx \\
&= 2\sqrt{2} - 2
\end{aligned}
$$

□

Some regions are treated by regarding x as a function of y. Suppose that the region S is bounded by curves with equation $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$ where f and g are continuous and $f(y) \geq g(y)$ for all $c \leq y \leq d$. The area of the region S is

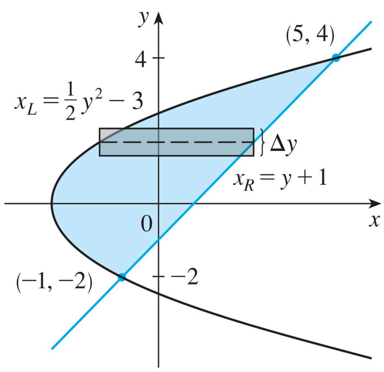
$$A = \int_c^d \left[f(y) - g(y)\right] dy.$$





Example 5.1.7. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Proof. The points of intersection is obtained by solving $y^2 = 2y + 8$. Hence, those points are $y = 4$ and $y = 2$. The area of the enclosed region is

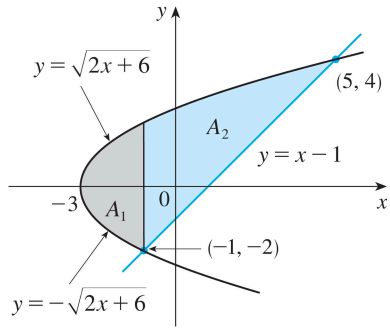


$$
\begin{aligned}
A &= \int_{-2}^{4} (y+1) - (\frac{1}{2}y^2 - 3) \, dy \\
&= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4 dy \\
&= 18
\end{aligned}
$$

□

Note. We can also obtain the area of the above region by integrating with respect to x instead of y.

Splitting the region into two subregions A_1 and A_2 and computing each area and adding them up. But it is very complicated.

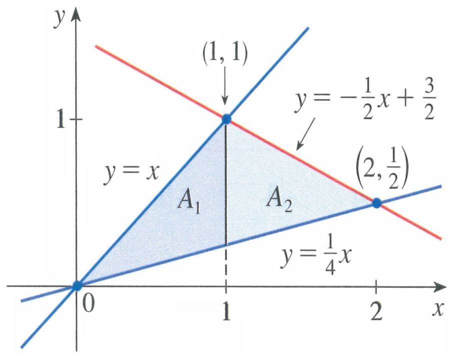


Example 5.1.8. Find the area of the region enclosed by the curves $x+2y = 3$, $y = x$, and $y = \frac{1}{4}x$

(a) Using x as as the variable of integration

Proof. We split the region into left and right parts, A_1 and A_2 as the figure. Then the area of the region is

$$\begin{aligned} A &= A_1 + A_2 = \int_0^1 (x - \frac{1}{4}x)\,dx + \int_1^2 (-\frac{1}{2}x + \frac{3}{2} - \frac{1}{4}x)\,dx \\ &= \Big[\frac{3}{8}x^2\Big]_0^1 + \Big[-\frac{3}{8}x^2 + \frac{3}{2}x\Big]_1^2 = \frac{3}{4}. \end{aligned}$$



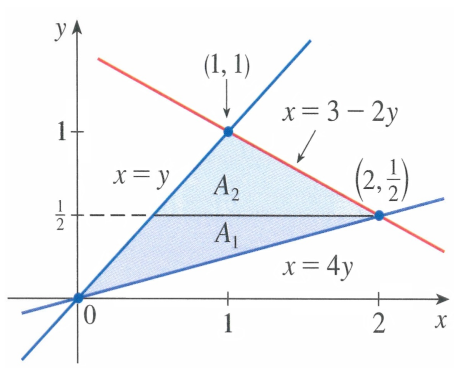
(a)

□

(b) Using y as the variable of integration

Proof. We split the region into top and bottom parts, A_1 and A_2 as the figure. Then the area of the region is

$$\begin{aligned} A &= A_1 + A_2 = \int_0^{1/2} (4y - y)\,dy + \int_1 1/2^1(3 - 2y - y)\,dy \\ &= \Big[\frac{3}{2}y^2\Big]_0^{1/2} + \Big[3y + \frac{3}{2}y^2\Big]_{1/2}^1 = \frac{3}{4}. \end{aligned}$$



(b)

□

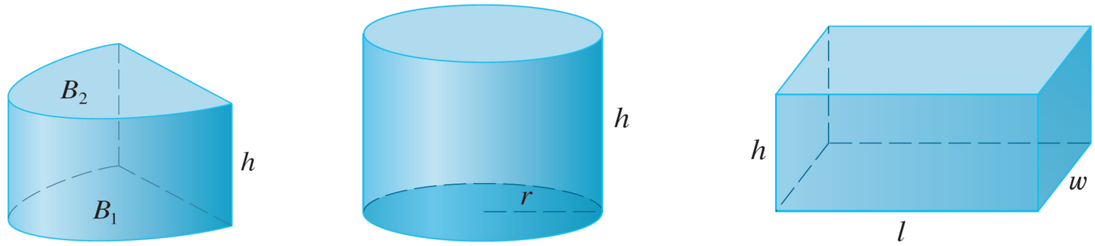
❑ Applications

(Skip)

Homework 5.1. 21, 26, 29, 32, 34, 37, 39, 41, 62, 64, 67, 70

5.2 Volumes

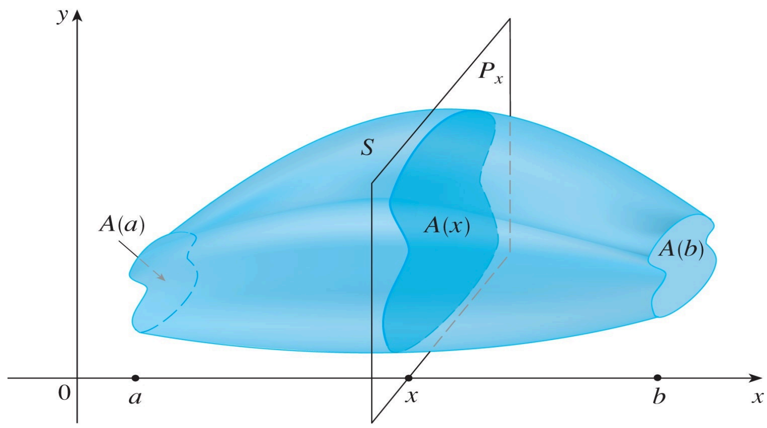
In the present section, we want to find the volume of a solid by using the techniques of integral to give an exact definition. We start with a simple type of solid called a "cylinder (right cylinder)".



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

For a general solid S (not a cylinder), we cut it into several slices and approximate each slice by regarding them as cylinders. We estimate the volume of S by adding the volumes of those approximating volumes of slabs.

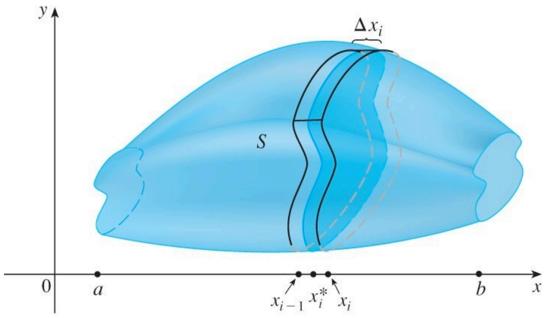
(i) The intersection of S with a plane and obtaining a plane region that is called a "*cross-section*" of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x where $a \le x \le b$.

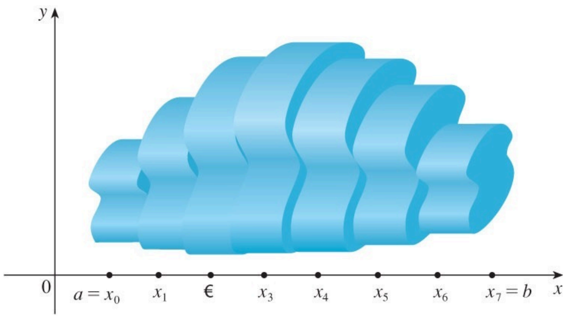


(ii) Dividing S into n "slabs" of equal width $\triangle x$ by using the planes $P_{x_1}, P_{x_2}, \cdots$ to slice the solid.

(iii) Choosing sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i by a cylinder with base $A(x_i^*)$ and "height" $\triangle x_i$. The volume of this cylinder is $A(x_i^*)\triangle x_i$. Hence, the volume of S_i is

$$V(S_i) = V_i \approx A(x_i^*)\triangle x_i.$$





(iv) Adding the volumes of these slabs, we get an approximation to the total volume of S,

$$V = \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} A(x_i^*)\triangle x_i.$$

(v) Let n tend to infinity, we define the volume of S as the limit of these sums.

Definition 5.2.1. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\triangle x_i = \int_a^b A(x)\, dx.$$

Note. For a (right) cylinder, $A(x) = A$ for all x. Then the volume is

$$V = \int_a^b A(x)\, dx = \int_a^b A\, dx = A(b-a).$$

Example 5.2.2. Find the volume of a sphere of radius r.

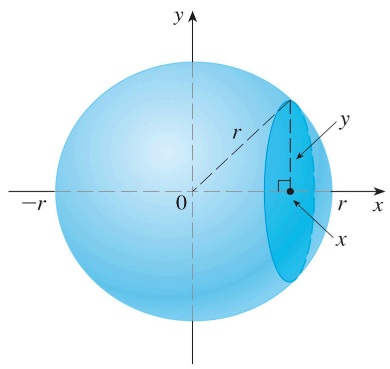
Proof. The plane P_x intersects the sphere in a circle whose radius is $y = \sqrt{r^2 - x^2}$.

Hence, the cross-sectional area is

$$A(x) = \pi(\sqrt{r^2 - x^2})^2 = \pi(r^2 - x^2).$$

The volume of the sphere is

$$\begin{aligned} V &= \int_{-r}^{r} A(x)\, dx = \int_{-r}^{r} \pi(r^2 - x^2)\, dx \\ &= \pi(r^2 x - \frac{1}{3}x^3)\Big|_{-r}^{r} = \frac{4}{3}\pi r^3. \end{aligned}$$



□

Remark. The slabs are circular cylinders, or disks, and the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\overline{x}_i)\triangle x = \sum_{i=1}^{n} \pi(1^2 - \overline{x}_i^2)\triangle x$$

when $n = 5,\ 10$ and 20 (as following figure) if we choose the sample points x_i^* to be the midpoints $\overline{x}_i$.

(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Approximating the volume of a sphere with radius 1

❑ Volumes of Solid of Revolution

If we revolve a region about a line, we obtain a "*solid of revolution*". In order to find the volume of the solid of revolution, we calculate the area of cross-section. The the volume is

$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy.$$

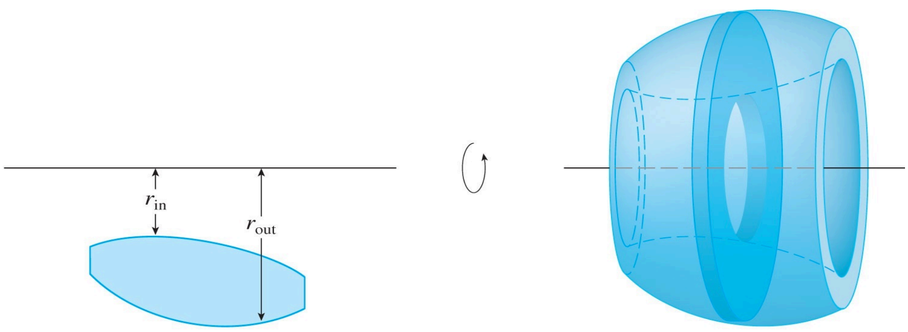
To find the area of each cross-section.

(i) If the cross-section is a disk, the area is

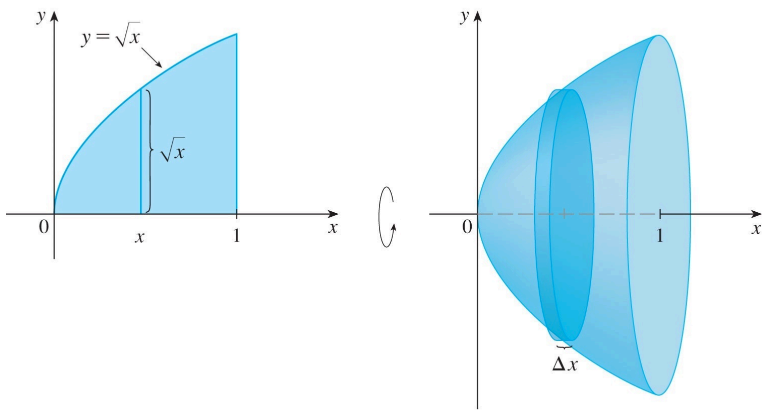
$$A = \pi(\text{radius})^2$$

(ii) If the cross-section is a washer, the area is

$$A = \pi r_{outer}^2 - \pi r_{inner}^2.$$



Example 5.2.3. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



Proof. The cross-sectional area is

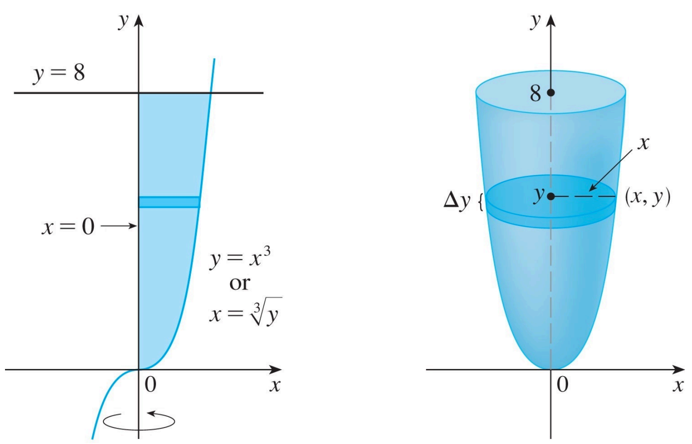
$$A(x) = \pi(\sqrt{x})^2 = \pi x.$$

The solid lies between $x = 0$ and $x = 1$ has volume

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \frac{\pi x^2}{2}\bigg|_0^1 = \frac{\pi}{2}.$$

□

Example 5.2.4. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



Proof. The region is rotated about y-axis. It makes to slice the solid perpendicular to the y-axis obtaining circular cross-sections. The area of a cross-section through y is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}.$$

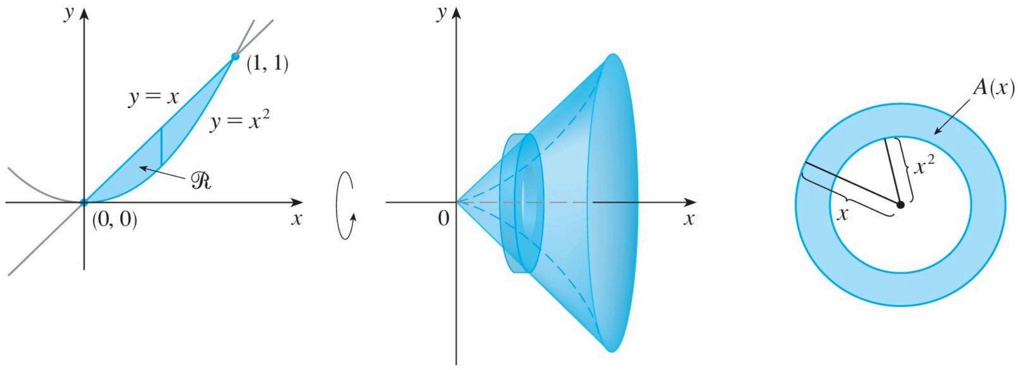
The volume of the solid is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \frac{3\pi}{5}y^{5/3}\bigg|_0^8 = \frac{96\pi}{5}.$$

□

❑ Washer Method (Method of Washer)

Example 5.2.5. Find the volume of the solid obtained by rotating the region which is enclosed by $y = x$ and $y = x^2$, about the x-axis.



Proof. The points of intersection is obtained by $x = x^2$ and hence those points are $x = 0$ and $x = 1$. The area of the cross-section perpendicular to x-axis is

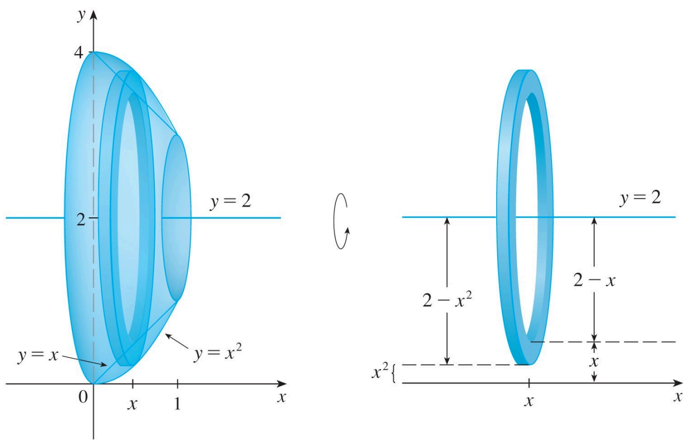
$$A(x) = \pi r_{outer}^2 - \pi r_{inner}^2 = \pi(x)^2 - \pi(x^2)^2 = \pi(x^2 - x^4).$$

The volume of the solid is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi(\frac{1}{3}x^3 - \frac{1}{5}x^5)\Big|_0^1 = \frac{2\pi}{15}.$$

□

Example 5.2.6. Find the volume of the solid obtained by rotating the region which is enclosed by $y = x$ and $y = x^2$, about the line $y = 2$.



Proof. The cross-section is a washer and its area is

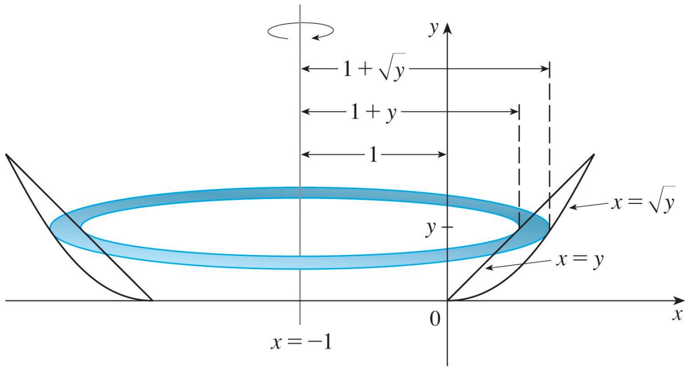
$$A(x) = \pi r_{outer}^2 - \pi r_{inner}^2 = \pi(2 - x^2)^2 - \pi(2 - x)^2 = \pi(x^4 - 5x^2 + 4x).$$

The volume of the solid is

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 x^4 - 5x^2 + 4x\,dx = \pi\left(\frac{1}{5}x^5 - \frac{5}{3}x^3 + 2x^2\right)\Big|_0^1 = \frac{8\pi}{15}.$$

□

Example 5.2.7. Find the volume of the solid obtained by rotating the region which is enclosed by $y = x$ and $y = x^2$, about the line $x = -1$.



Proof. The area of the cross-section is

$$\pi r_{outer}^2 - \pi r_{inner}^2 = \pi\left(\sqrt{y} - (-1)\right)^2 - \pi\left(y - (-1)\right)^2 = \pi(2\sqrt{y} - y - y^2).$$

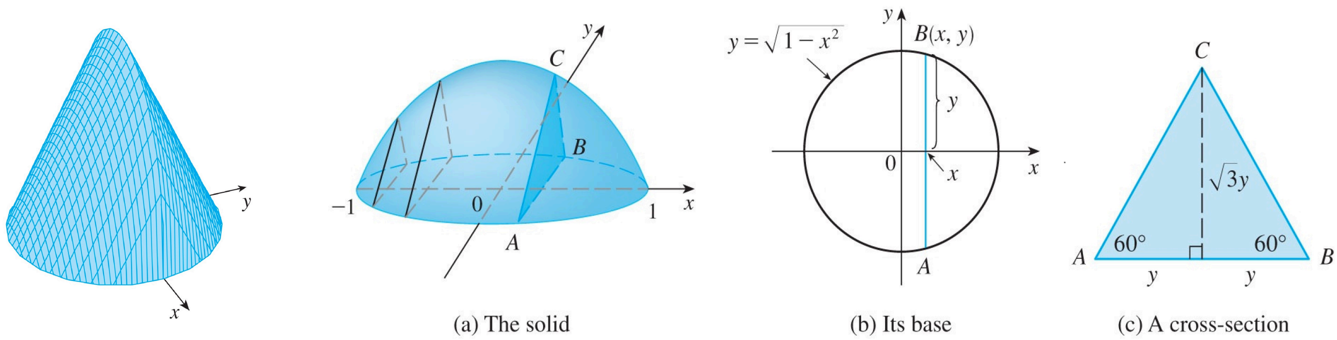
The volume of the solid is

$$V = \int_0^1 \pi(2\sqrt{y} - y - y^2)\, dy = \pi\left(\frac{4}{3}y^{3/2} - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1 = \frac{\pi}{2}.$$

□

❏ Finding Volume Using Cross-Sectional Area

Example 5.2.8. A solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



(a) The solid (b) Its base (c) A cross-section

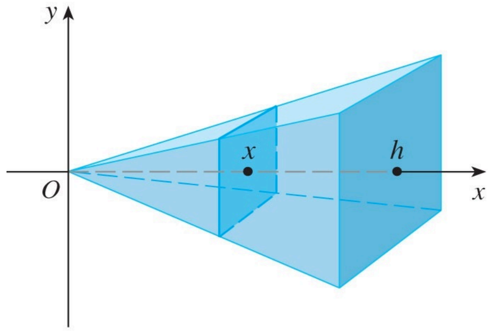
Proof. Each cross-section is an equilateral triangle, the base is $2y$ and the height is $\sqrt{3}y$. Hence the area of the cross-section is $A(x) = \sqrt{3}y^2 = \sqrt{3}(1 - x^2)$. The volume of the solid is

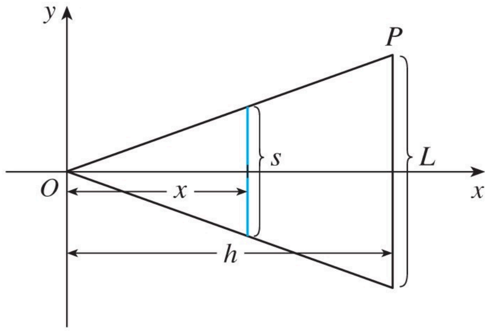
$$V = \int_{-1}^1 A(x)\, dx = \int_{-1}^1 \sqrt{3}(1 - x^2)\, dx = \frac{4\sqrt{3}}{3}.$$

□

Example 5.2.9. Find the volume of a pyramid whose base is a square with side L and whose height is h.

Proof. Placing the origin$O$ at the vertex of the pyramid and the x-axis along its central axis. Let P_x be the plane that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s.





By the similar triangle argument,

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L} \quad \Rightarrow \quad s = \frac{Lx}{h}.$$

Then the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2.$$

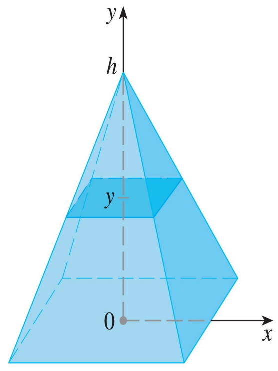
Hence, the volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx = \frac{L^2}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{L^2h}{3}$$

Alternating Method: We can place the center of the base at the origin and the vertex on the positive y-axis.

When the plane P_y when passes through y and is perpendicular to the y-axis intersecs the pyramid, the the cross-sectioinal area of the square $\frac{L^2}{h^2}(h-y)^2$. Then the volume of the pyramid is

$$V = \int_0^h \frac{L^2}{h^2}(h-y)^2 = \frac{L^2h}{3}.$$



□

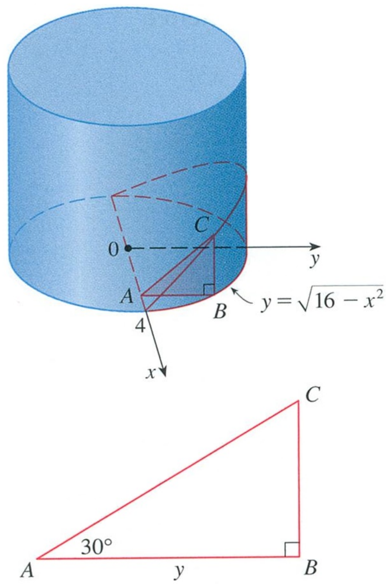
Example 5.2.10. A wedge is cut out of circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle 30° along a diameter of the cylinder. Find the volume of the wedge.

Proof. Each cross-section is a right triangle with base $y = \sqrt{16 - x^2}$. The intersection angle 30° implies that the height is $y \tan 30° = \frac{\sqrt{16-x^2}}{\sqrt{3}}$. The area of the cross-section is

$$A(x) = \frac{1}{2}\sqrt{16 - x^2} \cdot \frac{\sqrt{16 - x^2}}{\sqrt{3}} = \frac{1}{2\sqrt{3}}(16 - x^2).$$

The volume of the solid is

$$\begin{aligned} V &= \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{1}{2\sqrt{3}}(16 - x^2)\,dx \\ &= \frac{1}{2\sqrt{3}}(16x - \frac{1}{3}x^3)\Big|_{-4}^{4} = \frac{128}{3\sqrt{3}}. \end{aligned}$$



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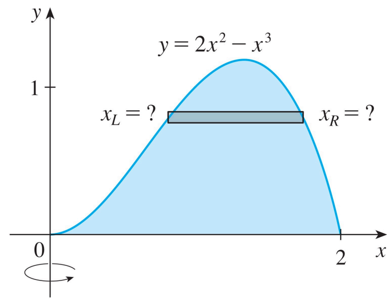
Homework 5.2. 16, 19, 22, 25, 28, 36, 39, 52, 59, 61, 67, 75, 81

5.3 Volumes by Cylindrical Shells

For some solids of revolution, it is difficult to find their volumes by using the washer method.

For example, the solid obtained by rotating the region which is enclosed by $y = 2x^2 - x^3$ and x-axis. If we want to use the washer method to find the volume of the solid, we have to evaluate the areas of each cross-section, $A(y)$, for every $0 \leq y \leq \frac{32}{27}$. But it is not easy to solve the equation $y = 2x^2 - x^3$.

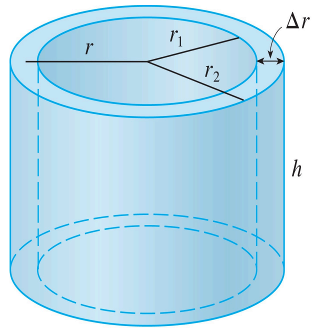


Hence, we study a different method, called the method of "cylindrical shells", to find its volume here.

■ **Method of Cylindrical Shells**

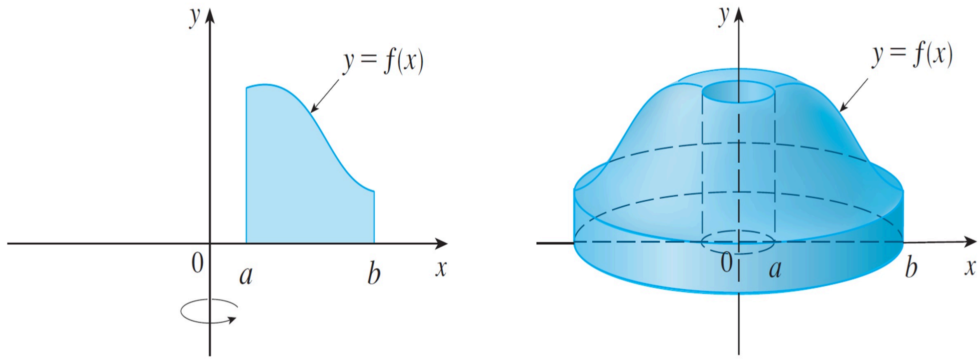
Consider a cylindrical shell with inner radius r_1, outer radius r_2 and height h. Then the thickness of the shell is $\triangle r = r_2 - r_1$. The volume of the shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h = 2\pi \cdot \underbrace{\frac{r_2 + r_1}{2}}_{\approx r} h \underbrace{(r_2 - r_1)}_{=\triangle r} \\ &= 2\pi \bar{r} h \triangle r \big(\approx 2\pi r h \triangle r \big). \end{aligned}$$



The approximating volume of the cylindrical shell is $2\pi rh\triangle r$.[†]

Let S be the solid obtained by rotating about the y-axis the region bound by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ where $0 \leq a < b$.

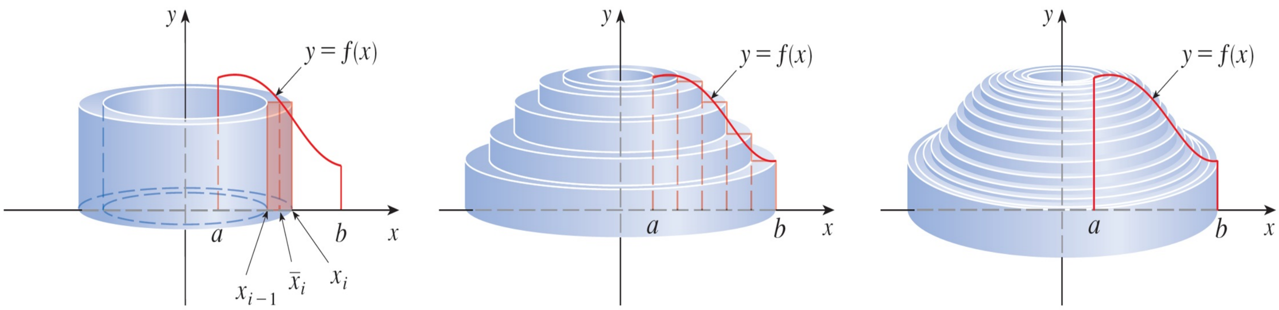


Dividing $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\triangle x$ and choose $\bar{x}$ as the midpoint of the ith subinterval. Consider the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x})$. The solid which is obtained by rotating the above region about the y-axis has volume

$$V_i \approx (2\pi\bar{x})\big(f(\bar{x}_i)\big)\triangle x.$$

The approximation to the volume of S is

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\triangle x.$$



Let $n \to \infty$, the volume of the solid is,

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\triangle x = \int_a^b 2\pi x f(x)\, dx.$$

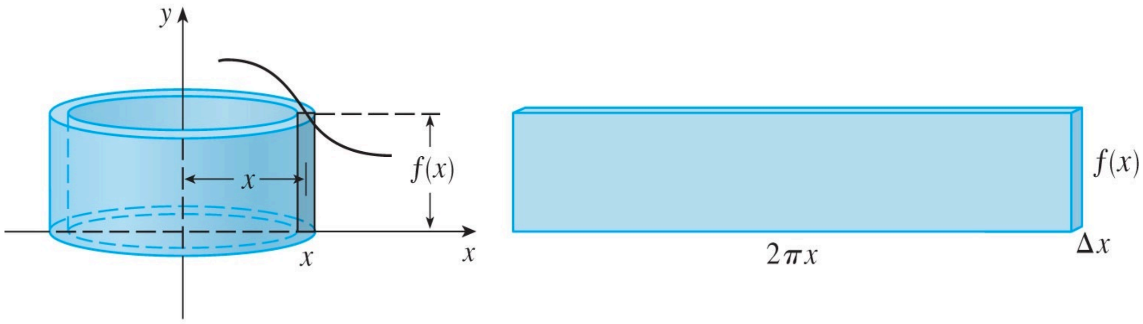
Theorem 5.3.1. *The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is*

$$V = \int_a^b 2\pi x f(x)\, dx.$$

[†]It can be remembered as $V \approx$ [circumference][height][thickness].

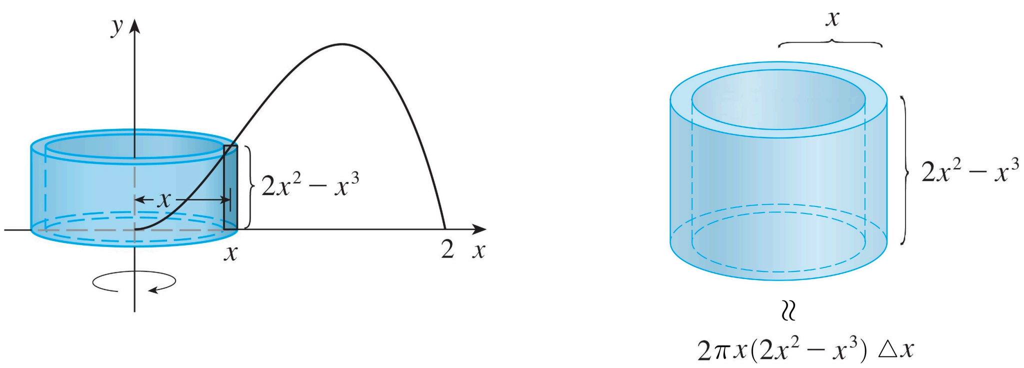
Note. Flattening a cylindrical shell with radius x, circumference $2\pi x$, height $f(x)$ and thickness $\triangle x$ (or dx). Hence, the volume of S is

$$V = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}.$$



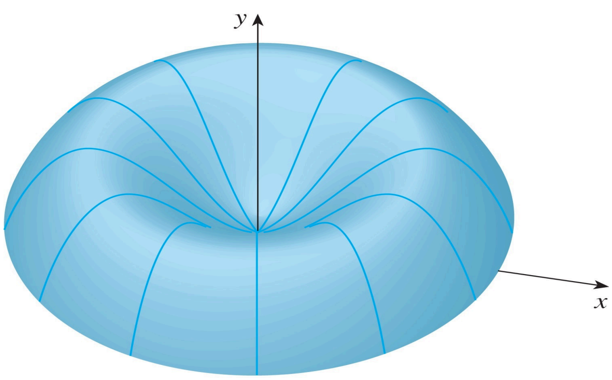
Example 5.3.2. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Proof.



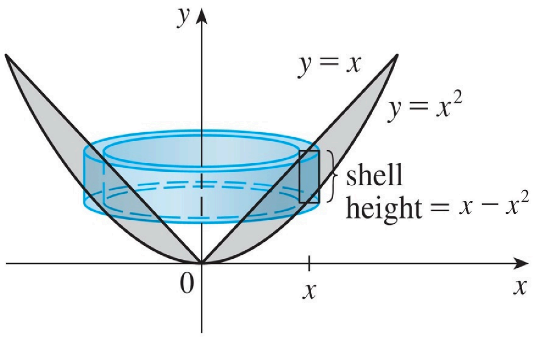
$$V = \int_0^2 2\pi x(2x^2 - x^3)\,dx = 2\pi(\frac{1}{2}x^4 - \frac{1}{5}x^5)\Big|_0^2 = \frac{16\pi}{5}.$$

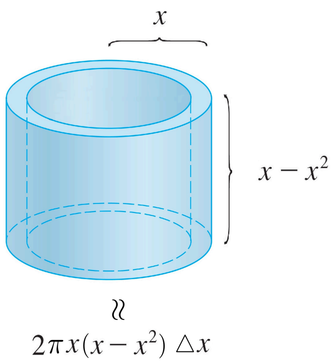
□



Example 5.3.3. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x$ and $y = x^2$.

Proof.





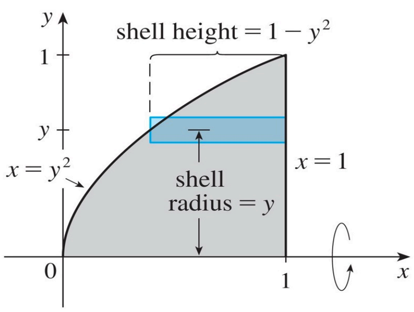
The points of intersection of $y = x$ and $y = x^2$ is $(0, 0)$ and $(1, 0)$. Therefore, the volume of the solid is

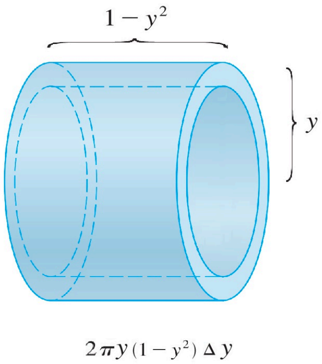
$$V = \int_0^1 2\pi x(x - x^2)\, dx = \frac{\pi}{6}.$$

□

Example 5.3.4. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Proof.



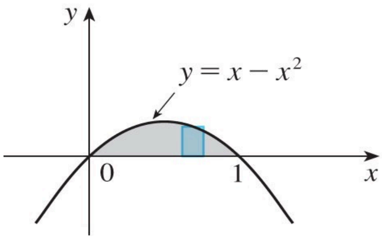


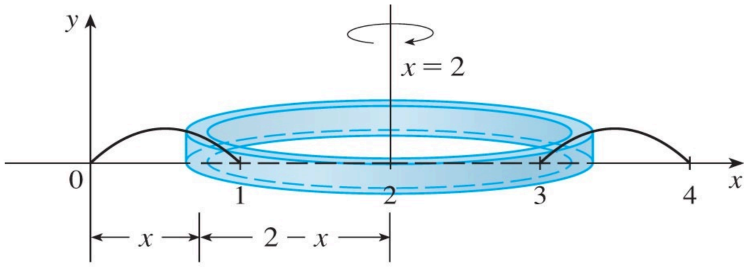
$$V = \int_0^1 2\pi y(1 - y^2)\, dy = 2\pi\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}.$$

□

Example 5.3.5. Find the volume of the solid obtained by rotating about the line $x = 2$ the region bounded by $y = x - x^2$ and $y = 0$.

Proof.





$$V = \int_0^1 2\pi(2-x)(x-x^2)\,dx = \frac{\pi}{2}.$$

□

❑ Disks and Washers versus Cylindrical Shells

Question: How do we know whether to use disks (or washers) or chylindrical shells?

Consideration:

(i) Is the region more easily described by top and bottom boundary curves of the form $y = f(x)$, or by left and right boundaries $x = g(y)$?

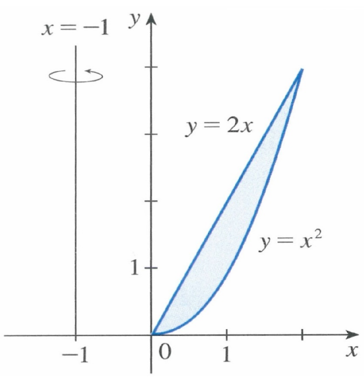
(ii) Which choice is easier to work with?

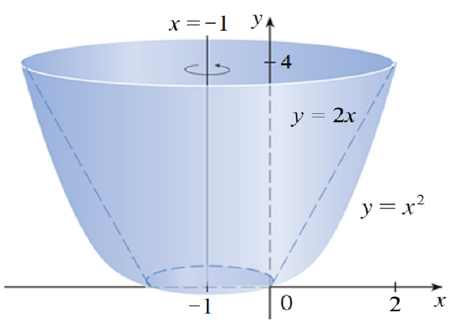
(iii) Are the limits of integration easier to find for one variable versus the others?

(iv) Does the region require two separate integrals when using x as the variable but only one integral in y?

(v) Are we able to evaluate the integral we set up with our choice of variable?

Example 5.3.6. A region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. A solid is formed by rotating the region about the line $x = -1$.



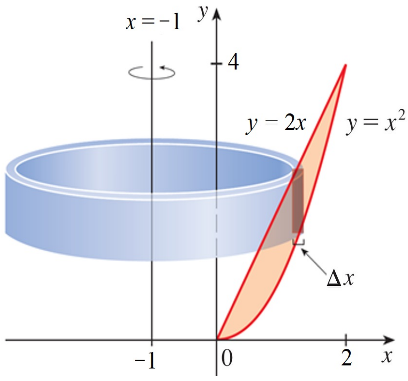


Find the volume of the solid by using

(a) x as the variable of integration

Proof. When x as the variable of integration and rotating the region about the line $x = -1$, we will use the cylindrical shells. Therefore, the volume is

$$\begin{aligned} V &= \int_0^2 2\pi(x+1)(2x - x^2)\,dx \\ &= 2\pi \int_0^2 (x^2 + 2x - x^3)\,dx \\ &= 2\pi\Big[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\Big]_0^2 = \frac{16\pi}{3}. \end{aligned}$$

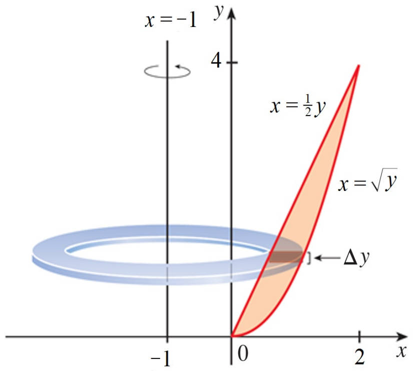


□

(b) y as the variable of integration.

Proof. When y as the variable of integration and rotating the region about the line $x = -1$, we will use the washer method. Therefore, the volume is

$$\begin{aligned} V &= \int_0^4 \Big[\pi(\sqrt{y}+1)^2 - \pi(\frac{1}{2}y + 1)^2\Big]\,dy \\ &= \pi \int_0^4 (2\sqrt{y} - \frac{1}{4}y^2)\,dy \\ &= \pi\Big[\frac{4}{3}y^{3/2} - \frac{1}{12}y^3\Big]_0^4 = \frac{16\pi}{3}. \end{aligned}$$

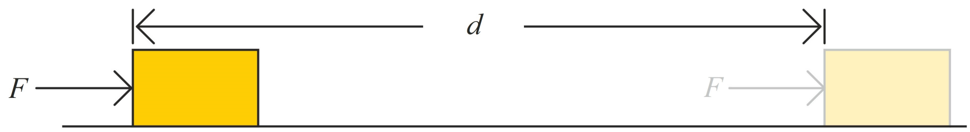


□

Homework 5.3. 11, 13, 16, 19, 21, 25, 29, 39, 42, 55, 59, 63

5.4 Work

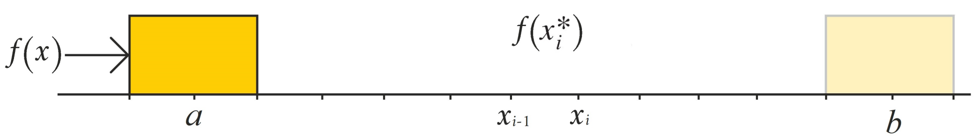
We can think of a force as describing a push or pull on an object. :



If the force F is constant and the work done is defined to be the product of the force F and the distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

Question: How about the force is not constant?



Dividing $[a, b]$ into n subintervals with equal width. The force which acts on the object from x_{i-1} to x_i gives the work

$$W_i = f(x_i^*)\triangle x.$$

The total work is

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

Let $n \to \infty$, $W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x$ (the work done in moving the object from a to b). We have

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \int_a^b f(x)\, dx.$$

Example 5.4.1. A particle is located a distance x meters from the origin. A force of $x^2 + 2x$ newton acts on it. How much work is done in moving it from $x = 1$ to $x = 3$.

Proof. The total work is

$$W = \int_1^3 x^2 + 2x\, dx = \frac{50}{3}\ J$$

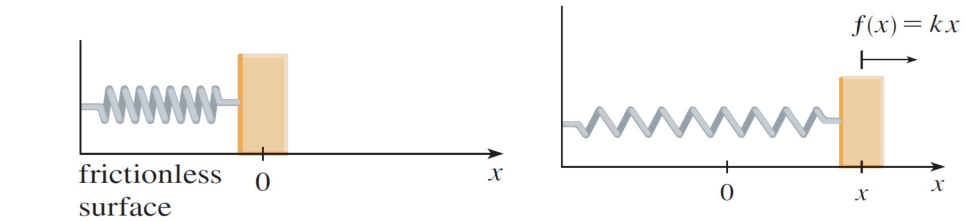
□

❑ Hooke's Law

The force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant called the spring constant.



(a) Natural position of spring (b) Stretched position of spring

Hooke's Law

Example 5.4.2. A force 40N is required to hold a spring that has been stretcvhed from its natural length of 10cm to a length 15cm. How much work is done in stretching the spring from 15cm to 18cm?

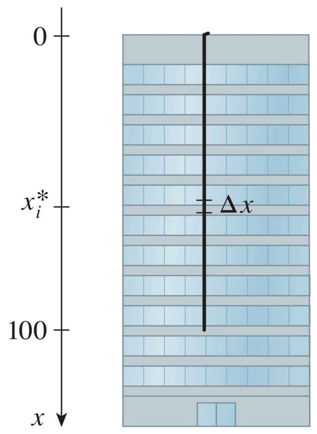
Proof. By the Hooke's Law, $40 = k \times (0.15 - 0.1)$. Then $k = 800$ and $f(x) = 800x$. The work done from 15cm to 18cm is

$$w = \int_{0.15}^{0.18} f(x)\, dx = \int_{0.15}^{0.18} 800x\, dx - 1.56\ J.$$

Then □

Example 5.4.3. A 200 lb cable is 100ft long and hangs vertically from the top of t tall building. How much work is required to lift the cable to the top of the building?

Proof.



Dividing the 100 ft cable into n piece with equal length $\triangle x = \frac{100}{n}$. Each piece has mass $\frac{200}{100} \times \triangle x = 2\triangle x$.

Move this piece vertically to the top of the building need work $W_i = x_i \cdot 2\triangle x = 2x_i\triangle x$. Hence, the total work acts on the cable is

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i\triangle x \\ &= \int_0^{100} 2x\,dx = 10000 \text{ ft-lb.} \end{aligned}$$

□

Example 5.4.4. A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It filled with water to a height of 8m. find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3).

Proof.

Dividing the water level $[2, 10]$ into n subintervals with equal width $\triangle x$. At the subinterval $[x_{i-1}, x_i]$, the mass of the water in the ith level is

$$1000V_i = 1000 \cdot \pi r_i^2 \triangle x$$

where $\dfrac{r_i}{10 - x_i^*} = \dfrac{4}{10}$. Then $r_i = \frac{2}{5}(10 - x_i^*)$ and

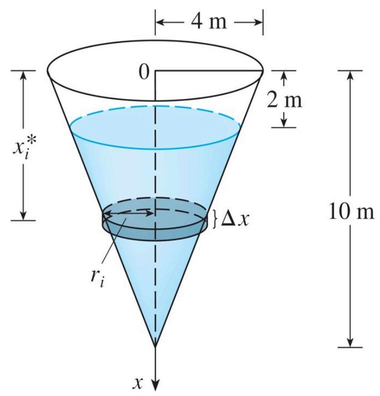
$$m_i = 1000\pi \cdot \left[\frac{2}{5}(10 - x_i^*)\right]^2 \triangle x = 160\pi(10 - x_i^*)\triangle x.$$

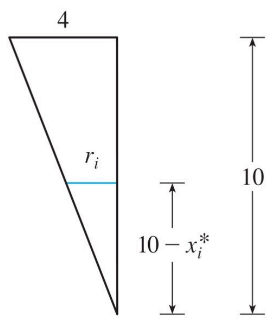
To move the level of water need work

$$W_i = 9.8 \cdot x_i^* \cdot 160\pi(10 - x_i^*)^2\triangle = 1568\pi x_i^*(10 - x_i^*)^2\triangle x.$$

The total work is

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^*(10 - x_i^*)^2 \triangle x \\ &= \int_2^{10} 1568\pi x(10 - x)^2\,dx = 1568\pi\left(\frac{2048}{3}\right) J \end{aligned}$$





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Homework 5.4. 7, 9, 13, 21, 23, 25, 29

5.5 Average Value of a Function

Observation: The average value of finitely many numbers $y_1, y_2, \cdots, y_n$ is

$$y_{avg} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

Question: How to compute the average value of a function $y = f(x)$, $a \le x \le b$?

Dividing $[a, b]$ into n subintervals with equal width $\triangle x = \dfrac{b-a}{n}$. Choose sample point $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$. The average value of f at these sample points is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{b-a}{\triangle x}} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\triangle x$$

Let $n \to \infty$, the average value of f on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\lim_{n\to\infty} f(x_i^*)\,\triangle x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

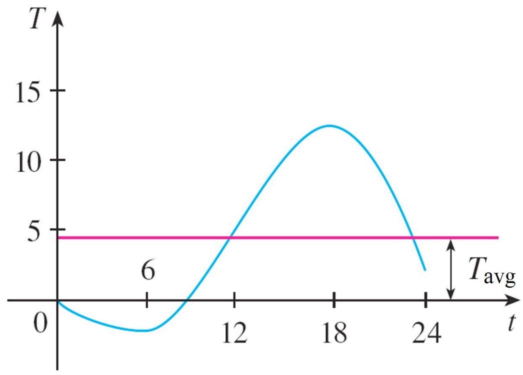
Note. $\dfrac{\text{area}}{\text{width}} = \text{average height}$

❑ The Mean Value Theorem for Integrals

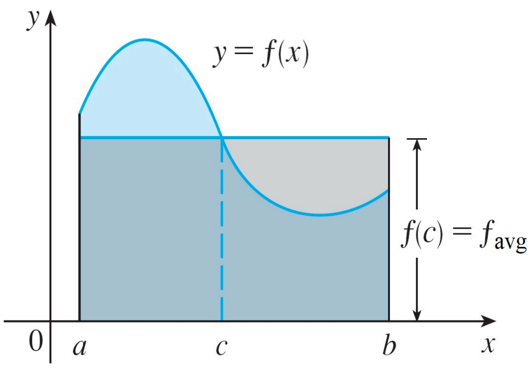
If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or $$\int_a^b f(x)\,dx = f(c)(b-a)$$



Note. Geometrically, the theorem means that the area below the graph $y = f(x)$ over $[a, b]$ is equal to the area of the rectangle with base $(b-a)$ and height f_{avg}.

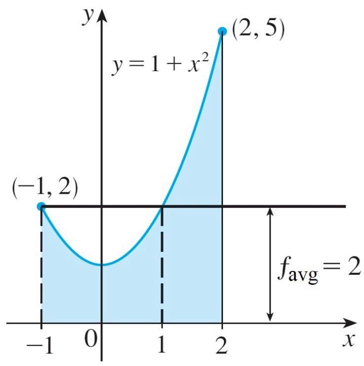


Example 5.5.1.

Let $f(x) = 1 + x^2$ be continuous on $[-1, 2]$. By the Mean Value Theorem for the integrals, there exists a nubmer $c \in [-1, 2]$ such that

$$f(c) = \frac{1}{2-(-1)} \int_{-1}^{2} 1+x^2\,dx = \frac{1}{3}\cdot(x+\frac{1}{3}x^3)\Big|_{-1}^{2} = 2.$$

Indeed, $1 + c^2 = 2$ and hence $c = \pm 1$.



Example 5.5.2. Let $s(t)$ be the displacement of the car at tiem t. Then

$$\text{average velocity} = \frac{\triangle s}{\triangle t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$\begin{aligned} v_{avg} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t)\,dt \\ &= \frac{1}{t_2 - t_1} \left[s(t_2) - s(t_1) \right] = \text{average velocity} \end{aligned}$$

Homework 5.5. 7, 8, 9, 13, 17, 22, 26

Chapter 7

Techniques of Integration

7.1 Integration by Parts

Differentiation	$\longleftrightarrow$	**Integration**
Chain Rule	$\longleftrightarrow$	Substitution Rule (Change of Variables)
Product Rule	$\longleftrightarrow$	Integration by Parts

❑ Integration by Parts

$$\begin{aligned} \frac{d}{dx}\big(f(x)g(x)\big) &= f(x)g'(x) + f'(x)g(x) \\ \Longrightarrow \quad f(x)g'(x) &= \frac{d}{dx}\big(f(x)g(x)\big) - f'(x)g(x) \\ \Longrightarrow \quad \int f(x)g'(x)\,dx &= f(x)g(x) - \int f'(x)g(x)\,dx \end{aligned}$$

■ **Another Expression**

Let $u = f(x)$ and $v = g(x)$. Then

$$\begin{cases} \dfrac{du}{dx} = f'(x) \\ \dfrac{dv}{dx} = g'(x) \end{cases} \implies \begin{cases} du = f'(x)\,dx \\ dv = g'(x)\,dx \end{cases}$$

We obtain

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Strategy:

(1) Observe the two funcitons.

(2) One will be differentiated and the other one will be integrated.

(3) Guess the next step $\int u\,dv$ or $\int f'(x)g(x)\,dx$.

Example 7.1.1. Find $\int xe^x\,dx$

Proof. (**Method 1:**)

$$\int \underbrace{x}_{f}\underbrace{e^x}_{g'}\,dx = \int \underbrace{x}_{f}\underbrace{e^x}_{g} - \int \underbrace{1}_{f'}\underbrace{e^x}_{g}\,dx = xe^x - e^x + C.$$

(**Method 2:**) $\int \underbrace{x}_{v}\underbrace{e^x\,dx}_{dv}$.

Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. We have

$$\int \underbrace{x}_{v}\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{dx}_{du} = xe^x - e^x + C.$$

□

Note. Using the integration by parts is to obtain a simpler integral than the begining integral. If we set different pair of funcitons, the process may be difficult. For example,

$$\int \underbrace{x}_{g'}\underbrace{e^x}_{f}\,dx = \int \underbrace{\frac{x^2}{2}}_{g}\underbrace{e^x}_{f} - \int \underbrace{\frac{x^2}{2}}_{g}\underbrace{e^x}_{f'}\,dx.$$

The last integral is difficult to compute.

Example 7.1.2. Evaluate $\int x^2e^x\,dx$.

Proof.

$$\begin{aligned}\int \underbrace{x^2}_{f}\underbrace{e^x}_{g'}\,dx &\overset{I.B.P}{=} \underbrace{x^2}_{f}\underbrace{e^x}_{g} - \int \underbrace{2x}_{f'}\underbrace{e^x}_{g}\,dx \\ &\overset{I.B.P}{=} x^2e^x - 2\Big(xe^x - \int e^x\,dx\Big) \qquad \text{(I.B.P twice)} \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

□

Example 7.1.3. Evaluate $\int \ln x\,dx$.

Proof. (**Method 1:**)

$$\int \ln x\,dx = \int \underbrace{\ln x}_{f}\cdot\underbrace{1}_{g'}\,dx = \underbrace{x}_{g}\underbrace{\ln x}_{f} - \int \underbrace{x}_{g}\underbrace{\frac{1}{x}}_{f'}\,dx = x\ln x - x + C.$$

(**Method 2:**) $\int \ln x\,dx = \int \underbrace{\ln x}_{u}\cdot\underbrace{1\,dx}_{dv}$. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. We have

$$\int \ln x \cdot 1\,dx = \underbrace{x}_{v}\underbrace{\ln x}_{u} - \int \underbrace{x}_{u}\underbrace{\frac{1}{x}\,dx}_{du} = x\ln x - x + C.$$

□

Example 7.1.4. Evaluate $\int e^x \sin x\,dx$.

Proof.

$$\begin{aligned}\int \underbrace{e^x}_{f}\underbrace{\sin x}_{g'}\,dx &\overset{I.B.P}{=} \underbrace{e^x}_{f}\underbrace{(-\cos x)}_{g} - \int \underbrace{e^x}_{f'}\underbrace{(-\cos x)}_{g}\,dx \\ &= -e^x\cos x + \int e^x\cos x\,dx \\ &\overset{I.B.P}{=} -e^x\cos x + \Big[e^x\sin x - \int e^x\sin x\,dx\Big]\end{aligned}$$

Then

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + C.$$

and we obtain

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

□

Example 7.1.5. Evaluate $\int \sin^n x\,dx$.

Proof. • $(n = 1)$

$$\boxed{\int \sin x\, dx = -\cos x + C}$$

• $(n = 2)$

$$\boxed{\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

• $(n \geq 3$ integer$)$

$$\begin{aligned}
\int \sin^n x\, dx &= \int \underbrace{\sin^{n-1} x}_{f}\,\underbrace{\sin x}_{g'}\, dx \\
&\overset{I.B.P}{=} \underbrace{\sin^{n-1} x}_{f}\,\underbrace{(-\cos x)}_{g} - \int \underbrace{(n-1)\sin^{n-2} x \cos x}_{f'}\,\underbrace{(-\cos x)}_{g}\, dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx \\
\Longrightarrow \quad n\int \sin^n x\, dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx
\end{aligned}$$

We obtain the "*reduction formulas*"

$$\boxed{\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx}$$

□

❑ Definite Integral

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\, dx.$$

Example 7.1.6.

$$\begin{aligned}
\int_0^1 \tan^{-1} x\, dx &\overset{I.B.P}{=} x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx \\
&= x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \qquad (u = 1+x^2) \\
&= \frac{\pi}{4} - \frac{1}{2}\ln u\Big|_1^2 \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

Homework 7.1. 3, 7, 11, 15, 19, 23, 26, 29, 30, 32, 36, 38, 40, 45, 48, 54, 57, 60, 64, 67, 72

7.2 Trigonometric Integrals

In this section, we will find the antiderivatives of the forms

$$\int \sin^m x \cos^n x \, dx \qquad \text{and} \qquad \int \tan^m x \sec^n x \, dx$$

for $m, n \in \{0\} \cup \mathbb{N}$.

❑ Integrals of Powers of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \in \{0\} \cup \mathbb{N}.$$

Recall:

- $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$
- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$
- $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$.

Case 1: Either m is odd or n is odd.

Strategy: If $m = 2k + 1$ is odd, set $u = \cos x$. If $n = 2k + 1$ is odd, set $u = \sin x$.

Example 7.2.1. Evaluate $\int \sin^3 x \, dx$

Proof. Let $u = \cos x$. Then $du = -\sin x \, dx$.

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx \\ &= -\int 1 - u^2 \, du = -(u - \frac{1}{3}u^3) + C \\ &= -\cos x + \frac{1}{3}\cos^3 x + C. \end{aligned}$$

□

Example 7.2.2. Evaluate $\int \sin^6 x \cos^5 x \, dx$.

Proof. Let $u = \sin x$. Then $du = \cos x\, dx$.

$$\begin{aligned}\int \sin^6 x \cos^5 x\, dx &= \int \sin^6 x \cos^4 x \cos x\, dx\\ &= \int \sin^6 x(1-\cos^2 x)^2 \cos x\, dx\\ &= \int u^6(1-u^2)^2\, du = \int u^6 - 2u^8 + u^{10}\, du\\ &= \frac{1}{7}u^7 - \frac{2}{9}u^9 + \frac{1}{11}u^{11} + C\\ &= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + C.\end{aligned}$$

□

Case 2: Both *m* and *n* are odd.

Strategy: Using the half-angle identity, either

(i) reducing the integral to Case1, or

(ii) converting the integral to another Case2 and using the half-angle identity until reducing the integral to Case1.

Example 7.2.3. Evaluate $\int \sin^4 x\, dx$.

Proof. (**Method 1**) Using the Integration by Parts to down the power by 2

(**Method 2**)

$$\begin{aligned}\int \sin^4 x\, dx &= \int \Big(\frac{1-\cos 2x}{2}\Big)^2\, dx\\ &= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\, dx\\ &= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1+\cos 4x}{2}\, dx\\ &= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\, dx\\ &= \frac{1}{4}\Big(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\Big) + C\end{aligned}$$

□

Example 7.2.4. Evaulate $\int \sin^4 x \cos^2 x\, dx$

Proof.

$$\begin{aligned}
\int \sin^4 x \cos^2 x \, dx &= \int (1-\cos^2 x)^2 \cos^2 x \, dx = \int \cos^2 x - 2\cos^4 x + \cos^6 x \, dx \\
&= \int \frac{1+\cos 2x}{2} - 2\Big(\frac{1+\cos 2x}{2}\Big)^2 + \Big(\frac{1+\cos 2x}{2}\Big)^3 \, dx \\
&= \int \frac{1+\cos 2x}{2} - \frac{1+2\cos 2x+\cos^2 2x}{2} + \frac{1+3\cos 2x+3\cos^2 2x+\cos^3 2x}{8} \, dx \\
&= \int \frac{1}{8} - \frac{1}{8}\cos 2x - \frac{1}{8}\cos^2 2x + \frac{1}{8}\cos^3 2x \, dx \\
&= \int \frac{1}{8} - \frac{1}{8}\cos 2x - \frac{1}{8}\Big(\frac{1+\cos 4x}{2}\Big) \, dx + \frac{1}{8}\int \cos^2 2x \cdot \cos 2x \, dx \\
\text{(set } u = \sin 2x\text{)} \quad &= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{8}\int (1-u^2)\cdot\frac{1}{2}\, du \\
&= \frac{1}{16}x - \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{16}(u - \frac{1}{3}u^3) + C \\
&= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

□

❑ Integrals of Powers of Secant and Tangent

$$\int \tan^m x \sec^n x \, dx \quad \text{for } m, n \in \{0\} \cup \mathbb{N}.$$

Recall:

- $\dfrac{d}{dx}(\tan x) = \sec^2 x$ and $\dfrac{d}{dx}(\sec x) = \tan x \sec x$
- $\sec^2 x = 1 + \tan^2 x$

Case 1: ***n* is even** ($n = 2k$)

Strategy: Let $u = \tan x$. Then $du = \sec^2 x \, dx$

Example 7.2.5. Evaluate $\int \tan^5 x \sec^6 x \, dx$

Proof. Let $u = \tan x$. Then $du = \sec^2 x \, dx$.

$$\begin{aligned}
\int \tan^5 x \sec^6 x \, dx &= \int \tan^5 x \sec^4 x \sec^2 x \, dx \\
&= \int \tan^5 x (1+\tan^2 x)^2 \sec^2 x \, dx \\
&= \int u^5 (1+u^2)^2 \, du = \frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10} + C \\
&= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^8 x + \frac{1}{10}\tan^{10} x + C.
\end{aligned}$$

□

Case 2: m **is odd** ($m = 2k + 1, n \neq 0$)

Strategy: Let $u = \sec x$. Then $du = \tan x \sec x \, dx$

Example 7.2.6. Evaluate $\int \tan^5 x \sec^6 x \, dx$

Proof. Let $u = \tan x$. Then $du = \sec^2 x \, dx$.

$$\begin{aligned}
\int \tan^5 x \sec^6 x \, dx &= \int \tan^4 x \sec^5 x \cdot \tan x \sec x \, dx \\
&= \int (\sec^2 x - 1)^2 \sec^5 x \cdot \tan x \sec x \, dx \\
&= (u^2 - 1)^2 u^5 \, du = \frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6 + C \\
&= \frac{1}{10}\sec^{10} x - \frac{1}{4}\sec^8 x + \frac{1}{6}\sec^6 x + C
\end{aligned}$$

□

Note. In order to solve the integral of other cases, we recall the integral

$$\int \tan x \, dx = \ln \left| \sec x \right| + C$$

Example 7.2.7. Evaluate $\int \tan^3 x \, dx$

Proof.

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x (\sec^2 x - 1) \, dx \\
&= \int \tan x \sec^2 x - \tan x \, dx \qquad \boxed{= \frac{1}{2}\tan^2 x - \ln \left| \sec x \right| + C} \\
&= \int \sec x \cdot \tan x \sec x \, dx - \int \tan x \, dx \\
(\text{set } u = \sec x) \quad &= \int u \, du - \ln \left| \sec x \right| + C \\
&= \frac{1}{2}\sec^2 x - \ln \left| \sec x \right| + C
\end{aligned}$$

□

Case 3: Others (m **is even or** n **is odd)**

Strategy: When $m = 2k$, we can convert the term $\tan^{2k} x$ into $(\sec^2 x - 1)^k$. Hence, we can convert the integral $\int \tan^{2k} x \sec^n x\, dx$ into $\int (\sec^2 x - 1)^k \sec^n x\, dx$. It suffices to consider the integral of the form

$$\int \sec^k x\, dx \quad \text{or} \quad \int \tan^k x\, dx \qquad \text{for every } k \in \mathbb{N}.$$

(i) $(k = 1)$

$$\begin{aligned} \int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ (\text{set } u = \sec x + \tan x) \quad &= \int \frac{1}{u}\, du = \ln|u| + C = \ln\big|\sec x + \tan x\big| + C \end{aligned}$$

(ii) $(k = 2)$

$$\int \sec^2 x\, dx = \tan x + C$$

(iii) ($k \geq 3$, integer) By the Integration by Parts,

$$\int \sec^k x\, dx = \frac{\tan x \sec^{k-2} x}{k-1} + \frac{k-2}{k-1} \int \sec^{k-2} x\, dx$$

❑ Using Product Identities

To evaluate the integarls

$$\int \sin mx \cos nx\, dx, \quad \int \sin mx \sin nx\, dx, \quad \int \cos mx \cos nx\, dx$$

we can use the following identities

(a) $\sin A \cos B = \frac{1}{2}\big[\sin(A - B) + \sin(A + B)\big]$

(b) $\sin A \sin B = \frac{1}{2}\big[\cos(A - B) - \cos(A + B)\big]$

(c) $\cos A \cos B = \frac{1}{2}\big[\cos(A - B) + \cos(A + B)\big]$

Example 7.2.8. Evaluate $\int \sin 4x \cos 5x\, dx$

Proof.

$$\begin{aligned}\int \sin 4x \cos 5x \, dx &= \int \frac{1}{2}\big[\sin(-x) + \sin 9x\big] \, dx \\ &= \frac{1}{2}\Big(\cos x - \frac{1}{9}\cos 9x\Big) + C\end{aligned}$$

□

Homework 7.2. 11, 17, 20, 25, 29, 38, 41, 48, 53, 56, 61, 66, 75

7.3 Trigonometric Substitution

Goal: To deal with the integral with the terms

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2} \quad \text{where } a > 0.$$

Question: The integral

$$\int \sqrt{a^2 - x^2} \, dx$$

interprets the area of a circle or an ellipse. How to compute it?

We can evaluate the integral $\int x\sqrt{a^2 - x^2} \, dx = \frac{1}{2}\int \sqrt{u} \, du$ by using the substitution method ($u = a^2 - x^2$)

Recall: (**Substitution Method**)

$$\int \underbrace{f\big(g(x)\big)}_{f(u)} \underbrace{g'(x) \, dx}_{du} \overset{u=g(x)}{=\!=\!=} \int f(u) \, du$$

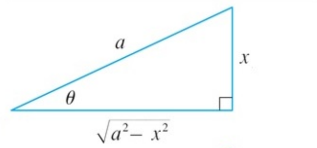
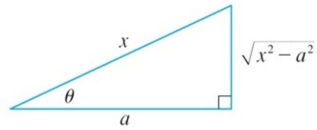
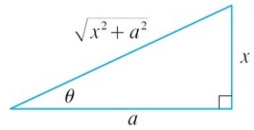
- When using the substitution method, "x" is old variable and "u" is a new variable. Moreover, the new variable $u = g(x)$ is a function of the old variable.
- Conversely, consider $\int f(x) \, dx$. Assume there exists an one-to-one function g such taht $x = g(t)$ [the old variable "x" is a function of the new variable "t"].

(**Inverse Substitution**)

$$\int f(x) \, dx \overset{u=g(t)}{=\!=\!=} \int \underbrace{f\big(g(t)\big)}_{f(x)} \underbrace{g'(t) \, dt}_{dx}.$$

Note. In general, the suitable function g is not easy to find. But, it is effective for the given radical expression because of the specified trigonometric identities.

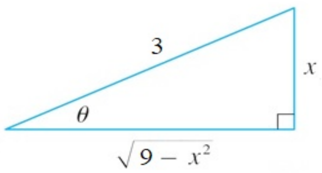
❏ Trigonometric Substitutions

Expression	Substitution	Identity	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$	

Example 7.3.1. Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$

Proof. Let $x = 3\sin\theta$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$.

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta = \int \cot^2\theta\,d\theta \\
&= \int \csc^2\theta - 1\,d\theta \\
&= -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9-x^2}}{x} - 3\sin^{-1}\left(\frac{x}{3}\right) + C
\end{aligned}$$



$\sin\theta = \frac{x}{3}$

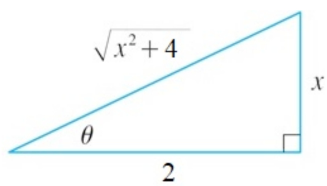
> $\sin\theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$
> It is why assuming g is 1-1.

□

Example 7.3.2. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Proof. Let $x = 2\tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{1}{4\tan^2\theta\cdot 2\sec\theta}\cdot 2\sec^2\theta\,d\theta \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du = -\frac{1}{4u} + C \\
&= -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C.
\end{aligned}$$



$\tan\theta = \frac{x}{2}$

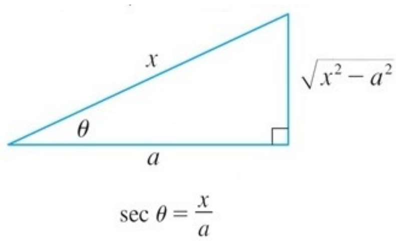
> Set $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$

□

Example 7.3.3. Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}\,dx,\ a>0$

Proof. Let $x = a\sec\theta$, $0<\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta<\pi$. Then $dx = a\tan\theta\sec\theta\,d\theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \int \frac{1}{a\tan\theta}\cdot a\tan\theta\sec\theta\,d\theta\\ &= \int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right| + C\\ &= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + C\\ &= \ln\left|x+\sqrt{x^2-a^2}\right| - \ln a + C\\ &= \ln\left|x+\sqrt{x^2-a^2}\right| + C\end{aligned}$$



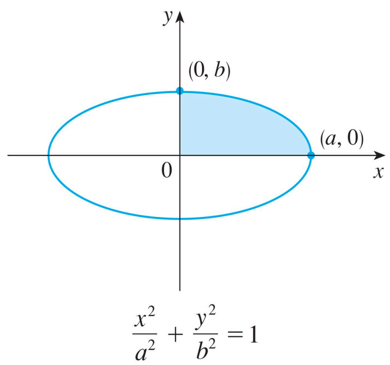
Notice that an alternating method is using the hyperbolic functions □

Example 7.3.4. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Proof. The area enclosed by the ellipse is equal to 4 multiple of the region in the first quadrant. Consider the curve $y = b\sqrt{1-\frac{x^2}{a^2}}$, $0\le x\le a$.

$$\begin{aligned}Area &= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx\\ &= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\cdot a\cos\theta\,d\theta\\ &= 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos^2\theta}{2}\,d\theta\\ &= 2ab\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}} = \pi ab.\end{aligned}$$



set $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

□

Example 7.3.5. Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

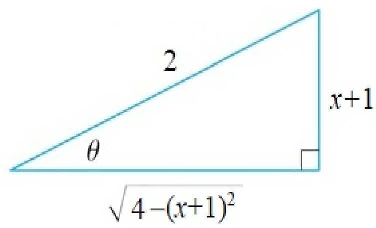
Proof. Let $x = \frac{3}{2}\tan\theta$, then $dx = \frac{3}{2}\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx &= \int_0^{\frac{\pi}{3}} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\cdot\frac{3}{2}\sec^2\theta\,d\theta \\
&= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta \\
\text{(Set } u=\cos\theta) \quad &= \frac{3}{16}\int_1^{\frac{1}{2}} \frac{1-u^2}{u^2}\big(-du\big) = \frac{3}{16}\int_{\frac{1}{2}}^1 u^{-2}-1\,du \\
&= \frac{3}{16}\big(-u^{-1}-u\big)\Big|_{\frac{1}{2}}^1 = \frac{3}{32}
\end{aligned}$$

□

Example 7.3.6.

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx \\
\text{(Set } x+1=2\sin\theta) \quad &= \int \frac{2\sin\theta-1}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\
&= \int 2\sin\theta-1\,d\theta \\
&= -2\cos\theta-\theta+C \\
&= -\sqrt{4-(x+1)^2}-\sin^{-1}\Big(\frac{x+1}{2}\Big)+C.
\end{aligned}$$



Homework 7.3. 13, 16, 19, 23, 28, 32, 36, 37(a), 39, 40, 46

7.4 Integration of Rational Functions by Partial Fractions

Observation:

$$\left\{\begin{array}{l} \int \frac{2}{x+1}\,dx = 2\ln|x+1|+C \\ \int \frac{1}{x-2}\,dx = \ln|x-2|+C \end{array}\right. \Longrightarrow \underbrace{\int \frac{2}{x+1}-\frac{1}{x-2}\,dx}_{=\int \frac{x-5}{x^2-x-2}\,dx} = 2\ln|x+1|-\ln|x-2|+C$$

Goal: In this section, we want to deal with the integration of the rational functions. Let

$$f(x) = \frac{P(x)}{Q(x)}$$

where

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \quad \text{and} \quad Q(x) = b_m x^m + \cdots + b_1 x + b_0 \quad \text{for } a_n, b_m \neq 0$$

are polynomials.

Definition 7.4.1. If $n < m$, we call $f(x)\left[=\dfrac{P(x)}{Q(x)}\right]$ a "*proper*" rational function ; if $n \geq m$, we call f a "*improper*" rational function.

Review: In high school algebra, we can use long-divison to express a rational function as the sum of a polynomial and a proper rational function.

$$\frac{P(x)}{Q(x)} = \underbrace{S(x)}_{polynomial} + \underbrace{\frac{R(x)}{Q(x)}}_{\substack{proper\ rational\\ function}}$$

Hence,

$$\int \frac{P(x)}{Q(x)}\,dx = \underbrace{\int S(x)\,dx}_{easy} + \underbrace{\int \frac{R(x)}{Q(x)}\,dx}_{\substack{partial\\ fraction}}$$

From now on, we assume all the below rational functions are proper and discuss the integration of proper rational functions by using the method of "*partial fraction*".

❑ Partial Fractions

$$\int \underbrace{\frac{R(x)}{Q(x)}}_{\substack{proper\ rational\\ function}}\,dx$$

■ Strategy

Step 1: Factorizing the denominator $Q(x)$ as far as possible.

Example 7.4.2.

$$\begin{aligned}
Q(x) &= x^4 - 16 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4)\\
Q(x) &= x^3 - 5x^2 + 7x - 2 = (x-2)(x^2-3x+1) = (x-2)\left(x - \frac{3+\sqrt{5}}{2}\right)\left(x - \frac{3-\sqrt{5}}{2}\right)\\
Q(x) &= x^5 - 2x^4 - 16x + 32 = (x-2)^2(x+2)(x^2+4)\\
Q(x) &= x^3 - 5x^2 + 12x - 12 = (x-2)(x^2-3x+6)
\end{aligned}$$

Remark. Every polynomail can be factorized as the product of several 1-degree and irreducible 2-degree polynomials. That is,

$$Q(x) = (a_1x+b_1)^{r_1}\cdots(a_nx+b_n)^{r_n}(c_1x^2+d_1x+e_1)^{s_1}\cdots(c_mx^2+d_mx+e_m)^{s_m}$$

Step 2: To express $\dfrac{R(x)}{Q(x)}$ as the sum of several terms of the forms

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

That is,

$$\begin{aligned}
\frac{R(x)}{Q(x)} &= \frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{A_{n1}}{(a_nx+b_n)} + \frac{A_{n2}}{(a_nx+b_n)^2} + \cdots + \frac{A_{nr_n}}{(a_nx+b_n)^{r_n}} \\
&+ \frac{C_{11}x+D_{11}}{(c_1x^2+d_1x+e_1)} + \frac{C_{12}x+D_{12}}{(c_1x^2+d_1x+e_1)^2} + \cdots + \frac{C_{1s_1}x+D_{1s_1}}{(c_1x^2+d_1x+e_1)^{s_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{C_{m1}x+D_{m1}}{(c_mx^2+d_mx+e_m)} + \frac{C_{m2}x+D_{m2}}{(c_mx^2+d_mx+e_m)^2} + \cdots + \frac{C_{ms_m}x+D_{ms_m}}{(c_mx^2+d_mx+e_m)^{s_m}}
\end{aligned}$$

Step 3: Take the integral on each of the above terms and use the techniques in the previous sections.

■ **Integration of each of the proper rational functions in Step 2.**

(I) **Case 1:** Let $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ where all (a_ix+b_i) are distinct. That is, $Q(x)$ has no factor repeated. Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \cdots + \frac{A_k}{(a_kx+b_k)}.$$

Example 7.4.3. Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

Proof. Since $2x^3+3x^2-2x = x(2x-1)(x+2)$, we have

$$\begin{aligned}
\frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \qquad (A=\frac{1}{2},\ B=\frac{1}{5},\ C=-\frac{1}{10}) \\
&= \frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx \\
&= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C
\end{aligned}$$

□

(II) **Case 2:** Let $Q(x) = (a_1x + b_1)^{r_1}(a_2x + b_2)^{r_2} \cdots (a_kx + b_k)^{r_k}$. Then

$$\begin{aligned}
\frac{R(x)}{Q(x)} &= \frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{A_{k1}}{(a_kx+b_k)} + \frac{A_{k2}}{(a_kx+b_k)^2} + \cdots + \frac{A_{kr_k}}{(a_kx+b_k)^{r_k}}
\end{aligned}$$

Example 7.4.4. Evaluate $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx$.

Proof. Since $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$, we have

$$\begin{aligned}
\frac{4x}{x^3 - x^2 - x + 1} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \qquad (A = 1,\ B = 2,\ C = -1) \\
&= \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{4x}{x^3 - x^2 - x + 1}\,dx &= \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx \\
&= \ln\left|x-1\right| - \frac{2}{x-1} - \ln\left|x+1\right| + C
\end{aligned}$$

□

(III) **Case 3:** Let $Q(x) = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)$ where all $(a_ix^2 + b_ix + c_i)$ are distinct and irreducible. Then

$$\frac{R(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}$$

Example 7.4.5. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

Proof. Since $x^3 + 4x = x(x^2 + 4)$, we have

$$\begin{aligned}
\frac{2x^2 - x + 4}{x^3 + 4x} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \qquad (A = 1,\ B = 1, C = -1) \\
&= \frac{1}{x} + \frac{x-1}{x^2 + 4}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx &= \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+4}\,dx \\
&= \int \frac{1}{x}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+4}\,dx \\
&= \ln|x| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}\left(\frac{x}{2}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + C}$$

□

Trick:

$$\begin{aligned}
\int \frac{Cu + D}{u^2 + a^2}\,du &= \frac{C}{2}\int \frac{2u}{u^2+a^2}\,du + D\int \frac{1}{u^2+a^2}\,du \\
&= \frac{C}{2}\ln|u^2+a^2| + D\tan^{-1}\left(\frac{x}{a}\right) + C.
\end{aligned}$$

Remark. As long as the denominator $ax^2 + bx + c$ cannot be factorized further (irreducible), $\frac{Ax+B}{ax^2+bx+c}$ must be expressed as

$$\begin{aligned}
\frac{Ax+B}{ax^2+bx+c} &= \frac{A}{2a}\cdot\frac{(2ax+b)}{ax^2+bx+c} + \left(B - \frac{Ab}{2a}\right)\cdot\frac{1}{ax^2+bx+c} \\
&= \frac{A}{2a}\cdot\frac{2ax+b}{ax^2+bx+c} + \left(B - \frac{Ab}{2a}\right)\cdot\frac{1}{(\alpha x+\beta)^2+\gamma^2}
\end{aligned}$$

Example 7.4.6.

$$\begin{aligned}
\int \frac{x-1}{4x^2-4x+3}\,dx &= \frac{1}{8}\int \frac{8x-4}{4x^2-4x+3}\,dx - \frac{1}{2}\int \frac{1}{(2x-1)^2+2} \\
(\text{set } u = 2x-1) \quad &= \frac{1}{8}\ln|4x^2-4x+3| - \frac{1}{4}\int \frac{1}{u^2+2}\,du \\
&= \frac{1}{8}\ln|4x^2-4x+3| - \frac{1}{4}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\
&= \frac{1}{8}\ln|4x^2-4x+3| - \frac{1}{4}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C
\end{aligned}$$

(IV) **Case 4:** Let $Q(x) = (a_1x^2 + b_1x + c_1)^{s_1}(a_2x^2 + b_2x + c_2)^{s_2}\cdots(a_kx^2 + b_kx + c_k)^{s_k}$ where all

$(a_i x^2 + b_i x + c_i)$ are distinct and irreducible. Then

$$\begin{aligned}
\frac{R(x)}{Q(x)} &= \frac{A_{11}x + B_{11}}{(a_1x^2 + b_1x + c_1)} + \frac{A_{12}x + B_{12}}{(a_1x^2 + b_1x + c_1)^2} + \cdots + \frac{A_{1s_1}x + B_{1s_1}}{(a_1x^2 + b_1x + c_1)^{s_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{A_{k1}x + B_{k1}}{(a_kx^2 + b_kx + c_k)} + \frac{A_{k2}x + B_{k2}}{(a_kx^2 + b_kx + c_k)^2} + \cdots + \frac{A_{ks_k}x + B_{ks_k}}{(a_kx^2 + b_kx + c_k)^{s_k}}
\end{aligned}$$

Example 7.4.7. Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\, dx.$

Proof.

$$\begin{aligned}
&\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} \\
&= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \qquad (A = 1,\ B = -1,\ C = -1,\ D = 1,\ E = 0) \\
&= \frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2}.
\end{aligned}$$

$$\begin{aligned}
\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\, dx &= \int \frac{1}{x}\, dx - \int \frac{x + 1}{x^2 + 1}\, dx + \int \frac{x}{(x^2 + 1)^2}\, dx \\
&= \int \frac{1}{x}\, dx - \frac{1}{2}\int \frac{2x}{x^2 + 1}\, dx - \int \frac{1}{x^2 + 1}\, dx + \frac{1}{2}\int \frac{2x}{(x^2 + 1)^2}\, dx \\
&= \ln|x| - \frac{1}{2}\ln|x^2 + 1| - \tan^{-1} x - \frac{1}{2(x^2 + 1)} + K
\end{aligned}$$

□

(V) **Case 5:** General case,

$$Q(x) = (a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_nx+b_n)^{r_n}(c_1x^2+d_1x+e_1)^{s_1}(c_2x^2+d_2x+e_2)^{s_2}\cdots(c_mx^2+d_mx+e_m)^{s_m},$$

then

$$\begin{aligned}
\frac{R(x)}{Q(x)} &= \frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{A_{n1}}{(a_nx+b_n)} + \frac{A_{n2}}{(a_nx+b_n)^2} + \cdots + \frac{A_{nr_n}}{(a_nx+b_n)^{r_n}} \\
&+ \frac{C_{11}x+D_{11}}{(c_1x^2+d_1x+e_1)} + \frac{C_{12}x+D_{12}}{(c_1x^2+d_1x+e_1)^2} + \cdots + \frac{C_{1s_1}x+D_{1s_1}}{(c_1x^2+d_1x+e_1)^{s_1}} \\
&+ \cdots \\
&\vdots \\
&+ \cdots \\
&+ \frac{C_{m1}x+D_{m1}}{(c_mx^2+d_mx+e_m)} + \frac{C_{m2}x+D_{m2}}{(c_mx^2+d_mx+e_m)^2} + \cdots + \frac{C_{ms_m}x+D_{ms_m}}{(c_mx^2+d_mx+e_m)^{s_m}}
\end{aligned}$$

❑ Rationalizing Substitutions

Example 7.4.8. Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$.

Proof. Let $u = \sqrt{x+4}$. Then $x = u^2 - 4$ and $du = \frac{1}{2\sqrt{x+4}}\,dx$. We have

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\,dx &= \int \frac{u^2}{u^2-4}\,du = 2\int 1 + \frac{4}{u^2-4}\,du \\
&= 2u + 2\int \frac{1}{u-2} - \frac{1}{u+2}\,du \\
&= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.
\end{aligned}$$

□

Homework 7.4. 9, 13, 17, 21, 25, 29, 33, 37, 45, 48, 51, 55, 61, 68

7.5 Strategy for Integration

■ Mermorized the Table

Table of Integration Formulas Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$	**2.** $\int \frac{1}{x}\,dx = \ln\lvert x\rvert$
3. $\int e^x\,dx = e^x$	**4.** $\int b^x\,dx = \frac{b^x}{\ln b}$
5. $\int \sin x\,dx = -\cos x$	**6.** $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$	**8.** $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$	**10.** $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$	**12.** $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$
13. $\int \tan x\,dx = \ln\lvert \sec x\rvert$	**14.** $\int \cot x\,dx = \ln\lvert \sin x\rvert$
15. $\int \sinh x\,dx = \cosh x$	**16.** $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	**18.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$
***19.** $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	***20.** $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert$

■ Strategy

(1) Simplify the integrand if possible

(2) Look for an obvious substitution

(3) Classify the integrand according to its form

(a) Trigonometric function: products of powers of $\sin x, \cdots, \csc x$.

(b) Rational function $\frac{P(x)}{Q(x)}$

(c) Integration by Parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

(d) Radicals:

- Trigonometric substitution: $\sqrt{x^2 \pm a^2}$, $\sqrt{a^2 \pm x^2}$

- Rationalizing substitution: $\sqrt[n]{ax+b}$ $\quad \left(\text{let } u = \sqrt[n]{ax+b}\right)$

(e) Try again!

Question: Can we integrate all continuous functions and find the explicit forms of their antiderivatives?

Answer: No! For example, we cannot find the explicit form of $\int e^{x^2}\,dx$, $\int \frac{e^x}{x}\,dx$, $\int \sin(x^2)\,dx$, $\int \cos(e^x)\,dx$, $\int \sqrt{x^3+1}\,dx$, $\int \frac{1}{\ln x}\,dx$, $\int \frac{\sin x}{x}\,dx$.

Homework 7.5. 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 94, 95

7.6 Integration Using Tables and Technology

Homework 7.6.

7.7 Approximate Integration

Sometimes, it is difficult to find the exact value of definite integarl. Two situations may be happened.

(1) We cannot find the explicit form of an antiderivative of f. For example, $\int_0^1 e^{x^2}\,dx$, $\int_{-1}^1 \sqrt{1+x^3}\,dx$.

(2) The function is determined from a scientific experiment. But there may be no formula for the function.

Goal: In this section, we want to approximate value of definite integrals.

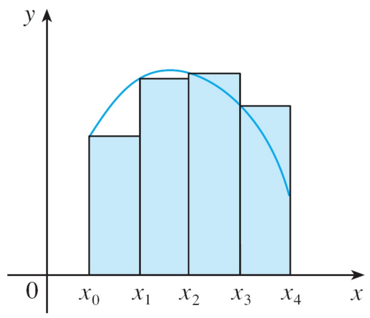
Recall: The Riemann integral is the limit of Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\triangle x.$$

Hence, as n is sufficiently large,

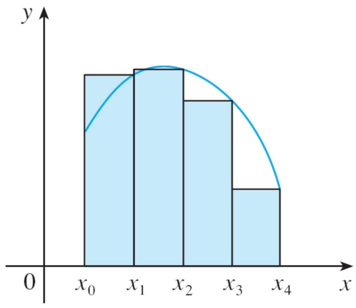
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n f(x_i^*)\triangle x$$

where x_i^* is any point in $[x_{i-1}, x_i]$.



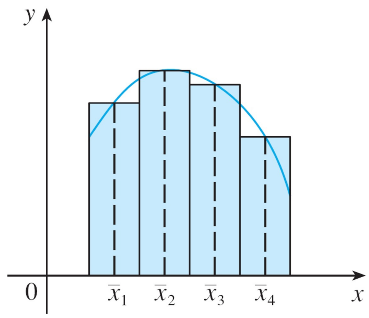
Left endpoint approximation

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^n f(x_{i-1})\,\Delta x$$



Right endpoint approximation

$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^n f(x_i)\,\Delta x$$



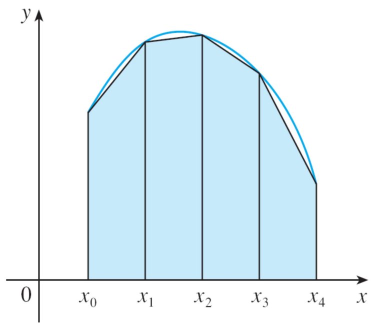
Midpoint approximation

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i)\,\Delta x$$

❑ Trapezoidal Rule

$$\begin{aligned}\int_a^b f(x)\,dx \approx T_n &= \frac{1}{2}\Big[\sum_{i=1}^n f(x_{i-1})\Delta x + \sum_{i=1}^n f(x_i)\Delta\Big] \\ &= \frac{\Delta x}{2}\big[f(x_{i-1}) + f(x_i)\big] \\ &= \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big]\end{aligned}$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.



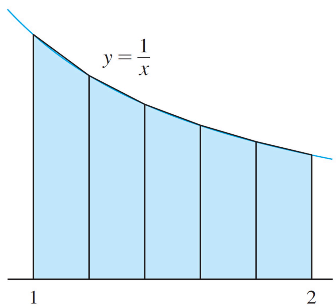
Trapezoidal approximation

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[\sum_{i=1}^n \big(f(x_{i-1}) + f(x_i)\big)\right]$$

Example 7.7.1. (a) Use the Trapezoidal Rule with $n = 5$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Proof. Compute that $\Delta x = \dfrac{2-1}{5} = 0.2$ and $x_i = 1 + 0.2i$ for $i = 0, 1, 2, 3, 4, 5$. Then

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx T_5 &= \frac{0.2}{2}\big[f(1) + 2f(1.2) + 2f(1.4) \\ &\qquad +2f(1.6) + 2f(1.8) + f(2)\big] \\ &\approx 0.695635.\end{aligned}$$



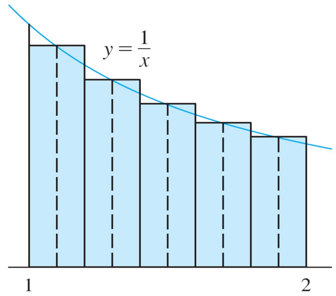
□

(b) Use the Midpoint Rule with $n = 5$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Proof.

$$\int_1^2 \frac{1}{x}\,dx \approx M_5 = 0.2\big[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\big] \approx 0.691908.$$

In fact, $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln 2 = 0.693147....$



□

Remark. Define the error of the Trapezoidal Rule and the error of the Midpoint Rule by

$$E_T = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad E_M = \int_a^b f(x)\,dx - M_n.$$

In Example 7.7.1, we have $E_T \approx -0.002488$ and $E_M \approx 0.001239$.

■ Observe the table for the approximation to $\displaystyle\int_1^2 \frac{1}{x}\,dx$

Approximations to $\int_1^2 \frac{1}{x}\,dx$

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Corresponding errors

n	E_L	E_R	E_T	E_M
5	−0.052488	0.047512	−0.002488	0.001239
10	−0.025624	0.024376	−0.000624	0.000312
20	−0.012656	0.012344	−0.000156	0.000078

(1) We get more accurate approximations when we increase the value n.

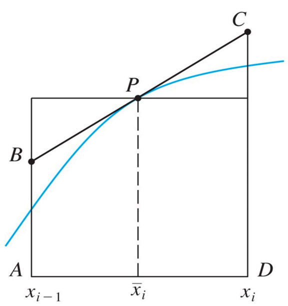
(2) The errors in the left and right endpoint approximations are opposite in sign ($E_L E_R < 0$) and ($E_{2n} \approx \frac{1}{2}E_n$)

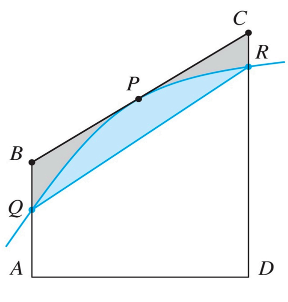
(3) $E_{T_n},\ E_{M_n} \le E_{R_n},\ E_{L_n}$

(4) $E_{T_n}E_{M_n} < 0$ and $E_{T_{2n}} \approx \frac{1}{4}E_{T_n}$

(5) $E_{M_n} \approx \frac{1}{2}E_{T_n}$ for $n \in \mathbb{N}$

❑ Compare with the Errors of Midpoint Rule and Trapezoidal Rule

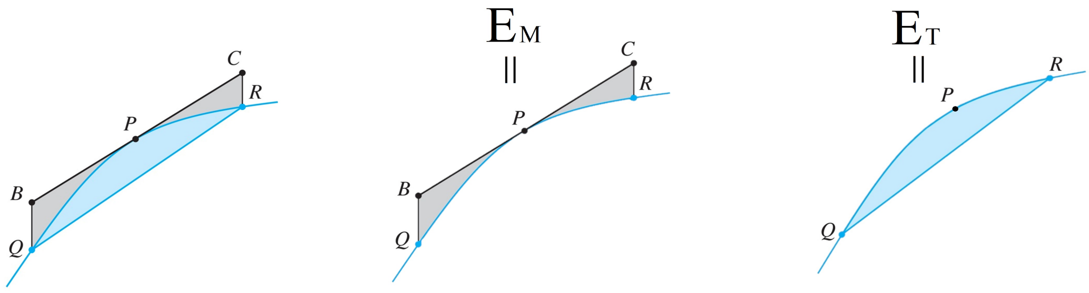




Note. The Midpoint Rule is more accurate than the Trapezoidal Rule ($E_M \le E_T$)

In the figure, the area of the rectangle □$AEFD$ is equal to the area of the trapezoid $ABCD$ where $\overline{BC}$ is the tangent line to the curve $y = f(x)$ at P.

Consider the polygon QRCB



The estimate of the error depends on $f''(x)$.

❏ Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \le \frac{K(b-a)^3}{24n^2}.$$

Example 7.7.2. Let $f(x) = \dfrac{1}{x}$ on $1 \le x \le 2$. How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rules approximateions for $\displaystyle\int_1^2 \frac{1}{x}\,dx$ are accurate to within 0.0001?

Proof. Compute $|f''(x)| = |2x^{-3}| \le 2$ for $1 \le x \le 2$. Then $K = 2$, $a = 1$ and $b = 2$. We obtain

$$\begin{aligned} E_T &\le \frac{2\cdot 1}{12n^2} < 0.0001 \quad &\Rightarrow \quad n > \frac{1}{\sqrt{0.0006}} \approx 40.8 \quad &\Rightarrow \quad n = 41 \\ E_M &\le \frac{2\cdot 1}{24n^2} < 0.0001 \quad &\Rightarrow \quad n > \frac{1}{\sqrt{0.0012}} \approx 29 \quad &\Rightarrow \quad n = 30. \end{aligned}$$

□

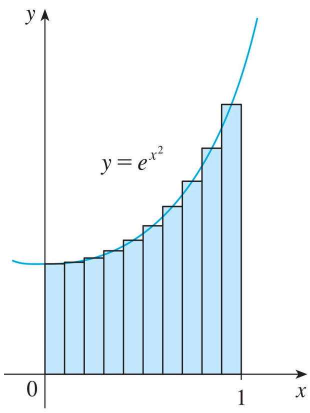
Example 7.7.3. (a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\displaystyle\int_0^1 e^{x^2}\,dx$.

(b) Give an upper bound for the error involved in this approximation.

Proof.

(a) Let $\triangle x = \dfrac{1-0}{10} = 0.1$. Then

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &\approx 0.1\left[f(0.05)+f(0.15)+\cdots_f(0.95)\right]\\ &\approx 1.460393.\end{aligned}$$



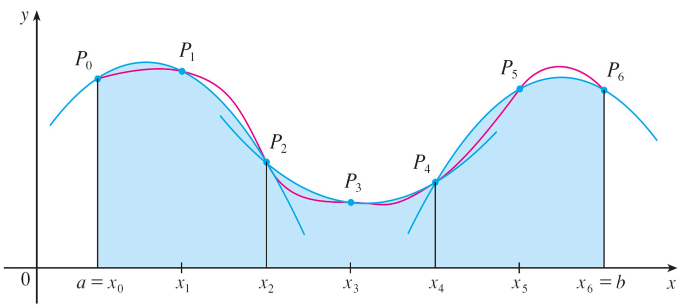
(b) $f(x) = e^{x^2}$, $f'(x) = 2xe^{x^2}$, $f''(x) = (2+4x^2)e^{x^2} \le 6e$ for $0 \le x \le 1$. Hence, $K = 6e$ and we have

$$E_M \le \frac{6e\cdot 1^3}{24\cdot 10^2} = \frac{e}{400} \approx 0.007.$$

□

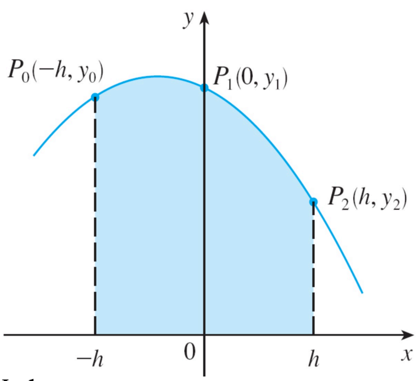
❏ Simpson's Rule

Idea: Use several pieces of parabolas to estimate the integral.



(1) Divide $[a, b]$ into n subintervals where n is an even number and $\triangle x = \dfrac{b-a}{n}$.

(2) Approximate the curve $y = f(x) \ge 0$ by a parabola passing through consecutive points P_i, P_{i+1} and P_{i+2}



A parabola $y = Ax^2 + Bx + C$ passing through $P_0(-h, y_0)$, $P(0, y_1)$ and $P_2(h, y_2)$ where $h = \triangle x = \dfrac{b-a}{n}$. Then

$$\begin{cases} y_0 = Ah^2 - Bh + C \\ y_1 = C \\ y_2 = Ah^2 + Bh + C \end{cases}$$

We have

$$\int_{-h}^{h} Ax^2 + Bx + C\,dx = 2\int_0^h Ax^2 + C\,dx = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}\left[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+\cdots+(y_{n-2}+4y_{n-1}+y_n)\right]\\ &= \frac{h}{3}\left[y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right]\\ &= \frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]\end{aligned}$$

Example 7.7.4. Use Simpson's Rule with $n = 10$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Proof. Let $f(x) = \dfrac{1}{x}$ and $\triangle x = 0.1$. Then by the Simpson's Rule,

$$\int_1^2 \frac{1}{x}\,dx \approx S_{10} = \frac{0.1}{3}\big[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)\big] \approx 0.693150.$$

□

Remark. $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx 0.693147$. Then $T_{10} \approx 0.693771$ and $M_{10} \approx 0.692835$. S_{10} is more accurate then T_{10} and M_{10}. In fact,

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

Usually, T_n and M_n have different signs and $|E_M| \approx \dfrac{1}{2}|E_T|$.

n	M_n	S_n
4	0.69121989	0.69315453
8	0.69266055	0.69314765
16	0.69302521	0.69314721

n	E_M	E_S
4	0.00192729	−0.00000735
8	0.00048663	−0.00000047
16	0.00012197	−0.00000003

Observe that $E_{S_{2n}} \approx \dfrac{1}{16}E_{S_n}$. Therefore the error bounds should have factor $\dfrac{1}{n^4}$.

■ **Error Bound for Simpson's Rule**

Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$

Example 7.7.5. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\displaystyle\int_1^2 \frac{1}{x}\,dx$ is accurate to within 0.0001?

Proof. Let $f(x) = \dfrac{1}{x}$ and $f^{(4)}(x) = \dfrac{24}{x^5}$. We have $|f^{(4)}(x)| < 24$ for $1 \le x \le 2$.

$$|E_S| \le \frac{24 \cdot 1}{180n^4} < 0.0001 \quad\Rightarrow\quad n^4 > \frac{24}{180 \cdot 0.0001} \quad\Rightarrow\quad n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04.$$

We take $n = 8$ since n must be an even number. □

Recall that for the same accuracy, $n = 41$ for trapezoidal Rule and $n = 29$ for Midpoint Rule.

Example 7.7.6. (a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in the approximation.

Proof. (a) Let $n = 10$ and $\triangle x = 0.1$.

$$\int_0^1 e^{x^2}\,dx \approx S_{10} = \frac{0.1}{3}\left[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)\right] \approx 1.42681.$$

(b) $f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2} \le 76e$ for $0 \le x \le 1$. Then

$$\left|E_S\right| \le \frac{76e \cdot 1^5}{180(10)^4} \approx 0.000115.$$

Hence,

$$\int_0^1 e^{x^2}\,dx \approx 1.463.$$

□

Homework 7.7. 9, 13, 17, 21, 41

7.8 Improper Integrals

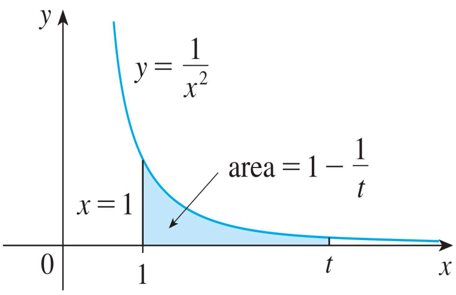
In the previous sections, we discuss the definite integral $\int_a^b f(x)\,dx$ of f under the assumptions that f is defined on a finite interval $[a, b]$ and f does not have an infinite discontinuity. In the presect section, we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an "*improper integral*".

❑ Type1: Infinite Intervals

Let f be a function defined on an infinite interval such as $[a, \infty]$, $(-\infty, a]$ or $(-\infty, \infty)$.

Example 7.8.1. Let $f(x) = \dfrac{1}{x^2}$ be defined on $[1, \infty)$.

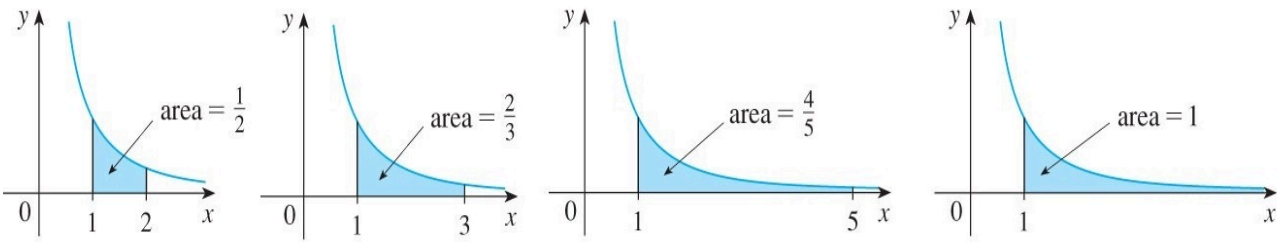
So far, we can only evaluate the integral of f on an finite interval. Fix $t > 1$, we have the area of the region bounded by $y = \dfrac{1}{x^2}$, x-axis, $x = 1$ and $x = t$



$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = 1 - \frac{1}{t}.$$

To evaluate the area of the region bounded by $y = \dfrac{1}{x^2}$, x-axis and $x = 1$, we let t tend to infinity and consider the limit

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}(1 - \frac{1}{t}) = 1.$$



Note. In the above process, the integral $\displaystyle\int_1^t \frac{1}{x^2}\,dx$ should be defined for all $t > 1$.

Definition 7.8.2. (Improper Integral of Type1)

(a) If f is defined on $[a, \infty)$ and $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists.

(b) If f is defined on $(-\infty, b]$ and $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} f(x)\,dx$$

proveided this limit exists.

We call the above improper integrals $\displaystyle\int_a^{\infty} f(x)\,dx$ and $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ "*convergent*" if the corresponding limit exists and "*divergent*" if the limit does not exists.

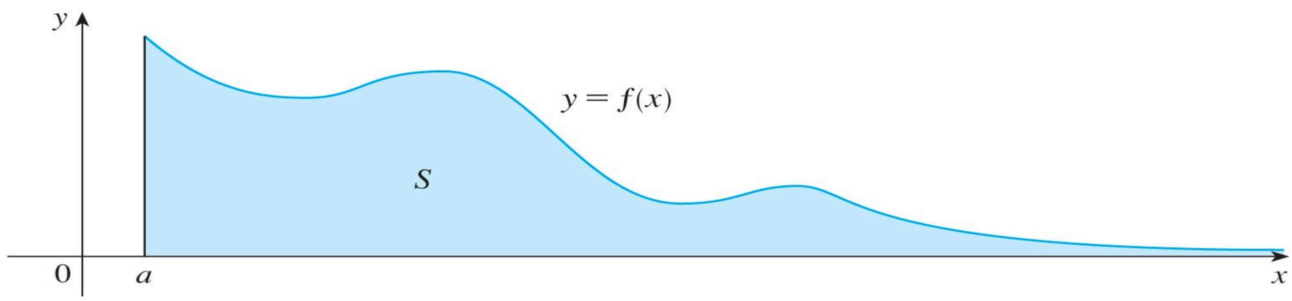
(c) If f is defined on $(-\infty, \infty)$ and both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we definte

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$

In part (c) any real number a can be used.

Remark. If $f(x) \geq 0$ and the integral $\int_a^{\infty} f(x)\,dx$ is convergent, we define the area of the region $S = \{(x, y) \mid x \geq a, 0 \leq y \leq f(x)\}$ to be

$$A(S) = \int_a^{\infty} f(x)\,dx.$$



Example 7.8.3.

(1) Discuss for what values of p the integral $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent or divergent.

Proof.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx$$

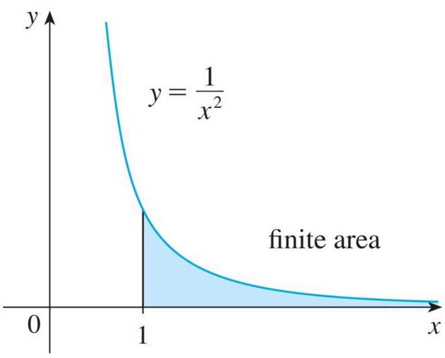
$$= \begin{cases} \lim\limits_{t\to\infty}\left(\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\right)\Big|_1^t & p\neq 1 \\ \lim\limits_{t\to\infty}\left(\ln|x|\right)\Big|_1^t & p=1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-p}\lim\limits_{t\to\infty}\left(\frac{1}{t^{p-1}}-1\right) & p\neq 1 \\ \lim\limits_{t\to\infty}\ln t & p=1 \end{cases}$$

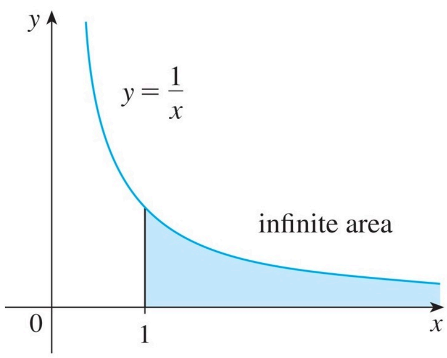
$$= \begin{cases} \frac{1}{1-p}\left(\lim\limits_{t\to\infty}\frac{1}{t^{p-1}}-1\right) = \begin{cases} \infty & p<1 \\ \frac{1}{p-1} & p>1 \end{cases} \\ \infty & p=1 \end{cases}$$

□

Conclusion: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergenet if $p>1$ and divergent if $p\leq 1$.



$\int_1^\infty (1/x^2)\,dx$ converges.



$\int_1^\infty (1/x)\,dx$ diverges.

(2) Evaluate $\int_{-\infty}^0 xe^x\,dx$.

Proof.

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\int_t^0 xe^x\,dx \overset{I.B.P}{=} \lim_{t\to-\infty}\left[xe^x\Big|_t^0 - \int_t^0 e^x\,dx\right] \\ &= \lim_{t\to-\infty}\left[-te^t - e^x\Big|_t^0\right] = \lim_{r\to-\infty}\left[-te^t - 1 + e^t\right] \\ &= -1.\end{aligned}$$

□

(3) Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$.

Proof.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx.$$

Consider

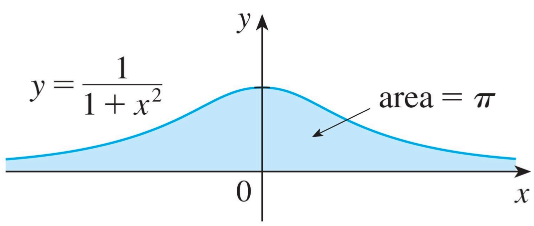
$$\begin{aligned}\int_0^{\infty}\frac{1}{1+x^2}\,dx &= \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\tan^{-1}x\Big|_0^t \\ &= \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}.\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0\frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\tan^{-1}x\Big|_t^0 \\ &= \lim_{t\to-\infty}(-\tan^{-1}t) = \frac{\pi}{2}.\end{aligned}$$

Hnece,

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

Note that $f(x) = \dfrac{1}{1+x^2}$ is an even function.



□

Remark. (Wrong Steps)

(1) We cannot replace "∞" by x directly. For example, $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty} = 0-(-1) = 1$

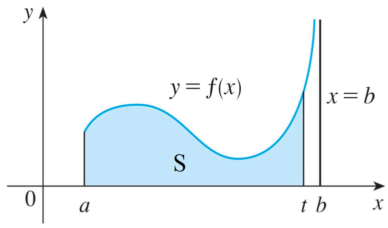
(2) When integrating a function over $(-\infty, \infty)$, the improer integral cannot write as $\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$. For example,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx &= \overbrace{\lim_{t\to\infty}\int_{-t}^{t}\frac{1}{1+x^2}\,dx}^{wrong} = \lim_{t\to\infty}\tan^{-1}x\Big|_{-t}^{t}\\ &= \lim_{t\to\infty}\tan^{-1}t-\tan^{-1}(-t)\\ &= \frac{\pi}{2}-(-\frac{\pi}{2}) = \pi\end{aligned}$$

❏ Type2: Discontinuous Integrands

Let f be a function defined on a finite interval $[a, b)$ but has a vertical asymptote at b.

In Type1 integrals, the regions extended indefinitely in a horizontal direction. In type2 integrals, the regioin is infinite in a vertical direction.



For $a \le t < b$, the area of the region S under the graph $y = f(x)$ from $x = a$ to $x = t$ is

$$A(t) = \int_a^t f(x)\,dx.$$

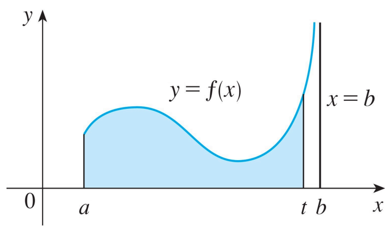
If the limit $\lim_{t\to b^-} A(t) = \lim_{t\to b^-}\int_a^t f(x)\,dx = A$ exists, we say that the area of the region S is A.

Definition 7.8.4. (Improper Integral of Type 2)

(a) If f is defined on $[a, b)$ and $\int_a^t f(x)\,dx$ exists for all $a \le t < b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

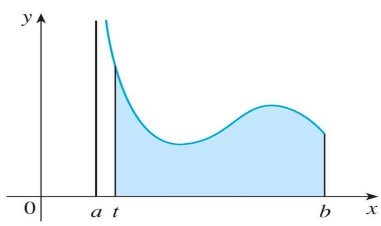
if this limit exists.



(b) If f is defined on $(a, b]$ and $\int_t^b f(x)\,dx$ exists for all $a < t \le b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

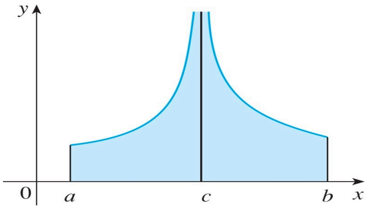
if this limit exists.



We call the improper integral $\int_a^b f(x)\,dx$ "*convergent*" if the corresponding limit exists and "*divergent*" if the limit does not exist.

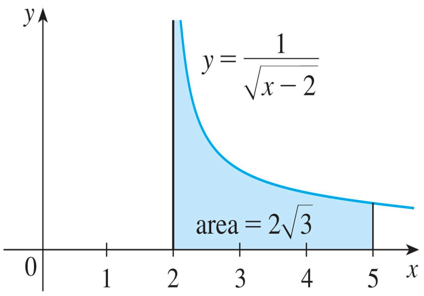
(c) For $a < c < b$, if f has an (infinite) discontinuity at c, if both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge then we say that $\int_a^b f(x)\,dx$ converges and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



Example 7.8.5.

(1) Evaluate $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

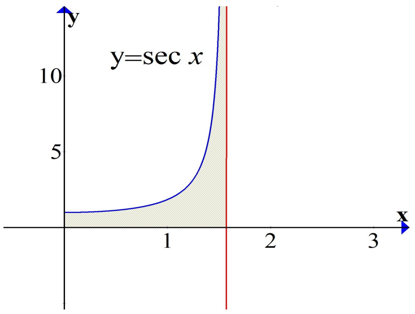


Proof. The function $f(x) = \dfrac{1}{\sqrt{x-2}}$ has the vertical asymptote $x = 2$. Thus,

$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}\,dx &= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx \\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5 \\ &= \lim_{t\to 2^+} 2(\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}.\end{aligned}$$

□

(2) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sec x\,dx$.

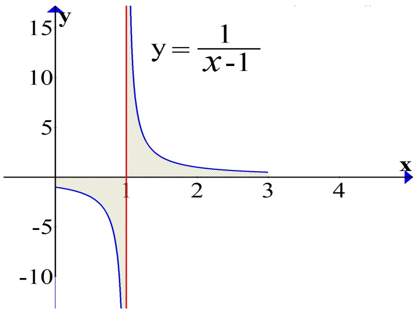


Proof. The function $f(x) = \sec x$ has the vertical asymptote $x = \frac{\pi}{2}$. Thus,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sec x\,dx &= \lim_{t\to(\frac{\pi}{2})^-} \int_0^t \sec x\,dx \\ &= \lim_{t\to(\frac{\pi}{2})^-} \ln|\sec x + \tan x|\Big|_0^t \\ &= \lim_{t\to(\frac{\pi}{2})^-} [\ln|\sec x + \tan x| - \ln 1] = \infty.\end{aligned}$$

□

(3) Evaluate $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$.



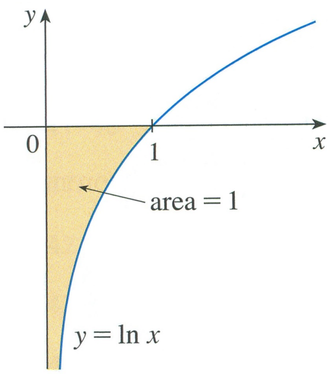
Proof. The function $f(x) = \dfrac{1}{x-1}$ has the vertical asymptote $x = 1$. Thus,

$$\begin{aligned}\int_0^1 \frac{1}{x-1}\,dx &= \lim_{t\to1^-}\int_0^t \frac{1}{x-1}\,dx\\ &= \lim_{t\to1^-}(\ln|x-1|)\Big|_0^t\\ &= \lim_{t\to1^-}\ln|t-1| = -\infty.\end{aligned}$$

Hence, $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ is divergent. □

Wrong method: $\displaystyle\int_0^3 \frac{1}{x-1}\,dx = (\ln|x-1|)\Big|_0^3 = \ln 2 - \ln 1 = \ln 2.$

(4) Evaluate $\displaystyle\int_0^1 \ln x\,dx.$



Proof. The function $f(x) = \ln x$ has the vertical asymptote $x = 0$. Thus,

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{t\to0^+}\int_t^1 \ln x\,dx\\ &= \lim_{t\to0^+}[x\ln x - x]\Big|_t^1\\ &= \lim_{t\to0^+}(-t\ln t - 1 + t) \overset{L.H.}{=} -1.\end{aligned}$$

□

(5) Discuss for what values of p the integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ is convergent or divergent.

Proof. When $p \le 0$, $f(x) = \dfrac{1}{x^p}$ is continuous on $[0,1]$. Hence, the integral is convergent and $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$. Consider the cases $p > 0$, then function $f(x) = \dfrac{1}{x^p}$ has a vertical asymptote $x = 0$. Then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p}\lim_{t\to 0^+}\dfrac{1}{x^{p-1}}\Big|_t^1 & p \neq 1 \\ \lim_{t\to 0^+}(\ln|x|)\Big|_t^1 & p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-p}\lim_{t\to 0^+}(1-t^{1-p}) = \begin{cases} \dfrac{1}{1-p} & p < 1 \\ \infty & p > 1 \end{cases} \\ \lim_{t\to 0^+}(-\ln t) = \infty & p = 1 \end{cases}$$

□

Conclusion: $\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $p < 1$ and divergent if $p \geq 1$.

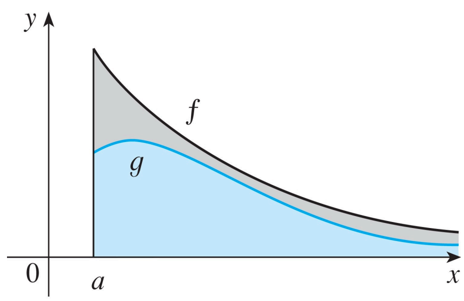
❑ Comparison Theorem

Note. For some definite integrals, it is impossible (difficult) to find their exact values but we can still determine whether these integrals are convergent or divergent.

Theorem 7.8.6. *(Comparison Theorem) Suppose that f and g satisfy $0 \leq g(x) \leq f(x)$ for every $x \geq a$.*

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

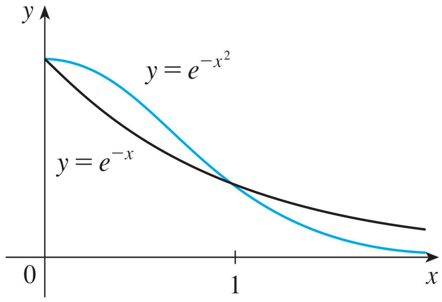
(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Example 7.8.7.

(1) Determine whether the improper integral $\int_0^\infty e^{-x^2}\,dx$ is convergent or divergent.

Proof.



Since $f(x) = e^{-x^2}$ is continuous on $[0, 1]$, it is integrable on $[0, 1]$. On the other hand, $0 \leq e^{-x^2} \leq e^{-x}$ for every $x \geq 1$ and

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(-e^{x})\Big|_1^t = e^{-1}.$$

By the Comparison Theorem, the improper integral $\displaystyle\int_1^\infty e^{-x^2}\,dx$ is convergent. Hence,

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

is also convergent. In fact, $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

□

(2) Determine whether the improper integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is convergent or divergent.

Proof.

Since $0 < \dfrac{1}{2x} < \dfrac{1+e^{-x}}{x}$ for every $1 \leq x < \infty$ and

$$\int_1^\infty \frac{1}{2x}\,dx = \frac{1}{2}\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \frac{1}{2}\lim_{t\to\infty}\ln t = \infty.$$

By the Comparison Theorem, the improper integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.

t	$\int_1^t [(1+e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

□

Homework 7.8. 15, 19, 24, 29, 32, 34, 36, 38, 46, 48, 50, 54, 57, 64, 69, 70, 80, 83

Type 1 (15, 19, 24, 29, 32, 34); Type 2 (36, 38, 46, 48)

Chapter 8

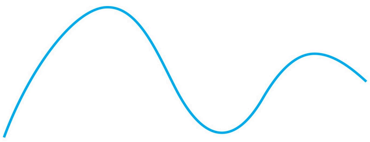
Further Applications of Integration

8.1 Arc Length

In the present section, we want to evaluate the arc length of a curve which is the graph of a smooth function.

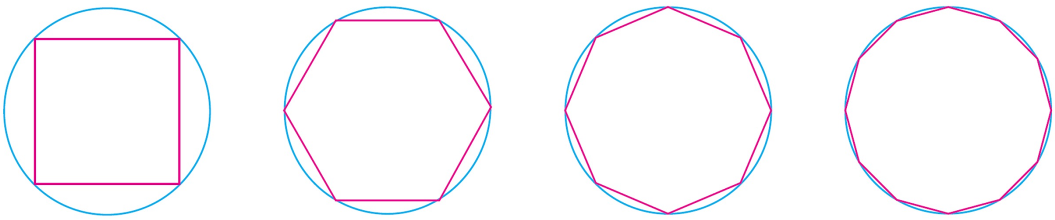
Question: For a given curve C, what is the length of C?
If the curve is a polygon, it is easy to find its length.

Question: How about the length of a general curve?



What is the length of this curve?

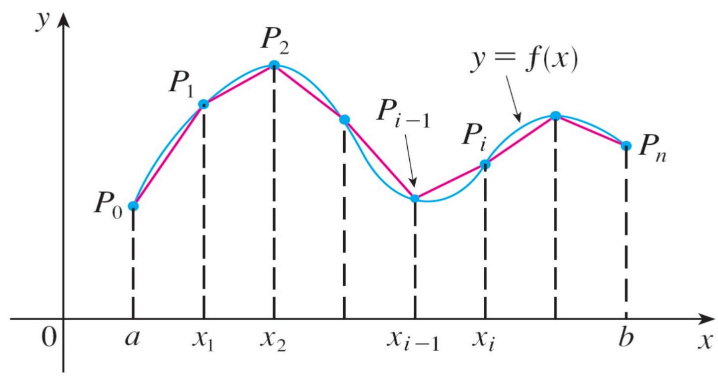
We try to approximate the length of a general curve by polygons and take a limit as the numbers of thy polygon is increased.

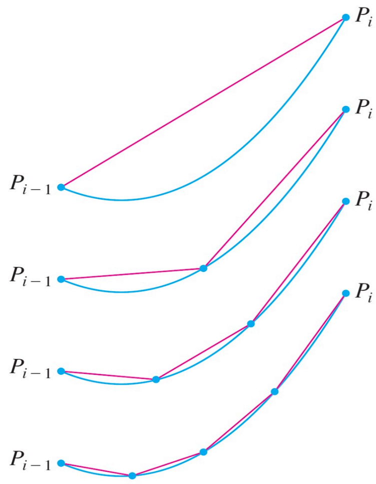


Suppose that f is a function defined on $[a, b]$ and C is the graph of f with equation $y = f(x)$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ and the point $P_i\big(x_i, f(x_i)\big)$ are points on C. Consider the polygon with vertices $P_0, P_1, \cdots, P_n$. Then the length L of the curve C is approximately the length of the polygon

$$\sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \big[f(x_i) - f(x_{i-1})\big]^2}.$$

As n increases, the approximation gets better





Definition 8.1.1. We define the"*length*" L of the curve with equation $y = f(x)$, $a \le x \le b$, as the limit of the lengths of these approximating polygonal paths (if the limit exists). That is,

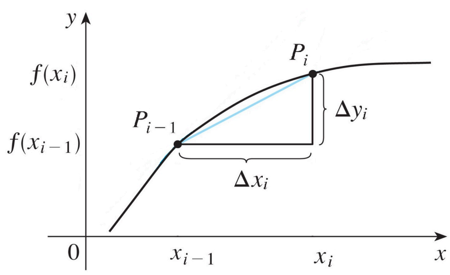
$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left|P_{i-1}P_i\right|$$

where $\left|P_{i-1}P_i\right|$ is the distance between the points P_{i-1} and P_i.

Unfortunately, for a general function f, the approximating length $\ell(P, f)$ is not easy to obtain. Therefore, from now on, we assume that f has a (continuous) derivative.

The length of the segment $P_{i-1}P_i$ is

$$\begin{aligned} \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2} &= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &\overset{M.V.T}{=} \sqrt{(x_i - x_{i-1})^2 + [f'(x_i^*)(x_i - x_{i-1})]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\ \triangle x. \end{aligned}$$



The length of the curve C with the equation $y = f(x)$ on $[a, b]$ is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \triangle x = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx.$$

The last equality is followed the hypothesis that f is continuously differentiable.

■ Arc Length Formula

If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx.$$

The expression in Leibniz notation is

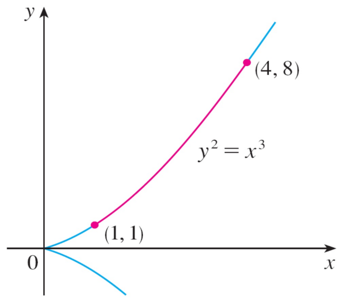
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

Example 8.1.2. Find the arc length of the semicubical parabola $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.

Proof.

The curve between $(1, 1)$ and $(4, 8)$ satisfies the equation $y = x^{3/2}$. Then $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2}$. The arc length of the curve is

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx = \frac{8}{27}u^{\frac{3}{2}}\Big|_{\frac{13}{4}}^{10} = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$



□

Suppose that the curve C has equation $x = g(y)$, $c \le y \le d$. Then the arc length of C is

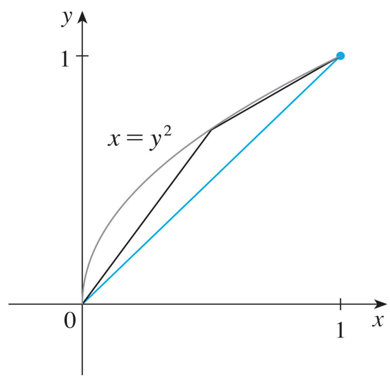
$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example 8.1.3. Find the arc length of the curve C with the equation $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Proof.

Since the curve has equation $x = y^2$, then $\dfrac{dx}{dy} = 2y$. The arc length of the curve is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (2y)^2}\, dy \\ &= \int_0^{\tan^{-1} 2} \sqrt{1 + \tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta \qquad (y = \frac{1}{2}\tan\theta) \\ &= \frac{1}{4}\Big(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big)\Big|_0^{\tan^{-1} 2} \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5} + 2). \end{aligned}$$



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Sometimes, the integral $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ is difficult to find. Hence, we may use the approximation for the integral.

Example 8.1.4. (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

Proof. Since $y = \dfrac{1}{x}$, we have $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$. Then the arc length is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx.$$

□

(b) Use Simpson's Rule with $n = 10$ to estimate the ar length.

Proof.

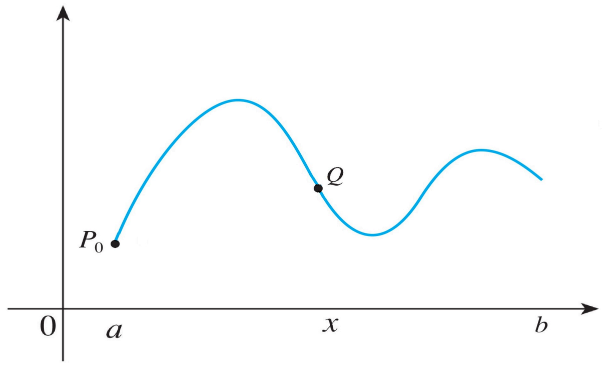
$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \approx \frac{0.1}{3}\left[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 4f(1.9) + f(2)\right] \approx 1.1321$$

□

■ Arc Length Function

Suppose that a smooth curve C has the equation $y = f(x)$, $a \le x \le b$. Let $s(x)$ be the distance along C from the initial point $P_0\big(a, f(a)\big)$ to the point $Q\big(x, f(x)\big)$. Then s is a function, called the "*arc length function*" and

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

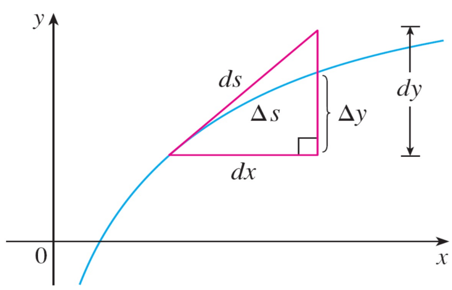


By the Fundamental Theorem of Calculus,

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This shows that the rate of change of s with respect to x is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0. The differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$



It is sometimes written in the symmetric form

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Similarly,

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Hence, the arc length along the curve C from $\big(a, f(a)\big)$ to $\big(t, f(t)\big)$ is

$$L = \int_a^t \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}_{ds} = \int_a^t 1\, ds = s(x)\Big|_a^t = s(t) - s(a) = s(t).$$

Example 8.1.5. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

Proof. The rate of change of y with respect to x is

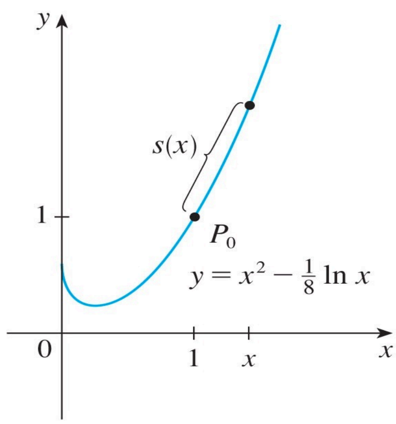
$$\frac{dy}{dx} = 2x - \frac{1}{8x}.$$

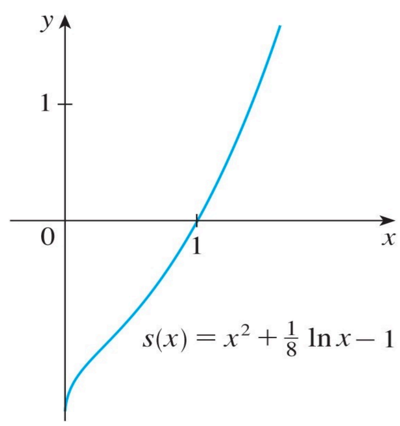
The arc length function is

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + (2t - \frac{1}{8t})^2}\, dt = \int_1^x \sqrt{(2t + \frac{1}{8t})^2}\, dt \\ &= \int_1^x 2t + \frac{1}{8t}\, dt = x^2 + \frac{1}{8}\ln x - 1. \end{aligned}$$

The arc length from $(1, 1)$ to $(3, f(3))$ is

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8}.$$





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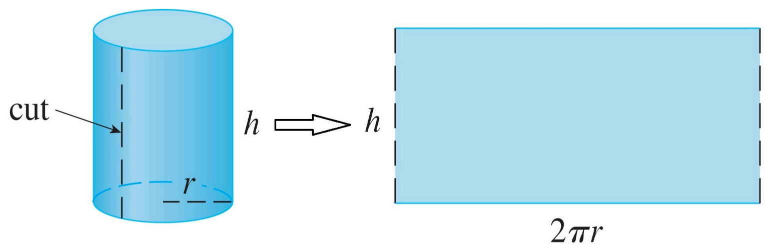
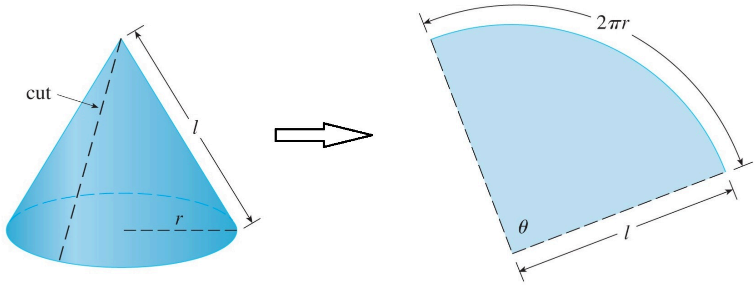
Homework 8.1. 9, 13, 17, 21, 25, 41, 43, 46, 53

8.2 Area of a Surface of Revolution

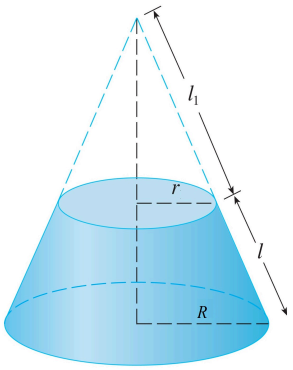
In the present section, we want to evaluate the area of a surface of revolution which is formed when a curve is rotated about a line. Let's look at some simple cases.



Area $= 2\pi rh$.



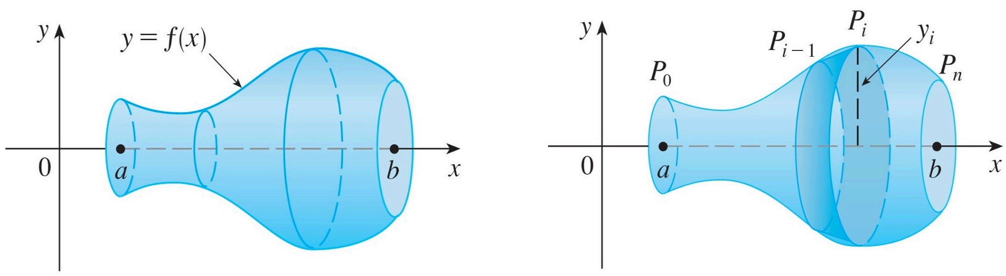
$$\theta = \frac{2\pi r}{\ell}$$
$$\text{Area} = \frac{1}{2}\ell^2\theta = \pi r\ell.$$



$$\frac{r}{R} = \frac{\ell_1}{\ell+\ell_1}$$
$$\Rightarrow \ell_1 = \frac{r\ell}{R-r}$$

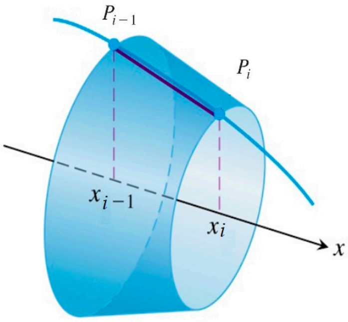
$$\begin{aligned}\text{Area} &= \pi R(\ell+\ell_1) - \pi r\ell \\ &= \pi(R-r)\ell_1 + \pi R\ell \\ &= \pi(R+r)\ell.\end{aligned}$$

Consider the surface which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis where f is positive and has a continuous derivative. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$. The points $P_0\big(x_0, f(x_0)\big), \cdots, P_n\big(x_n, f(x_n)\big)$ are points on the curve $y = f(x)$.



(a) Surface of revolution (b) Approximating band

The surface of revolution S is divided into several "bands". The surface area of a band can be calculated in terms of its radius and its arc length.



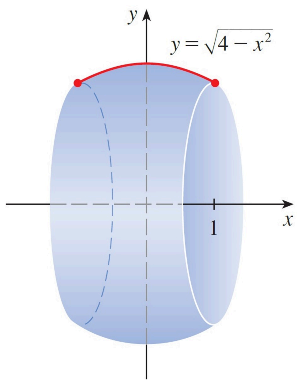
$$\begin{aligned} \text{Area} &= \pi\big(f(x_i)+f(x_{i-1})\big)\sqrt{(x_i-x_{i-1})^2\big[f(x_i)-f(x_{i-1})\big]^2} \\ &\overset{M.V.T}{=} \pi\big(f(x_i)+f(x_{i-1})\big)\sqrt{(\triangle x_i)^2+\big[f'(x_i^*)\triangle x_i\big]^2} \\ &= \pi\big(f(x_i)+f(x_{i-1})\big)\sqrt{1+[f'(x_i^*)]^2}\,\triangle x_i \\ &\approx 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\triangle x_i \end{aligned}$$

Hence, the sufrace area of the revolution is

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\triangle x_i = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx \\ \text{(Leibniz notation)} &= \int_a^b 2\pi y\sqrt{1+\Big(\frac{dy}{dx}\Big)^2}\,dx \\ \text{(arc length notation)} &= \int_a^b 2\pi y\,ds \qquad \text{(where } ds=\sqrt{1+\Big(\frac{dy}{dx}\Big)^2}\,dx\ ) \end{aligned}$$

Example 8.2.1. The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2+y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis.

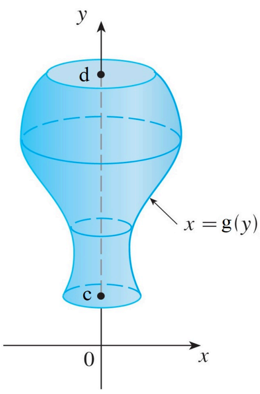
Proof. Since $y = \sqrt{4-x^2}$, then $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{4-x^2}}$. The surface area is



$$\begin{aligned} S &= \int_{-1}^{1} 2\pi y\sqrt{1+\Big(\frac{dy}{dx}\Big)^2}\,dx \\ &= 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{1+\Big(\frac{-x}{\sqrt{4-x^2}}\Big)^2}\,dx \\ &= 2\pi\int_{-1}^{1} 2\,dx = 8\pi. \end{aligned}$$

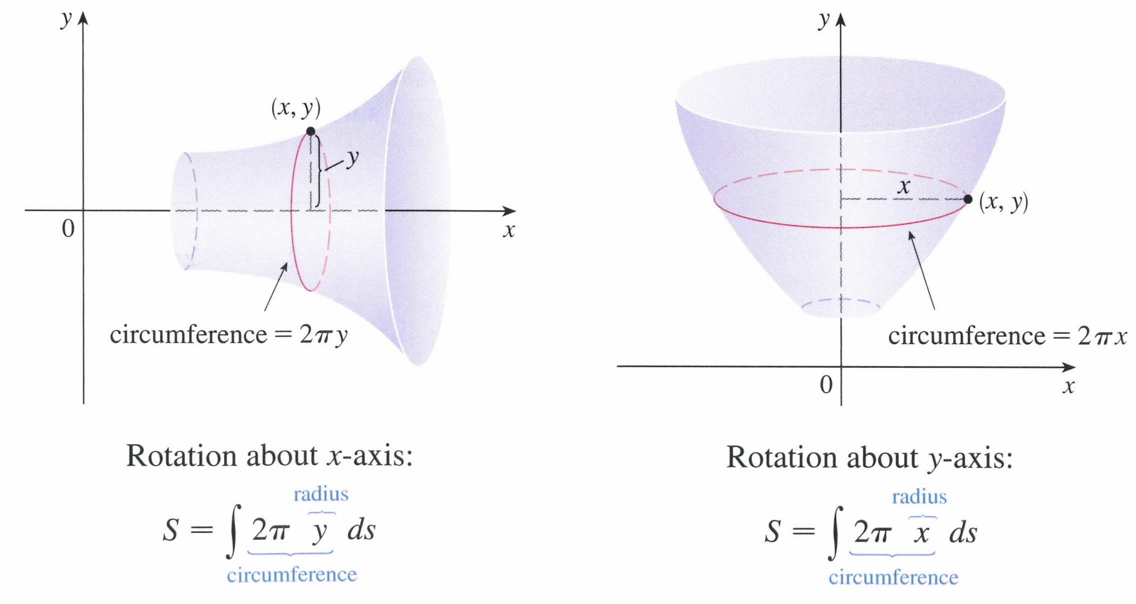
□

Similarly, the surface is obtained by rotating the curve $x = g(y)$, $c \le y \le d$, about the y-axis. The surface area is



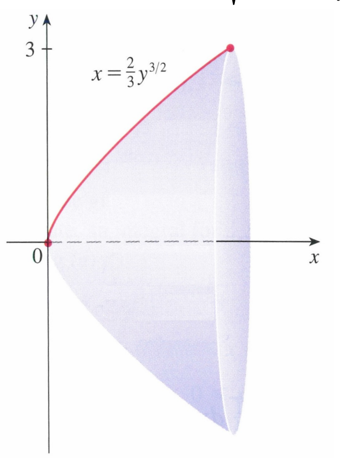
$$\begin{aligned} S &= \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy \\ &= \int_c^d 2\pi x\sqrt{1+\Big(\frac{dx}{dy}\Big)^2}\,dy \\ &= \int_c^d 2\pi x\,ds \qquad (ds = \sqrt{1+\Big(\frac{dx}{dy}\Big)^2}\,dy\ ) \end{aligned}$$

Note. Thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis respectively.



Example 8.2.2. The portion of the curve $x = \frac{2}{3}y^{3/2}$ between $y = 0$ and $y = 3$ is rotated about the x-axis. Find the Area of the resulting surface.

Proof. Observe the equation that x is given as a function of y, we will use y as the variable of integration and $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$. The surface area is

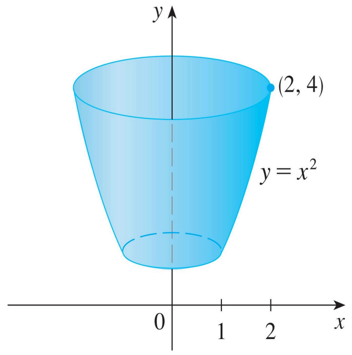


$$
\begin{aligned}
S &= \int_0^3 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= 2\pi \int_0^3 y \sqrt{1 + \left(y^{1/2}\right)^2}\, dy = 2\pi \int_0^3 y\sqrt{1+y} \\
&= 2\pi \int_1^4 (u-1)\sqrt{u}\, du \qquad (\text{set } u = 1 + y) \\
&= 2\pi \int_1^4 (u^{3/2} - u^{1/2})\, du \\
&= 2\pi \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \frac{232}{15}\pi
\end{aligned}
$$

□

Example 8.2.3. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

Proof. **Method 1:** Since $y = x^2$, then $\frac{dy}{dx} = 2x$. The surface area is



$$
\begin{aligned}
S &= \int 2\pi x\, dx = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_1^2 x\sqrt{1+4x^2}\, dx \\
&= \frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).
\end{aligned}
$$

Method 2 : Since $x = \sqrt{y}$, then $\dfrac{dx}{dy} = \dfrac{1}{2\sqrt{y}}$. The surface area is

$$
\begin{aligned}
S &= \int 2\pi x\, ds = \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \pi \int_1^4 \sqrt{4y+1}\, dy \\
&= \frac{\pi}{4}\int_5^{17} \sqrt{u}\, du = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).
\end{aligned}
$$

□

Example 8.2.4. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

Proof. Since $y = e^x$, then $\dfrac{dy}{dx} = e^x$. The surface area is

$$
\begin{aligned}
S &= \int 2\pi y\, ds = \int_0^1 2\pi e^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx \\
(u = e^x) \quad &= 2\pi \int_1^e \sqrt{1+u^2}\, du \\
(u = \tan\theta) \quad &= 2\pi \int_{\pi/4}^{\tan^{-1} e} \sec^3\theta\, d\theta \\
&= \pi \Big[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big]_{\pi/4}^{\tan^{-1} e} \\
&= \pi\left[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2}) - \sqrt{2} - \ln(\sqrt{2}+1)\right] \\
&\approx 22.943.
\end{aligned}
$$

□

Homework 8.2. 7, 13, 14, 16, 17, 19, 33, 38(原題有誤，as in Exercise 5.2.75), 41

Chapter 10

Parametric Equations and Polar Coordinates

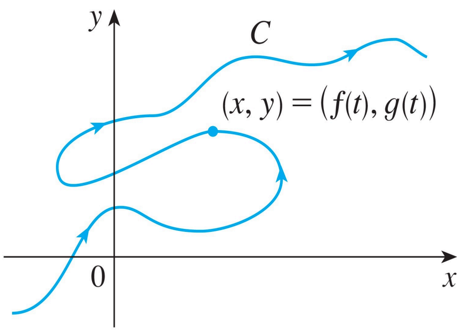
So far, we have studied the plane curves which area the graphs of explicit function ($y = f(x)$ or $x = g(y)$) or implicit functions ($f(x, y) = 0$). In the present chapter, we wiil discuss those curves which are given in terms of a third variable t ($x = f(t)$ and $y = g(t)$).

10.1 Curves Defined by Parametric Equations

When a particle moves on a plane along the curve C, in general, the path may not be described as an equation of the form $y = f(x)$ (or $x = g(y)$). Suppose that x and y are both given as functions of a third variable t (called a "*parameter*"). The equation

$$x = f(t), \quad y = g(t)$$

is called a "*parametric equation*".

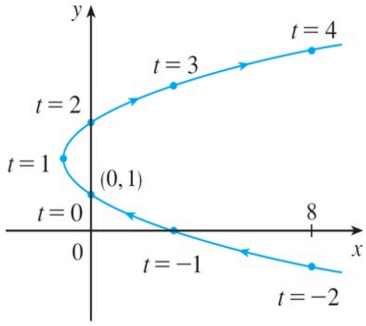


Each value of t determines a point (x, y) which we can plot in a coordinate plane. As t varies, the point $(x, y) = \big(f(t), g(t)\big)$ varies and traces out a curve C. We call the curve C : $(x, y) = \big(f(t), g(t)\big)$ a "*parametric curve*".

Example 10.1.1. Sketch and identify the curve defined by the parametric equation

$$x = t^2 - 2t \quad y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

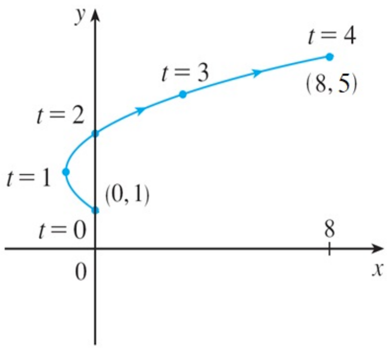


$$t = y - 1 \Rightarrow x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3 \quad \text{(Cartesian equation)}$$

We sometimes restrict t to lie in a finite interval.

Example 10.1.2.

$$x = t^2 - 2t \quad y = t + 1 \quad 0 \le t \le 4$$

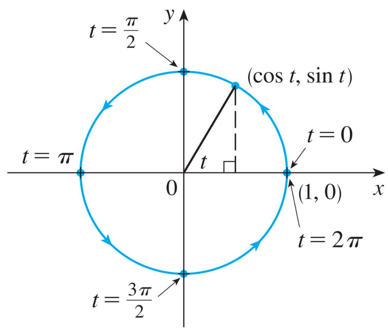


Example 10.1.3.

Observe the parametric equation

$$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$$

represents the circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

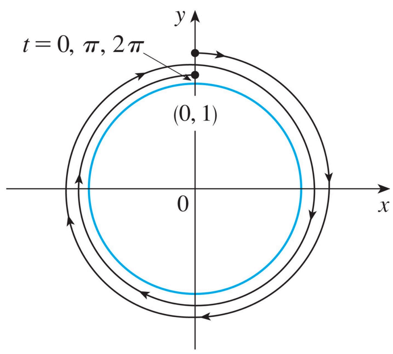


Example 10.1.4.

The parametric equation

$$x = \sin 2t \quad y = \cos 2t \quad 0 \le t \le 2\pi$$

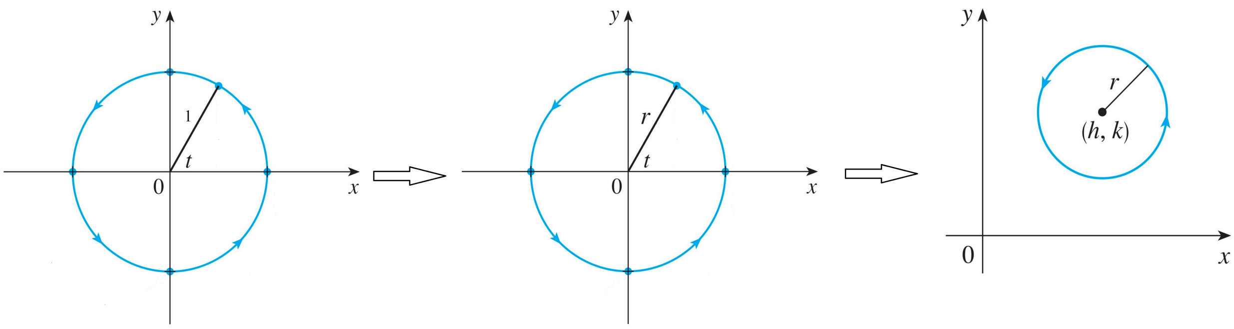
still represents the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves twice around the circle in the clockwise direction.



Remark. If we regard a curve as a set of points, it can be represented by different parametric equations. Thus, we distinguish between a "*curve: a set of points*" and a "*parametric curve: the points are traced in a particular way.*"

Example 10.1.5. Find parametric equations for the circle with center (h, k) and radius r.

Proof. We start from the circle $x = \cos t$, $y = \sin t$. Multiply the expressions for x and y by r, we get $x = r\cos t$, $y = r\sin t$ and it represents a circle with radius r and center the origin traced counterclockwise. Then we shift $h$units in the x-direction and k units in the y-direction and obtain parametric equations of the circle with center (h, k) and raidus r.



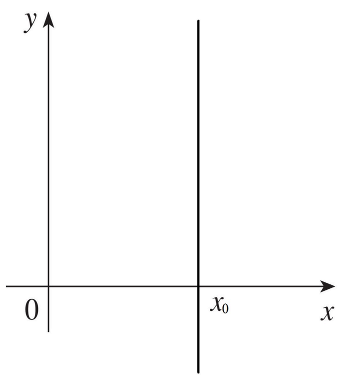
$x = \cos t$, $y = \sin t$ $\qquad$ $x = r\cos t$, $y = r\sin t$ $\qquad$ $x = h + r\cos t$, $y = k + r\sin t$

□

Example 10.1.6. (Straight Line)

The parametric equation of a straight line perpendicular the x-axis and passing $(x_0, 0)$ is

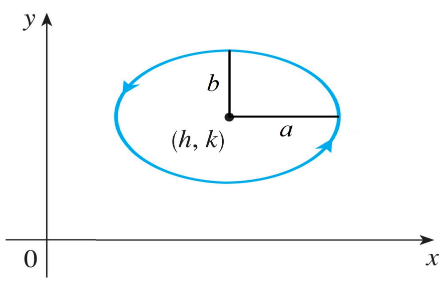
$$x = x_0 \quad y = t.$$



Example 10.1.7. (Ellipsoid)

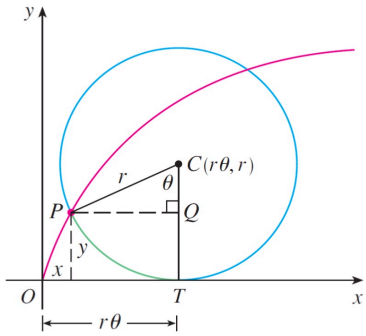
The parametric equation of an ellipsoid with center (h, k) and two axes wth lengths a and b is

$$x = h + a\cos t \quad y = k + b\sin t \quad 0 \leq t \leq 2\pi.$$

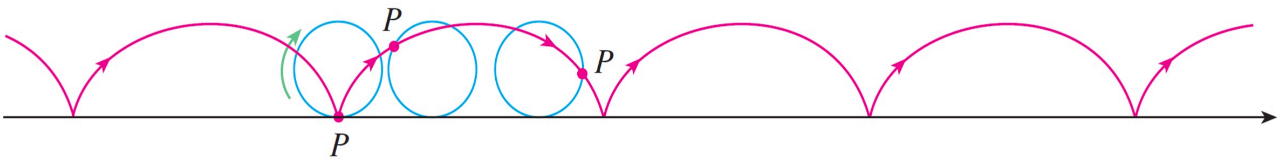


$x = h + a\cos t$, $y = k + b\sin t$

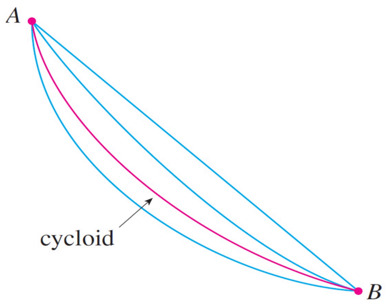
Example 10.1.8. (The Cycloid) The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a "*cycloid*".



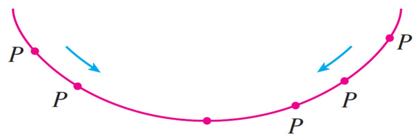
$$\begin{aligned} x &= |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta) \\ y &= |CT| - |CQ| = r - r\cos\theta = r(1-\cos\theta). \end{aligned}$$



- Physical problems



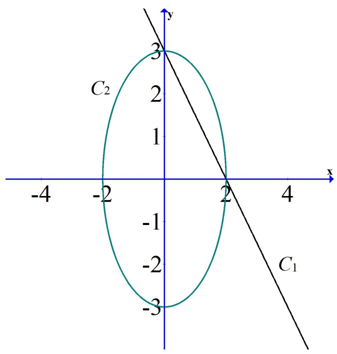
brachistochrone problem
最短時線(最速降線)



tautochrone problem
等時(降)線

Example 10.1.9. Two particles move along the curves C_1 and C_2, respectively, with parametric equations

$$C_1 : \begin{cases} x = \dfrac{16}{3} - \dfrac{8}{3}t \\ y = 4t - 5 \end{cases} \quad t \geq 0, \qquad C_2 : \begin{cases} x = 2\sin(\dfrac{1}{2}\pi t) \\ y = -3\cos(\dfrac{1}{2}\pi t) \end{cases} \quad t \geq 0$$



(a) Do the two curves intersect?

Proof. The Cartesian equations of C_1 and C_2 are $C_1 : 3x + 2y - 6 = 0$ and $C_2 : \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. We can solve the two equations and find the points where the the curves intersect at $(2, 0)$ and $(0, 3)$. □

(b) Do the two particles collide?

Proof. Find $t \geq 0$ such that both $\frac{16}{3} - \frac{8}{3}t = 2\sin(\frac{1}{2}\pi t)$ and $4t - 5 = -3\cos(\frac{1}{2}\pi t)$. We have $t = 2$ and the two particle collide at $(0, 3)$ when $t = 2$. □

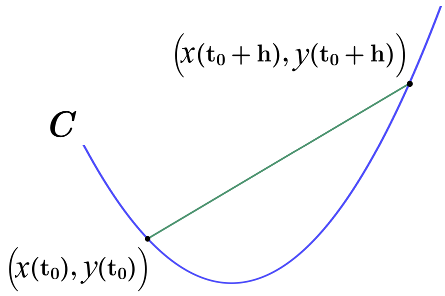
Homework 10.1. 4, 10, 13, 21, 27, 30, 34, 37, 46

10.2 Calculus with Parametric Curves

In the present section, we will apply the methods of calculus to the parametric curves. We will solve problems involving tangents, areas, arc length, and surface area.

❑ Tangents

Suppose that f and g are differentiable functions and C is a curve with parametric equation $x = f(t)$, $y = g(t)$. We want to find the tangent line of the curve C at a given point. In order to find the equation of the tangent line, it suffices to obtain its slope $\frac{dy}{dx}$.



The slope of the secant line connecting $\big(x(t_0), y(t_0)\big)$ and $\big(x(t_0 + h), y(t_0 + h)\big)$ is

$$\frac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} = \frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} \xrightarrow{h\to 0} \frac{y'(t_0)}{x'(t_0)} = \frac{dy/dt}{dx/dt}\Big|_{t=t_0}$$

By the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $\frac{dx}{dt} \neq 0$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Remark.

(1) The rate of change of y with respect to x, $\frac{dy}{dx}$, is followed by the Chain Rule. It is not necessary to express y in terms of x.

(2) The curve has a horizontal tangent line when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

(3) The curve has a vertical tangent line when $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$.

(4) How about $\dfrac{dx}{dt} = 0 = \dfrac{dy}{dt}$? It may need further discussion.

(5) To discuss the concavity of a curve, we consider

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Note that

$$\frac{d^2y}{dt^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}.$$

Example 10.2.1. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.

Proof. Find the value(s) of t at which the curves passes $(3, 0)$.

$$t^2 = 3 \;\Rightarrow\; t = \pm\sqrt{3} \quad \text{and} \quad t^3 - 3t = 0 \;\Rightarrow\; t = 0, \pm\sqrt{3}.$$

Hence, when $t = \pm\sqrt{3}$, the curve passes $(3, 0)$. Also, $\dfrac{dy}{dt} = 3t^2 - 3$ and $\dfrac{dx}{dt} = 2t$. Then

$$\frac{dy}{dt}\Big|_{t=-\sqrt{3}} = \frac{dy/dt}{dx/dt}\Big|_{t=-\sqrt{3}} = \frac{3}{2}(t - \frac{1}{t})\Big|_{t=-\sqrt{3}} = -\sqrt{3}.$$

The equation of the tangent line is $\underline{y = -\sqrt{3}(x-3)}$. Similarly, $\dfrac{dy}{dx}\Big|_{\sqrt{3}} = \dfrac{3}{2}(t - \dfrac{1}{t})\Big|_{t=\sqrt{3}} = \sqrt{3}$. The equation of the tangent line is $\underline{y = \sqrt{3}(x-3)}$. □

(b) Find the points on C where the tangent is horizontal or vertical.

Proof. (i) **Horizontal tangent line:** Let $\dfrac{dy}{dt} = 3t^2 - 3 = 0$. Then $t \pm 1$. Also, $\dfrac{dx}{dt} = 2t \neq 0$ when $t = \pm 1$. Hence, when $t = 1$, $\big(x(1), y(1)\big) = (1, -2)$. The curve has a horizontal tangent line $\underline{y = -2}$. When $t = -1$, $\big(x(-1), y(-1)\big) = (1, 2)$. The curve has a horizontal tangent line $\underline{y = 2}$.

(ii) **Vertical tangent line:** Let $\dfrac{dx}{dt} = 2t = 0$. Then $t = 0$. Also, $\dfrac{dy}{dt} = 3t^2 - 3 \neq 0$ when $t = 0$ and $\big(x(0), y(0)\big) = (0, 0)$. The curve has a vertical tangent line $\underline{x = 0}$.

□

(c) Determine where the curve is concave upward or downward.

Proof. Consider

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}[\frac{3}{2}(t - \frac{1}{t})]}{2t} = \frac{3(t^2+1)}{4t^3}.$$

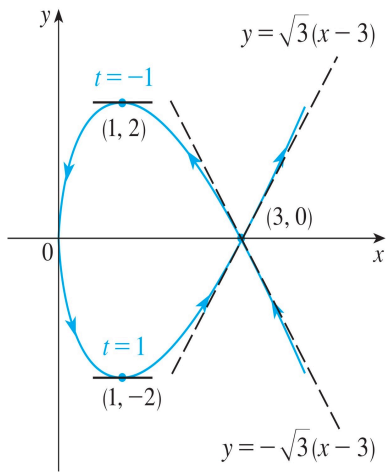
Then

$$\frac{d^2y}{dx^2} > 0 \quad \text{as} \quad t > 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} < 0 \quad \text{as} \quad t < 0.$$

The curve is concave upward when $t > 0$ and concave downward when $t < 0$. □

(d) Sketch the curve

Proof.



□

Example 10.2.2.

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \frac{\pi}{3}$.

Proof. Consider

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}.$$

When $\theta = \frac{\pi}{3}$, $\big(x(\theta), y(\theta)\big) = \big(r(\frac{\pi}{3} - \frac{\sqrt{3}}{2}), \frac{r}{2}\big)$ and $\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$. Therefore, when $\theta = \frac{\pi}{3}$, the tangent line is

$$y - \frac{r}{2} = \sqrt{3}\Big(x - r(\frac{\pi}{3} - \frac{\sqrt{3}}{2})\Big).$$

□

(b) At what points is the tangent horizontal? When is it vertical?

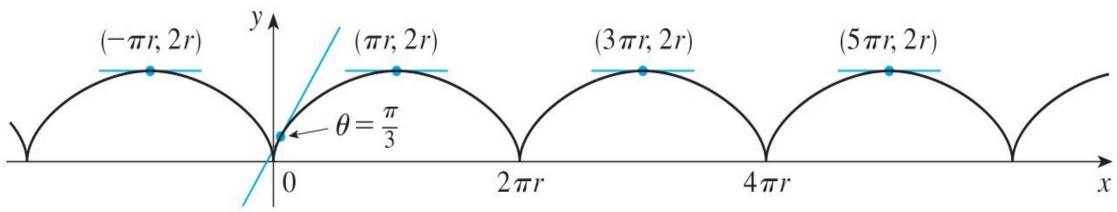
Proof. The function $\sin\theta = 0$ or $1 - \cos\theta = 0$ occurs only when $\theta = n\pi$, $n \in \mathbb{Z}$.

(i) When $n = 2m - 1$ is odd and $\theta = n\pi$, $\dfrac{dx}{d\theta} = r(1 - \cos\theta) \neq 0$. The curve has horizontal tangent lines at $\big(x((2m-1)\pi), y((2m-1)\pi)\big) = \big((2m-1)\pi r, 2r\big)$, $m \in \mathbb{Z}$.

(ii) When $n = 2m$ is even and $\theta = n\pi$, $\dfrac{dx}{d\theta} = 0$. Consider the limit

$$\lim_{\theta \to 2m\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2m\pi^+} \frac{\sin\theta}{1 - \cos\theta} \overset{L.H}{=} \lim_{\theta \to 2m\pi^+} \frac{\cos\theta}{\sin\theta} = \infty.$$

Similarly, $\displaystyle\lim_{\theta \to 2m\pi^-} \frac{dy}{dx} = -\infty$. The curve has vertical tangent line at $\big(x(2m\pi), y(2m\pi)\big) = (2m\pi r, 0)$.



□

❑ Areas

Recall that, for a function $F(x) \geq 0$, the area under the cruve $y = F(x)$ from a to b is $A = \displaystyle\int_a^b F(x)\,dx$. Suppose that a curve has the parametric equation $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, we want to calculate an area formula. Let $a = f(\alpha)$ and $b = f(\beta)$. Then the area of the region under the curve is

$$A = \int_a^b y\,dx = \int_\alpha^\beta y\,\frac{dx}{dt}\,dt = \int_\alpha^\beta g(t)f'(t)\,dt.$$

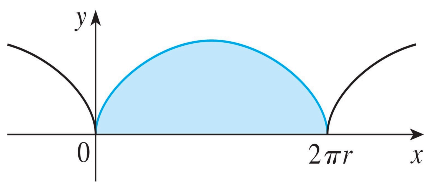
Example 10.2.3. Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

Proof.

Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, the area of one arch is

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\,d\theta \\ &= r^2(\frac{3}{2} \cdot 2\pi) = 3\pi r^2. \end{aligned}$$



□

❏ Arc Length

Let C be a curve with equation $y = F(x)$, $a \le x \le b$. If $F'(x)$ is continuous, the arc length of C is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx.$$

We want to calculate the arc length of C with parametric equation $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$.

(i) If C can be expressed as the graph of a function $y = F(x)$, it is traversed once from left to right as t increases (i.e. $\frac{dx}{dt} = f'(t) > 0$). The arc length is

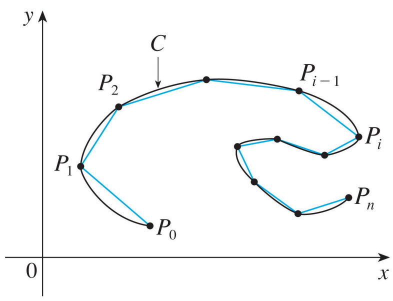
$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\left(\frac{dx}{dt}\right) dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned}$$

(ii) If C cannot be expressed in the form $y = F(x)$, we take a partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[\alpha, \beta]$. Let $P_i\big(f(t_i), g(t_i)\big)$, $i = 1, \cdots, n$, be point on the curve C. Then the length of the segment $P_{i-1}P_i$ is

$$\sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}$$

By the polygonal approximations and the mean value theorem,

$$\begin{aligned} \sum_{i=1}^{n} |P_{i-1}P_i| &= \sum_{i=1}^{n} \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2} \\ &= \sum_{i=1}^{n} \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**}\Delta t)]^2} \\ &= \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t \end{aligned}$$



The arc length of C is

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t \\ &= \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dt \end{aligned}$$

Theorem 10.2.4. *If a curve C is described by the parametric equation $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the arc length of C is*

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Note. The formula is consisent with the general formulas $L = \int 1\, ds$ and $(ds)^2 = (dx)^2 + (dy)^2$.

Example 10.2.5. Compute the circumference of a unit circle by expressing it as the parametric equation

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

Proof. We have $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$. Then the arc length is

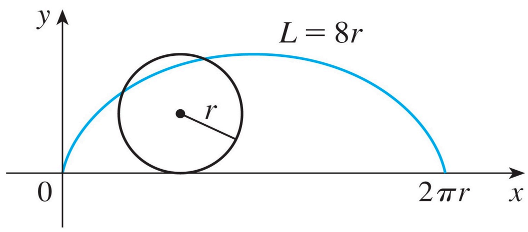
$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = 2\pi.$$

□

Example 10.2.6. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$.

Proof. We have $\frac{dx}{d\theta} = r(1 - \cos\theta)$ and $\frac{dy}{d\theta} = r\sin\theta$. The arc length of one arch is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\ &= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \\ &= r\int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta \\ &= 8r. \end{aligned}$$



□

Recall: Consider the arc length function

$$s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

which represents the arc length along C from an initial point $\big(f(\alpha), g(\alpha)\big)$ to a point $\big(f(t), g(t)\big)$. If parametric equation describes the position of a moving particle, then the "*speed*" of the particle at time t, $v(t)$, is the rate of change of distance traveled (arc length) with repect to time: $s'(t)$. By the Fundamental Theorem of Calculus, we have

$$v(t) = s'(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Example 10.2.7. The position of a particle at time t is given by the parametric equations $x = 2t + 3$, $y = 4t^2$, $t \ge 0$. Find the speed of the particle when it is at the point $(5, 4)$.

Proof. The speed of the particle at any time t is

$$v(t) = \sqrt{2^2 + (8t)^2} = 2\sqrt{1 + 16t^2}.$$

At $(5, 4)$ when $t = 1$, its speed at that point is $v(1) = 2\sqrt{17}$. □

❑ Surface Area

Recall that the surface area of the surface obtained by rotating a curve, $C : y = F(x)$ where $F(x) \geq 0$ for $a \leq x \leq b$, about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Suppose that C has the parametric equation $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ where f' and g' are continuous and $g(t) \geq 0$. Then rotating the curve C about x-axis and the surface area is

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_\alpha^\beta 2\pi g(t) \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \left(\frac{dx}{dt}\right) dt \\ &= \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \end{aligned}$$

Note. Let $s(t)$ be the arc length function. Then

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The surface area formula is

$$S = \int 2\pi y\, ds$$

Example 10.2.8. Find the surface area of a sphere of radius r.

Proof. The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leq t \leq \pi$$

about x-axis. The surface area of the sphere is

$$\begin{aligned} S &= \int_0^\pi 2\pi r\sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t \cdot r\, dt = 4\pi r^2 \end{aligned}$$

□

Homework 10.2. 9, 11, 17, 19, 23, 31, 33, 36, 38, 40, 44, 47, 50, 55, 59, 67, 73, 75

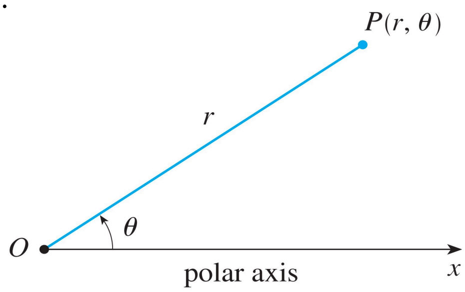
10.3 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. In the present section, we will study a coordinate system which is called the "*polar coordinate system*". The coordinate is established by the following steps

(i) We choose a point in the plane that is called the "*pole*" (or origin) and is labeled O.

(ii) We draw a ray starting at O called the "*polar axis*" . It is usually horizontal to the right and corresponds to the positive x-axis in Cartesian coordinates.

(iii) If $P \neq O$ is an point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.

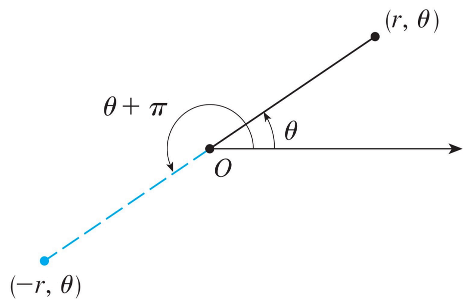


Then the point P is represented by the ordered pair (r, θ) and r, θ are called "*polar coordinates*" of P.

Note. The origin $O = (0, \theta)$ for any θ.

Now, we extend (r, θ) to the case that in which r is negative. The point $(-r, \theta)$ means the point which is opposite to (r, θ) about the origin. Hence, $(-r, \theta) = (r, \theta + \pi)$. Moreover, we can also extend (r, θ) to the case where $r \in \mathbb{R}$ (not only on $[0, 2\pi]$). We have

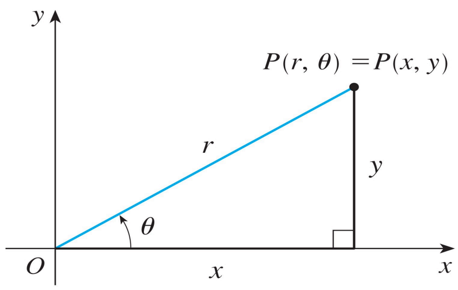
$$\begin{aligned}(r, \theta) &= (-r, \theta + \pi) = (r, \theta + 2\pi) \\ &= (-r, \theta + 3\pi) = (r, \theta + 4\pi) \\ &= \big(-r, \theta + (2k+1)\pi\big) \\ &= \big(r, \theta + 2k\pi\big) \quad \text{for every } k \in \mathbb{Z}.\end{aligned}$$



Remark. In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system, each point has infinitely many representations.

■ The connection between polar and Cartesian coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} \cos\theta = \dfrac{x}{r} \\ \sin\theta = \dfrac{y}{r} \end{cases} \implies \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



Note. The equation $\tan\theta = \frac{y}{x}$ do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leq \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinatesm, it is not good enough just to find r and θ that satisfy the above equation.

Example 10.3.1. (i) Convert $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

Proof. From the above formulas, $x = 2\cos\frac{\pi}{3} = 1$ and $y = 2\sin\frac{\pi}{3} = \sqrt{3}$. Then $(x, y) = (1, \sqrt{3})$. □

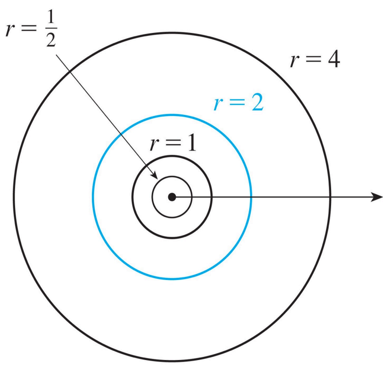
(ii) Convert $(1, -1)$ from Cartesian to polar coordinates.

Proof. Again, $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $\tan\theta = \frac{-1}{1} = -1$. Then $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Since $(1, -1)$ is a point in the fourth quadrant, $\theta = \frac{7\pi}{4}$ and $(r, \theta) = (\sqrt{2}, \frac{7\pi}{4})$. □

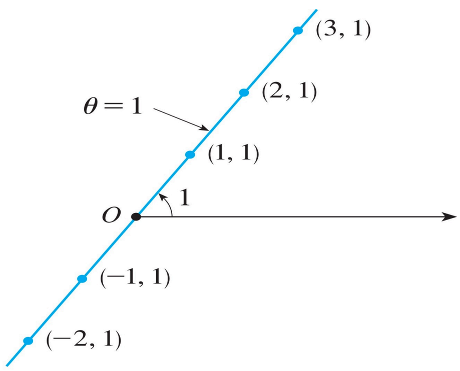
❏ Polar Curves

Definition 10.3.2. A polar curve is the graph of a polar equation, $r = f(\theta)$ or $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example 10.3.3.



$r = 2$



$\theta = 1$

Example 10.3.4. (a) Sketch the curve with polar equation $= 2\cos\theta$.

Proof.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

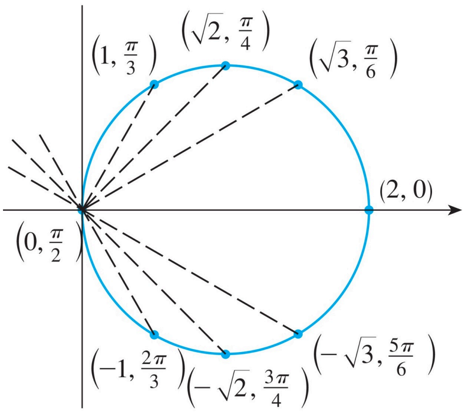


Table of values and graph of $r = 2\cos\theta$

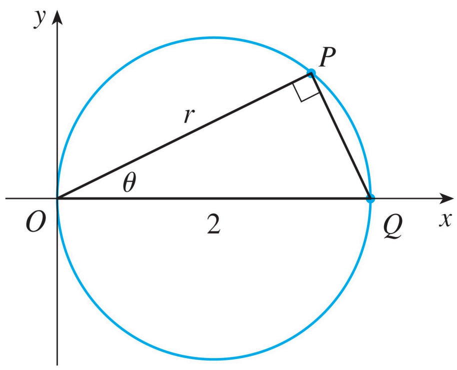
□

(b) Find a Cartesian equation for this curve.

Proof.

Consider $r = 2\cos\theta$. Then $r^2 = 2r\cos\theta$. Convert this polar equation into Cartesian equation $x^2 + y^2 = 2x$ and we have

$$(x-1)^2 + y^2 = 1.$$

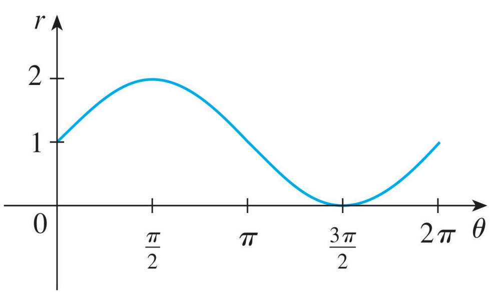


□

Example 10.3.5. Sketch the curve $r = 1 + \sin\theta$.

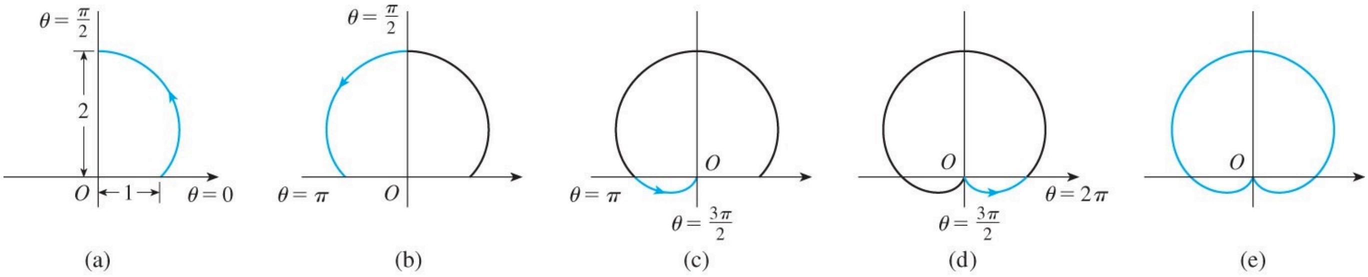
Proof.

(1) Sketch the graph of $r = 1 + \sin\theta$ in Cartesina coordinates (θ-r plane). That is a shift the curve of sine function up by one unit.



$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

(2) Sketch the polar curve as θ increases $0 \to \frac{\pi}{2} \to \pi \to \frac{3\pi}{2} \to 2\pi$.
(Cardioid)

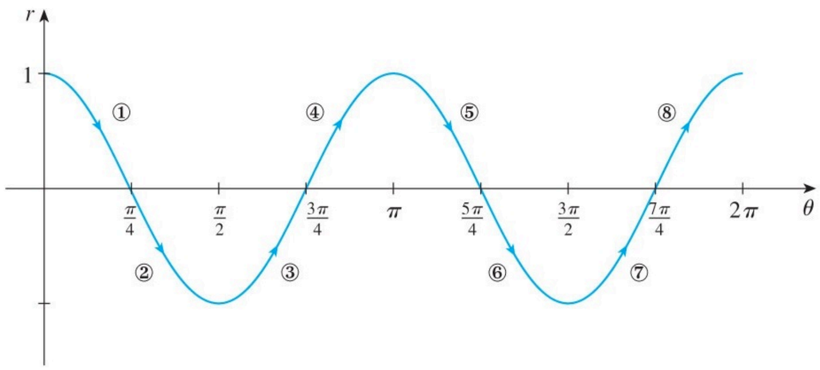


Stages in sketching the cardioid $r = 1 + \sin\theta$

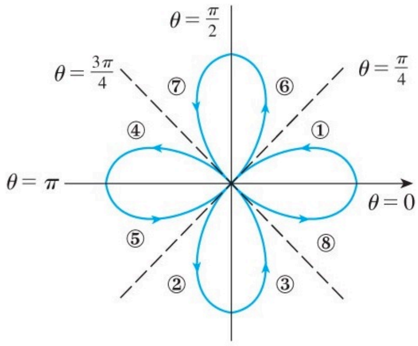
□

Example 10.3.6. Sketch the curve $r = \cos 2\theta$.

Proof.



$r = \cos 2\theta$ in Cartesian coordinates



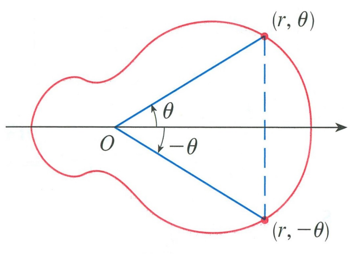
Four-leaved rose $r = \cos 2\theta$

□

■ **Symmetry**

(a)

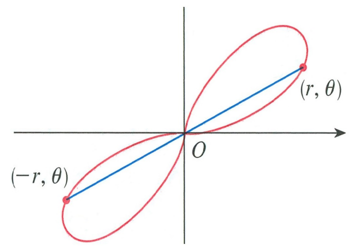
If $f(\theta) = f(-\theta)$ or $F(r, \theta) = F(r, -\theta)$, then the curve is symmetric about the polar axis.



(a)

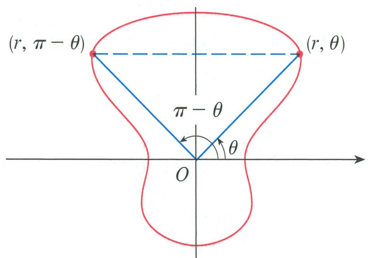
(b)

If $f(\theta) = f(\theta+\pi)$ or $F(r,\theta) = F(r,\theta+\pi)$, then the curve is symmetric about the pole.



(b)

(c) If $f(\theta) = f(\pi-\theta)$ or $F(r,\theta) = F(r,\pi-\theta)$, then the curve is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$.



(c)

❏ Graphing Polar Curves with Technology

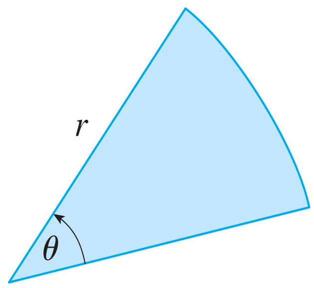
(Skip)

Homework 10.3. 2, 4, 6, 8, 11, 16, 20, 22, 25, 34, 38, 44, 51, 56, 58

10.4 Areas and Lengths in Polar Coordinates

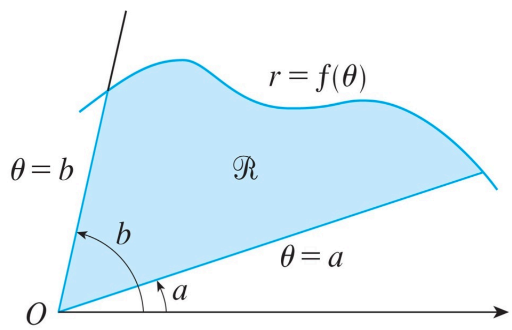
❏ Areas

We try to find the area of a region whose boundary is given by a polar equation. Let's start with an easy case that the area of an sector of a circle with radius r and central angle θ.

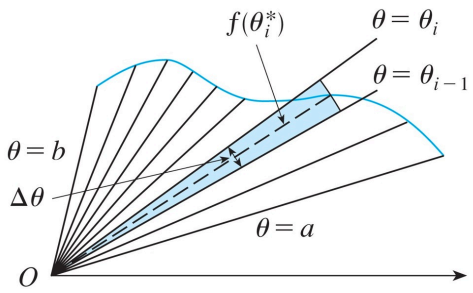


$$\text{Area} = \frac{1}{2}r^2\theta.$$

Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a < 2\pi$. We will use the approximating sectors to estimate the area of $\mathcal{R}$.



Let $P = \{\theta_0, \theta_1, \cdots, \theta_n\}$ be a partition of $[a, b]$ with $\triangle\theta = \theta_i - \theta_{i-1}$. The region $\mathcal{R}$ is divided into n subregions by the rays $\theta = \theta_i$. The area of each subregion denotes $\triangle A_i$. Choose a sample point $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Then



$$\triangle A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \triangle\theta.$$

Then an approximation to the total area A of $\mathcal{R}$ is

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \triangle\theta$$

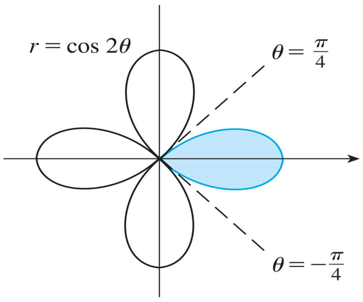
Taking $n \to \infty$, then

$$\begin{aligned} \text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \triangle\theta = \frac{1}{2}\int_a^b [f(\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_a^b r^2\, d\theta \quad \text{where } r = f(\theta). \end{aligned}$$

Note. The area formula is to compute the area of the region whose area enclosed by a polar curve and two straight lines connecting the origin and their intersections of the polar curve.

Example 10.4.1. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

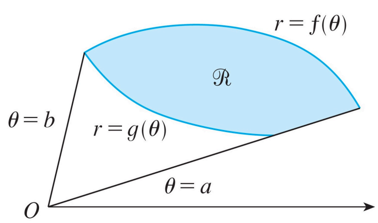
Proof.

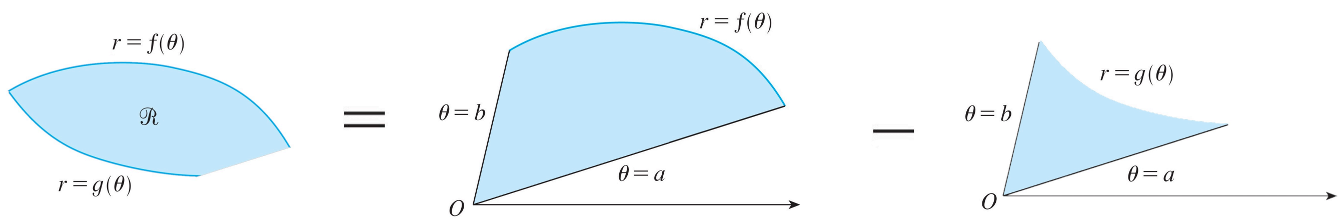


$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{\pi}{8}. \end{aligned}$$

□

■ **Region enclosed by two polar curves**



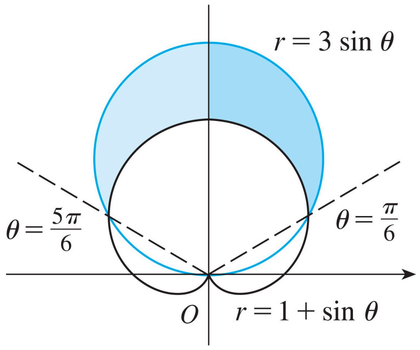


The area of $\mathcal{R}$ is

$$\int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2\,d\theta = \frac{1}{2}\int_a^b f^2(\theta) - g^2(\theta)\,d\theta.$$

Example 10.4.2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Proof.



The points of intersection of the two polar curves are obtained by solving $3\sin\theta = 1 + \sin\theta$ and hence $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The area of the region is

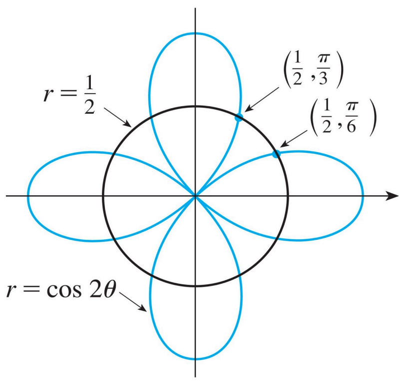
$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(3\sin\theta)^2 - \frac{1}{2}(1+\sin\theta)^2\,d\theta = \pi.$$

□

Note. The origin O is also a point of intersection of the two polar curves. But it cannot be obtained by solving the equation $3\sin\theta = 1 + \sin\theta$ since $r = 3\sin\theta = 0$ when $\theta = 0$ and π and $r = 1 + \sin\theta = 0$ when $\theta = \dfrac{3\pi}{2}$.

Remark. It is usually difficult to find the points of intersection of two polar curves since a single point may have many representation in polar coordinates. Suppose we want to find the points of intersection by solving $f_1(\theta) = r = f_2(\theta)$. The point of intersection has polar coordinate $\big(f_1(\theta_1), \theta_1\big) = \big(f_2(\theta_2), \theta_2\big)$. But, in general, the angles θ_1 may not equal θ_2.

Example 10.4.3. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \dfrac{1}{2}$.

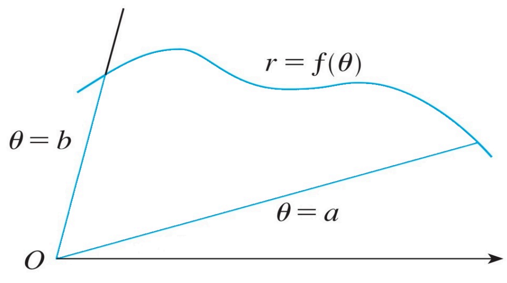


Proof. Let $\cos 2\theta = \dfrac{1}{2}$. Then $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$. The points of intersection are $(\frac{1}{2}, \frac{\pi}{6})$, $(\frac{1}{2}, \frac{5\pi}{6})$, $(\frac{1}{2}, \frac{7\pi}{6})$ and $(\frac{1}{2}, \frac{11\pi}{6})$.

However, the points $(\frac{1}{2}, \frac{\pi}{3})$, $(\frac{1}{2}, \frac{2\pi}{3})$, $(\frac{1}{2}, \frac{4\pi}{3})$ and $(\frac{1}{2}, \frac{5\pi}{3})$ are also points of intersection of the two polar curves. Those points can be found by solving $\cos 2\theta = -\dfrac{1}{2}$. □

❏ Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$, we regard θ as the parameter if we write the polar equation of the curve as

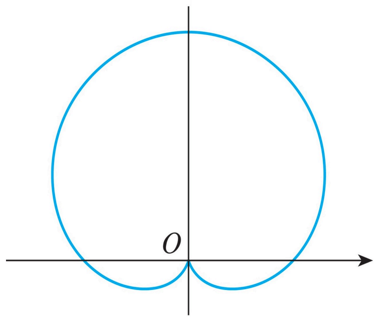


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta \\ \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta \end{cases}$$

The arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 10.4.4. Find the length of the cardioid $r = 1 + \sin\theta$.



$r = 1 + \sin\,\theta$

Proof. The arc length fo the cardioid is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\cos\theta)^2 + (1+\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2 - 2\sin\theta}} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}}\, d\theta - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}}\, d\theta \\ &= 8. \end{aligned}$$

□

❑ Tangents

We want to use the techniques of finding the tangent lines of parametric curves to obtain the tangents of polar curves. Consider the curve with polar equation $r = f(\theta)$. Then

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases} \Longrightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

(i) **Horizontal tangent line:** When $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$, the polar curve has a horizontal tangent line.

(ii) **Vertical tangent line:** When $\dfrac{dy}{d\theta} \neq 0$ and $\dfrac{dx}{d\theta} = 0$, the polar curve has a vertical tangent line.
(Special case: $\dfrac{dy}{d\theta} = 0 = \dfrac{dx}{d\theta}$, we should further consider the limit $\lim\limits_{\theta\to\theta_0} \dfrac{dy/d\theta}{dx/\theta}$).

(iii) **Tangent line at pole:**

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta}{\frac{dr}{d\theta}\cos\theta} = \tan\theta, \qquad \text{if } \frac{dr}{d\theta} \neq 0.$$

Example 10.4.5. The cardioid has polar equation $r = 1 + \sin\theta$.

(a) Find the slope of the tangent line when $\theta = \dfrac{\pi}{3}$.

Proof. Consider $\dfrac{dr}{d\theta} = \cos\theta$. Then

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

Hence, the slope of the tangent line when $\theta = \dfrac{\pi}{3}$ is $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -1$. □

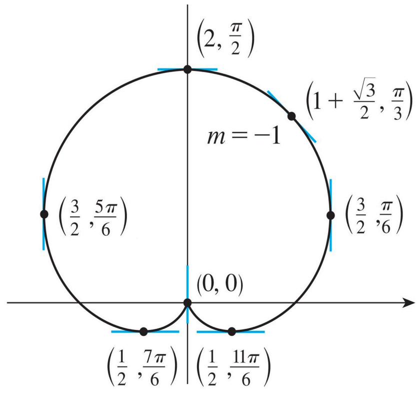
(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Proof. We have

$$\frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \Rightarrow \quad \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

The curve has horizontal tangent lines at $(2, \pi/2)$, $(1/2, 7\pi/6)$, $(1/2, 11\pi/6)$ and has vertical tangent lines at $(3/2, \pi/6)$, $(3/2, 5\pi/6)$.



Tangent lines for $r = 1 + \sin\theta$

For $\theta = \dfrac{3\pi}{2}$, $\dfrac{dy}{d\theta} = \dfrac{dx}{d\theta} = 0$. Consider

$$\lim_{\theta\to(3\pi/2)^-} \frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-} \frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-} \frac{\cos\theta}{1+\sin\theta}\right) \overset{L.H.}{=} -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-} \frac{-\sin\theta}{\cos\theta} = \infty.$$

Similarly, $\displaystyle\lim_{\theta\to(3\pi/2)^+} \frac{dy}{dx} = -\infty$. Hence, the cardioid has a vertical tangent line at $(0, 3\pi/2)$.

□

Homework 10.4. 4, 6, 11, 17, 21, 22, 24, 27, 31, 34, 40, 45, 49, 51, 53, 65, 68, 72

10.5 Conic Sections

(Skip)

Homework 10.5.

10.6 Conic Sections in Polar Coordinates

(Skip)

Homework 10.6.

Chapter 11

Sequences, Series, and Power Series

We have learned topics which are related the infinite sequence. For example,

- Zeno's paradoxes
- Decimal representation of numbers
- Newton's idea of representing functions as sums of infinite series
- Integrating a function by first expressing it as a series and then integrating each term of the series. (e.g $f(x) = e^{-x^2}$)

11.1 Sequences

A *sequence* (of numbers) can be thought of as a list of numbers written in a definite order
It can be regarded as a list of values of a function defined on $\mathbb{N}$.

$$\begin{array}{cccccccc} & \mathbb{N} & 1 & 2 & 3 & 4 & \ldots \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \mathbb{R} & f(1) & f(2) & f(3) & f(4) & \ldots \end{array}$$

We usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

Note. From now on, we say "a sequence" instead of "a sequence of numbers" for the convenience.

Definition 11.1.1. An (infinite) *sequence* (of numbers), denoted by $\{a_n\}$ $\left(\text{or } \{a_n\}_{n=1}^{\infty}\right)$, is a function whose domain is a set of positive numbers. The functional values $a_1, a_2, \cdots, a_n, \cdots$ are the "*terms*" of the sequence, and the term a_n is called the "*n*th term" of the sequence.

Remark.

(i) In the textbook, a sequence can be thought of as a list of numbers written in a definite order

$$\begin{array}{cccccc} a_1, & a_2, & a_3, & \cdots & a_n, & \cdots \\ \uparrow & \uparrow & \uparrow & \cdots & \uparrow & \cdots \\ \substack{1st\\term} & \substack{2nd\\term} & \substack{3rd\\term} & \cdots & \substack{nth\\term} & \cdots \end{array}$$

(ii) To distinguish the notation of a set with the one of a sequence, we use $\left\{a_n \mid n \in \mathbb{N}\right\}$ to represent a set and $\{a_n\}$ for a sequence.

Example 11.1.2.

(1) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$, $\rightsquigarrow$ $a_n = \dfrac{n}{n+1}$ $\quad\Rightarrow\quad$ $\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots\right\}$.

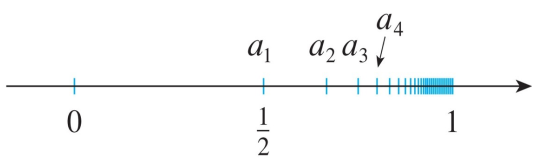
(2) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$, $\rightsquigarrow$ $a_n = \cos\dfrac{n\pi}{6}, n \geq 0$ $\Rightarrow$ $\left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots\right\}$.

(3) (Fibonacci sequence)
$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ $\quad\Rightarrow\quad$ $\left\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\right\}$.

■ Visualization of sequence

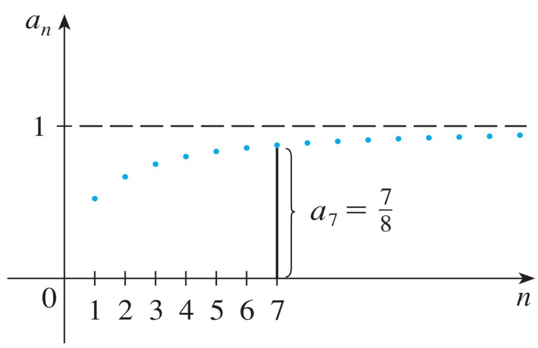
(i) Plot all terms of a sequence on number line.

Example: $a_n = \dfrac{n}{n+1}$.



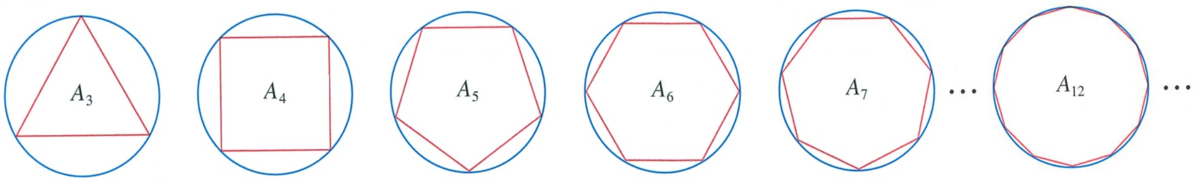
(ii) Regard a sequence as a function. $f : \mathbb{N} \mapsto \mathbb{R}$ by $a_n = f(n)$. Plot the graph of f. $(1, a_1)$, $(2, a_2), \ldots,$ (n, a_n).

Example: $f(n) = \dfrac{n}{n+1}$



Observation: From the above figures, the functional values a_n approaches as close to 1 as possible when n becomes large.

Note. People studied the limit of sequences over thousands of years. For example, to compute the area of a circle.



Question: Does A_n approach a number as n becomes large?

❑ Limit and Convergence

■ **Intuitive Definition:** Let $\{a_n\}$ be a sequence. We say that "*the limit of* $\{a_n\}$ *exists*" if there exists a real number $L \in \mathbb{R}$ such that we can make the term a_n as close to L as we like by taking n sufficiently large. Denote

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \quad \text{as} \quad n \to \infty$$

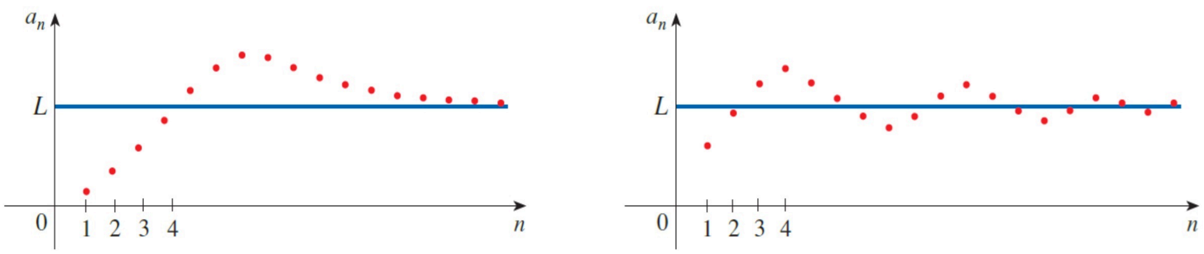
Definition 11.1.3. Let $\{a_n\}$ be a sequence.

(a) We say that $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

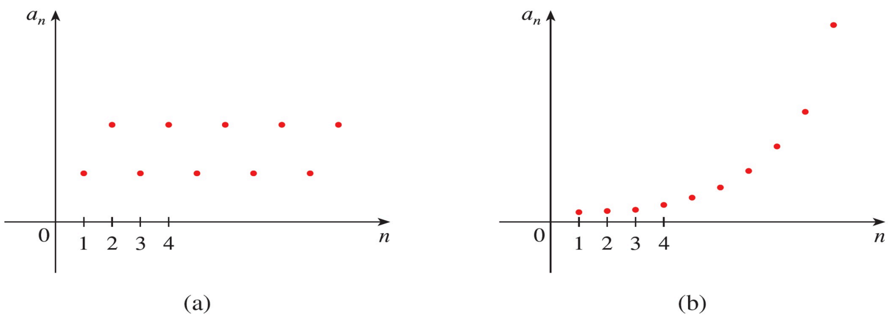
if we can make the term a_n as close to L as we like by taking n sufficiently large.

(b) If $\{a_n\}$ has a limit (i.e. $\lim_{n\to\infty} a_n$ exists), we say that the sequence converges. Otherwise, we say that the sequence diverges.



Graphs of two sequence with $\lim_{n\to\infty} a_n = L$

(c) We say that the sequence $\{a_n\}$ diverges to ∞ $(-\infty)$ and denote $\lim_{n\to\infty} a_n = \infty$ $(-\infty)$ if we can make the term a_n as (negatively) large as we like by taking n sufficiently large.



Graphs of two divergent sequences

Example 11.1.4. (1) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$, $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 0$.

(2) $\left\{1, -1, 1, -1, \ldots\right\}$, $a_n = (-1)^{n-1}$. Then $\lim_{n\to\infty} a_n$ does not exist (DNE).

(3) $\{n\}_{n=1}^{\infty}$ where $a_n = n$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$.

Definition 11.1.5. (Precise) Let $\{a_n\}$ be a sequence.

(a) We say that "*the limit of* $\{a_n\}$ *exists*" if there exists a real number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a corresponding integer N such that

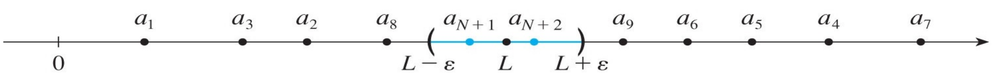
$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N.$$

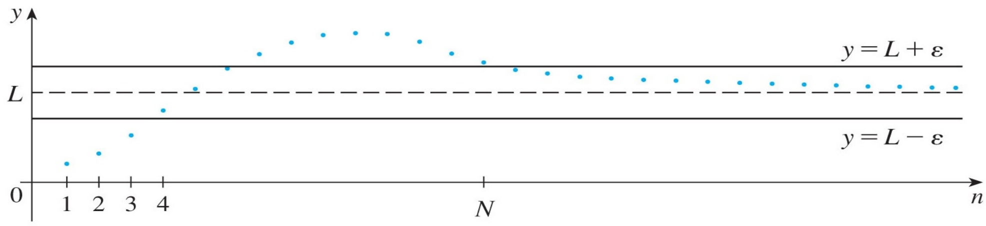
The value L is called "*the limit of* $\{a_n\}$" and we write

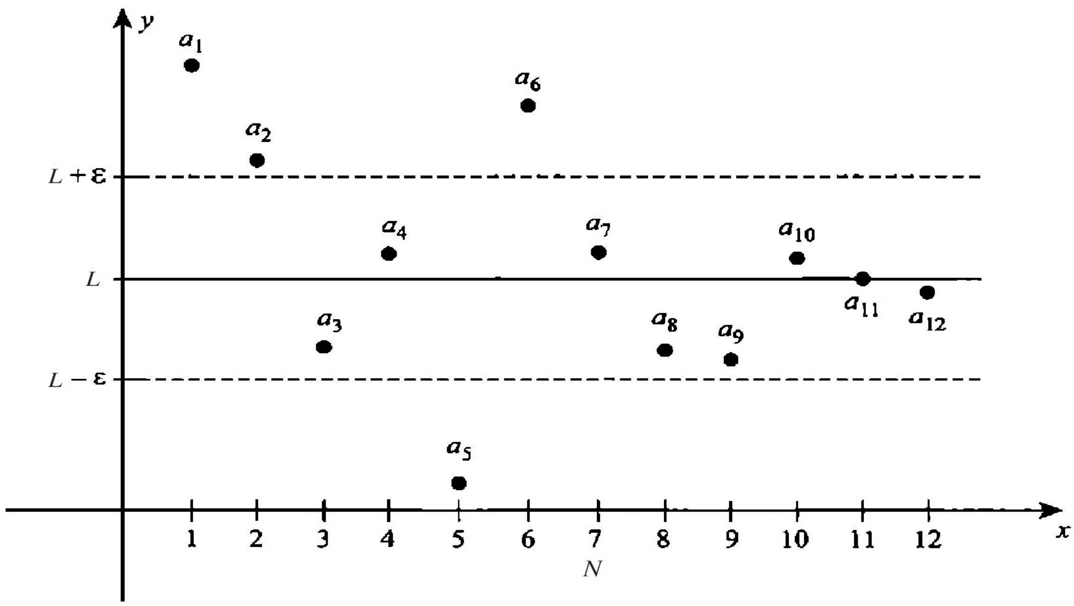
$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \quad \text{as} \quad n \to \infty$$







(b) If $\{a_n\}$ has a limit L. (i.e. $\lim\limits_{n\to\infty} a_n = L$), we say that the sequence "*convergs to L*". Otherwise, we say that the sequence "*diverges*".

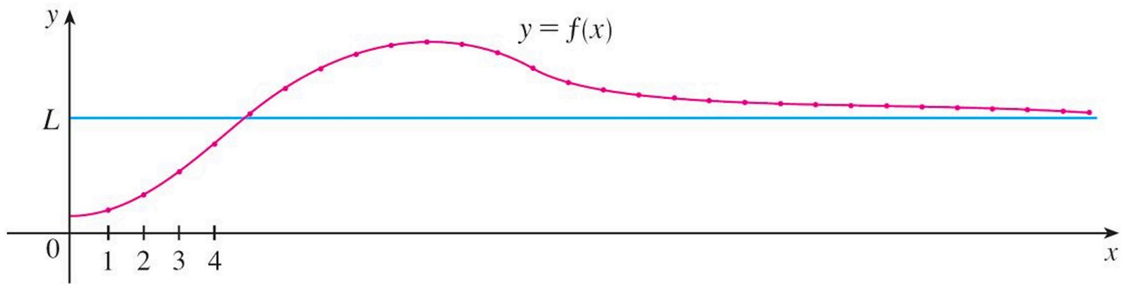
(c) $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positive number M there exists an integer N such that

$$\text{if } n > N \quad \text{then} \quad a_n > M.$$

■ $\lim\limits_{n\to\infty} a_n$ v.s. $\lim\limits_{x\to\infty} f(x)$

The difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 11.1.6. *If $f : [1,\infty) \to \mathbb{R}$ is a function and $a_n = f(n)$ for $n = 1,2,3,\cdots$. Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then $\lim\limits_{n\to\infty} a_n = L$.*



Remark. (1) This theorem also holds if the limit $L = \pm\infty$.

(2) The converse of the theorem is false.

(i)

$$\lim_{n\to\infty} a_n = L \quad \not\Longrightarrow \quad \lim_{x\to\infty} f(x) = L.$$

For example $f(x) = \sin(\pi x)$. Then $a_n = \sin(n\pi) = 0$ and $\lim_{n\to\infty} a_n = 0$. But $\lim_{x\to\infty} f(x) = 0$.

(ii)

$$\lim_{x\to\infty} f(x) \quad \text{DNE} \quad \not\Longrightarrow \quad \lim_{n\to\infty} a_n \quad \text{DNE}.$$

Example 11.1.7. Prove that $\lim_{n\to\infty} \frac{1}{n^r} = 0$ when $r > 0$.

Proof. Let $f(x) = \frac{1}{x^r}$. Then $f(n) = \frac{1}{n^r}$. Since $\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{for } r > 0.$$

□

Example 11.1.8. Find $\lim_{n\to\infty} \frac{\ln n}{n}$.

Proof. Let $f(x) = \frac{\ln x}{x}$. Then $f(n) = \frac{\ln n}{n}$. Since $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, we have

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

□

❑ Limit Laws (for Sequences)

Theorem 11.1.9. *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

(1) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.

(2) $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.

(3) $\lim_{n\to\infty} c = c$.

(4) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$.

(5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *if* $\lim_{n\to\infty} b_n \neq 0$.

(6) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ *if* $p > 0$ *and* $a_n > 0$.

Remark. The hypothesis that "$\{a_n\}$ and $\{b_n\}$ are convergent" is important.

Example 11.1.10.

$$\lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} \left(\frac{n}{n+1} \times \frac{\frac{1}{n}}{\frac{1}{n}}\right) = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = 1.$$

(Wrong process)

$$\lim_{n\to\infty} \frac{n}{n+1} \not= \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty}(n+1)} = \frac{\infty}{\infty} = ?.$$

Example 11.1.11.

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2}+\frac{1}{n}}} = \infty$$

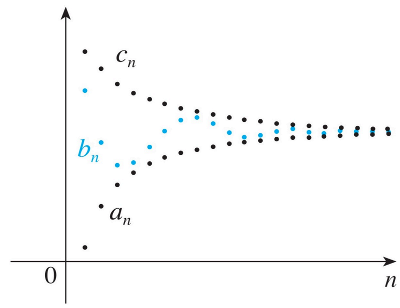
because the numerator is 1 and the denominator approaches 0.

❑ Squeeze Theorem (for sequences)

Theorem 11.1.12. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences. If there exists $n_0 \in \mathbb{N}$ such that $a_n \le b_n \le c_n$ for every $n \ge n_0$. Suppose that*

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $\lim_{n\to\infty} b_n = L$.



The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

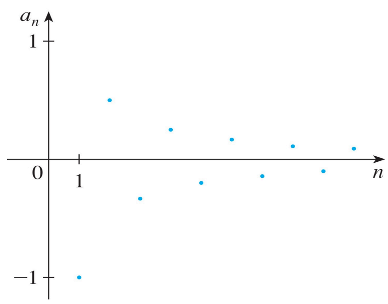
Theorem 11.1.13. $\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

Example 11.1.14. Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof.

Since $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$, we have

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$



□

Example 11.1.15. Discuss the convergence of the sequence $a_n = \dfrac{n!}{n^n}$.

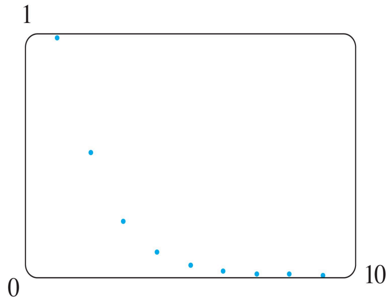
Proof. Observe that

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \left(\frac{1}{n}\right)\underbrace{\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right)}_{<1} < \frac{1}{n}.$$

Define $r_n = 0$ and $s_n = \frac{1}{n}$ for $n = 1, 2, \cdots$. Then

$$r_n \leq a_n \leq s_n \qquad \text{for every } \ n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$, by the Squeeze Theorem the limit $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = 0$.



□

Example 11.1.16. For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Proof. For $r > 0$, consider the exponential function $f(x) = r^x$, $r > 0$,

$$\lim_{x\to\infty} f(x) = \begin{cases} 0 & \text{if } 0 < r < 1 \quad \text{(convergent)} \\ 1 & \text{if } r = 1 \quad \text{(convergent)} \\ \infty & \text{if } r > 1 \quad \text{(divergent)} \end{cases}$$

Consider $r \leq 0$.

(i) For $r = 0$, $\lim_{n\to\infty} r^n = 0$ (**convergent**)

(ii) For $-1 < r < 0$, we have $0 < |r| < 1$ and

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0.$$

Hence, $\lim_{n\to\infty} r^n = 0$ (**convergent**).

(iii) For $r = -1$, $a_n = (-1)^n$, $\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 \cdots\}$ is an oscillatory sequence and hence it is **divergent**.

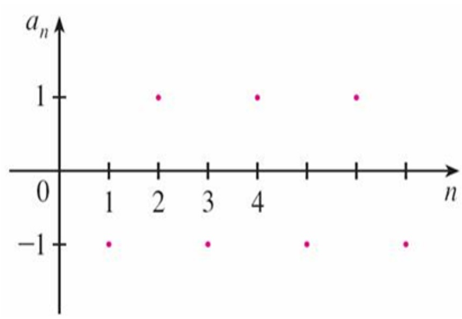
(iv) For $r < -1$,

$$a_n = (-1)^n |r|^n = \begin{cases} -|r|^n < -1 & \text{if } n \text{ is odd} \\ |r|^n > 1 & \text{if } n \text{ is even} \end{cases}$$

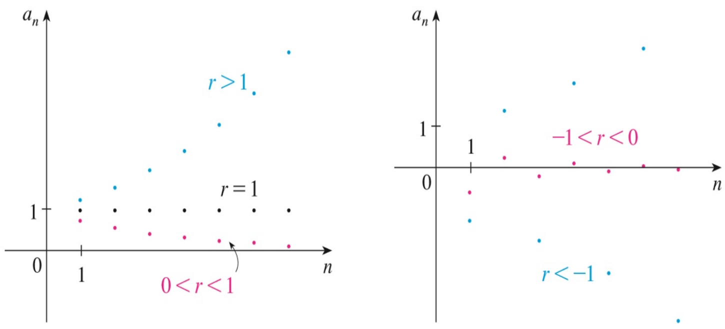
Hence, we cannot find a number L such that a_n is close to L within $\dfrac{1}{2}$ for every n. Thus, the sequence is **divergent**.

Conclusion: The sequence $\{r^n\}_{n=1}^{\infty}$ converges when $-1 < r \le 1$ and diverges when $r \le -1$ or $r > 1$. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$



The sequence $\{(-1)^n\}$



The sequence $a_n = r^n$

□

■ Continuous Functions

Theorem 11.1.17. *If* $\lim_{n\to\infty} a_n = L$ *and the function is continuous at L, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example 11.1.18. Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$.

Proof. Since the function $\sin x$ is continuous at 0 and $\lim_{n\to\infty} \frac{\pi}{n} = 0$, we have

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0.$$

□

❑ Monotonic Sequence and Bounded Sequences

■ Monotonic Sequences

Definition 11.1.19. (1) A sequence $\{a_n\}$ is called "*increasing*" ("*decreasing*") if

$$a_n < a_{n+1} \quad (a_n > a_{n+1})$$

for all $n \ge 1$.

(2) A sequence $\{a_n\}$ is "*monotonic*" if it is either increasing or decreasing.

Example 11.1.20. Show that $\left\{\frac{3}{n+5}\right\}$ is decreasing.

Proof. Since

$$a_{n+1} - a_n = \frac{3}{(n+1)+5} - \frac{3}{n+5} < 0$$

for all $n \geq 1$, the sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing. □

Example 11.1.21. Show that $\left\{\frac{n}{n^2+1}\right\}$ is decreasing.

Proof.
(**Method 1:**)

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} < 0.$$

(**Method 2:**) Let $f(x) = \frac{x}{x^2+1}$. Then $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x^2 > 1$. Thus, $f(x)$ is decreasing. Then

$$a_{n+1} = f(n+1) < f(n) = a_n.$$

□

■ Bounded Sequences

Definition 11.1.22. (1) A sequence $\{a_n\}$ is "*bounded above*" ("*bounded below*") if there exists a number M such that

$$a_n \leq M \quad (a_n \geq M)$$

for all $n \geq 1$.

(2) A sequence is "*bounded*" if it is both bounded above and below.

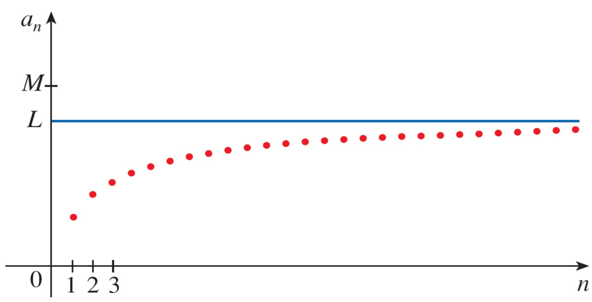
Example 11.1.23. (1) $\{n\}_{n=1}^{\infty}$ is bounded below but not above.

(2) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is bounded (both above and below).

■ Boundedness, Monotonicity and Convergence

Note. Not every bounded sequence is convergent. For example, $a_n = (-1)^n$. Then the sequence $\{a_n\}$ is bounded and divergent.

Observation: If $\{a_n\}$ is monotonic and boudned, then the terms are forced to crowd together and approach some number L.



Theorem 11.1.24. *(Monotonic Sequence Theorem) Every bounded and monotonic sequence is convergent.*

Example 11.1.25. Let $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n > 1$. Then $a_2 = 4, a_3 = 5, \cdots$.

(i)

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{2}(a_n + 6) - \frac{1}{2}(a_{n-1} + 6) = \frac{1}{2}(a_n - a_{n-1}) \\ &= \frac{1}{2}\Big[\frac{1}{2}(a_{n-1} - a_{n-2})\Big] = \frac{1}{4}(a_{n-1} - a_{n-2}) \\ &= \cdots \\ &= \frac{1}{2^{n-1}}(a_2 - a_1) = \frac{1}{2^{n-2}} > 0. \end{aligned}$$

Then $\{a_n\}_{n=1}^{\infty}$ is increasing.

(ii) **Claim:** $a_n < 6$ for all $n \in \mathbb{N}$.
Proof of the claim: For $n = 1$, $a_1 = 2 < 6$.
If $a_k < 6$ for $n = k$, then $a_{k+1} = \frac{1}{2}(a_k + 6) < \frac{1}{2}(6 + 6) = 6$. By the mathematical induction, $a_n < 6$ for all $n \in \mathbb{N}$. Hence, $\{a_n\}$ is bounded above.

Since $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above, it is convergent. In fact, $\lim_{n\to\infty} a_n = 6$.

Remark. To determine the convergence of a sequence $\{a_n\}_{n=1}^{\infty}$, it suffices to consider the convergenc of its "tails" $\{a_n\}_{n=n_0}^{\infty}$ for some $n_0 \in \mathbb{N}$. Hence, in general, we usually concern the above theorem on the subsequence $\{a_n\}_{n=n_0}^{\infty}$.

Homework 11.1. 14, 21, 26, 29, 34, 39, 42, 45, 50, 54, 56, 60, 70, 74, 75, 79, 84, 87

11.2 Series

Motivation:

(i) Every real number can be expressed as a digital number. Especially, most numbers have the expression of infinite deciamls. For example,

$$\begin{aligned} \pi &= 3.1415926\ldots \\ &= \underbrace{3}_{a_1} + \underbrace{\frac{1}{10}}_{a_2} + \underbrace{\frac{4}{10^2}}_{a_3} + \underbrace{\frac{1}{10^3}}_{a_4} + \underbrace{\frac{5}{10^4}}_{a_5} + \underbrace{\frac{9}{10^5}}_{a_6} + \underbrace{\frac{2}{10^6}}_{a_7} + \underbrace{\frac{6}{10^7}}_{a_8} + \cdots \\ &= a_1 + a_2 + a_3 + \cdots \end{aligned}$$

(ii)

$$\begin{aligned} & 1 + 2\left(\tfrac{2}{3}\right) + 2\left(\tfrac{2}{3}\right)^2 + 2\left(\tfrac{2}{3}\right)^3 + \cdots \\ = \ & a_0 + a_1 + a_2 + a_3 + \cdots \\ = \ & \text{sum of an infinite sequence} \end{aligned}$$

Heuristically, for a given sequence $\{a_n\}$, we want to consider whether the sum of all terms makes sense.

Definition 11.2.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We call the sum of the infinite sequence $\{a_n\}$ an "(*infinite*) *series*" and is denoted by "$\sum_{n=1}^{\infty} a_n$" or "$\sum a_n$".

Note. In mathematics, adding infinite numbers is not doable. Hence, the sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

does not make sense.

Question: How to define the sum of infinite numbers (terms)?

Consider the "*partial sum*" of $\{a_n\}$

$$\begin{aligned}
s_1 &= a_1 && \text{(first partial sum)}\\
s_2 &= a_1 + a_2 && \text{(second partial sum)}\\
s_3 &= a_1 + a_2 + a_3 && \text{(third partial sum)}\\
&\vdots\\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k && \text{(nth partial sum).}
\end{aligned}$$

Then, for every $n \in \mathbb{N}$, s_n is well-defined and $\{s_n\}_{n=1}^{\infty}$ forms a new sequence. Suppose that sum of the infinite terms of $\{a_n\}$ is well-defined. It is supposed to be the limit of $\{s_n\}$.

Definition 11.2.2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and denote its nth partial sum

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

(1) We call the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ an "infinite series" and denote

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

(2) If the sequence $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series is called "*convergent*" and write $\sum_{n=1}^{\infty} a_n = s$. The number s is called the "*sum*" of the series.

(3) If the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent, then we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 11.2.3.

(1) Let $a_n = \frac{1}{2^n}$. Then $s_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

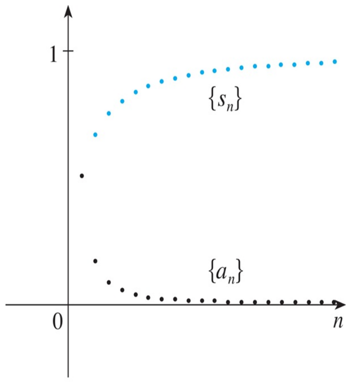
n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Hence,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1.$$

(2) (Telescoping series) Let $a_n = \frac{1}{n(n+1)}$. Then the partial sum

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$



Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

(3) Let $a_n = (-1)^n$. Then the partial sum

$$\begin{aligned} s_{2n} &= (-1) + 1 + (-1) + 1 + \cdots + 1 = 0 \\ s_{2n+1} &= (-1) + 1 + (-1) + 1 + \cdots + 1 + (-1) = -1 \end{aligned}$$

Hence, the limit $\lim_{n\to\infty} s_n$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n$ is divergent.

■ Geometric Series

A geometric series with ratio r is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \qquad a \neq 0$$

Note: The series starts with the 0th term rather than the 1st term.

(1) For $r = 1$, $s_n = \underbrace{a + a + \cdots + a}_{n} = na \to \pm\infty$ as $n \to \infty$. Hence $\lim_{n\to\infty} s_n$ is divergent.

(2) For $r \neq 1$,

$$\begin{aligned} s_n &= a + ar + \cdots + ar^n \\ rs_n &= \phantom{a + {}} ar + \cdots + ar^n + ar^{n+1} \end{aligned}$$

We have $(r-1)s_n = a(r^{n+1}-1)$ and hence

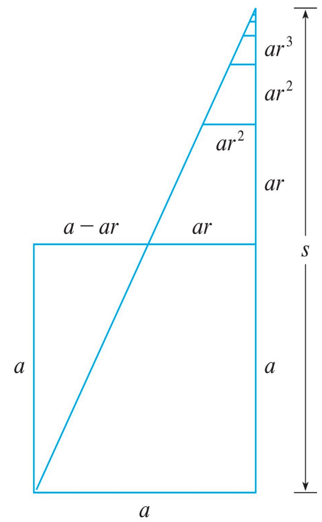
$$s_n = \frac{a(r^{n+1}-1)}{r-1}.$$

Consider the limit $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(r^{n+1}-1)}{r-1}$ provided $r \neq 1$.

(i) If $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$. Hence, $\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n = \frac{a}{1-r}$.

(ii) If $|r| > 1$, then $\lim_{n\to\infty} r^{n+1}$ diverges. Hence, $\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n$ diverges.

(iii) If $r = -1$, $s_n = a-a+a-a+\cdots+(-1)^{n-1}a = \begin{cases} 0 & n \text{ is even} \\ a & n \text{ is odd.} \end{cases}$ Hence, $\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n$ diverges.



Conclusion: The geometric series $\sum_{n=0}^{\infty} ar^n$, $a \neq 0$

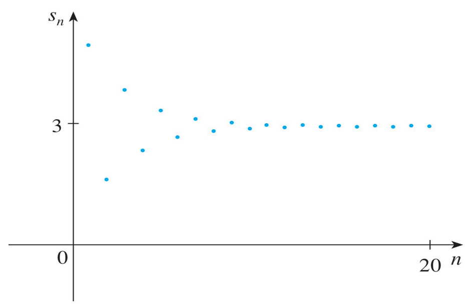
(i) converges if $|r| < 1$ and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

(ii) diverges if $|r| \geq 1$.

In the figure, $\frac{s}{a} = \frac{a}{a-ar}$. Then $s = \frac{a}{1-r}$.

Example 11.2.4.

(1) Evaluate $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$.



Proof. For the series, the first term $a = 5$ and the ratio $r = -\frac{2}{3}$. Since $|r| = |-\frac{2}{3}| = \frac{2}{3} < 1$, the series is convergent and

$$\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n = \frac{5}{1-(-\frac{2}{3})} = 3.$$

□

(2) Evaluate $\sum_{n=0}^{\infty} 2\left(\frac{5}{3}\right)^n$.

Proof. Since the ratio of the geometric series is $r = \frac{5}{3} > 1$. The series is divergent. □

(3) Write $0.1232323\cdots = 0.1\overline{23}$ as a ratio of integers.

Proof.

$$\begin{aligned} 0.1\overline{23} &= 0.1 + 0.023 + 0.00023 + 0.0000023 + \cdots \\ &= \frac{1}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \cdots \\ &= \frac{1}{10} + \frac{23}{10^3}\Big(\underbrace{1}_{a} + \underbrace{\frac{1}{10^2}}_{r} + \frac{1}{10^4} + \cdots\Big) \\ &= \frac{1}{10} + \frac{23}{10^3}\cdot\frac{1}{1-\frac{1}{10^2}} = \frac{122}{99}. \end{aligned}$$

□

(4) Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

Proof.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

The first term of the series is $a = 1$ and the ratio $r = x$ with $|r| = |x| < 1$. Hence, the series is convergent and $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$. □

■ **Harmonic Series**

A harmonic series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

We claim that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. It suffices to show that for any number $M > 0$, $\sum_{n=1}^{\infty} \frac{1}{n} > M$. Consider

$$\begin{aligned} \sum_{n=1}^{2^k} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{8} + \cdots + \frac{1}{16} + \cdots + \frac{1}{2^k} \\ &> 1 + \frac{1}{2} + \Big(\frac{1}{4} + \frac{1}{4}\Big) + \Big(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\Big) + \Big(\underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{8\text{ terms}}\Big) + \cdots \\ &\quad + \Big(\underbrace{\frac{1}{2^k} + \cdots + \frac{1}{2^k}}_{2^{k-1}\text{ terms}}\Big) \\ &> 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k\text{ terms}} \\ &= 1 + \frac{k}{2} \end{aligned}$$

Choose $k > 2M$. Then $\displaystyle\sum_{n=1}^{2^k} \frac{1}{n} > M$. Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=1}^{2^k} \frac{1}{n} > M$. Since M is an arbitrary positive number,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

❑ Test for Divergence

For most series, it is difficult to find their limits even if they have nice patterns. Therefore, we usually don't expect to compute the exact limit of a convergent series. Instead of this, we want to study some tests for convergence or divergence of a series and estimate their limits if they converge.

Theorem 11.2.5. *If the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Proof. Consider $a_n = s_n - s_{n-1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

The second equality holds since the sequence $\{s_n\}_{n=1}^{\infty}$ converges. □

Note. With any series $\sum a_n$ we associate two *sequences:* the sequence s_n of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 11.2.5 asserts, the limit of the sequence $\{a_n\}$ is 0.

Remark. The converse of Theorem 11.2.5 is false. That is, even if $\displaystyle\lim_{n\to\infty} a_n = 0$, it cannot imply that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. That is,

$$\sum_{n=1}^{\infty} a_n \ \text{(or } \lim_{n\to\infty} s_n\text{) converges} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \lim_{n\to\infty} a_n = 0$$

For example, $a_n = \dfrac{1}{n}$. Then $a_n \to 0$ as $n \to \infty$ but $\displaystyle\sum_{n=1}^{\infty} a_n = \infty$.

■ Test for Divergence

Theorem 11.2.6. *(Test for Divergence) If* $\displaystyle\lim_{n\to\infty} a_n$ *does not converge to* 0 *(either* $\displaystyle\lim_{n\to\infty} a_n$ *DNE or* $\displaystyle\lim_{n\to\infty} a_n = L \neq 0$*), then the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 11.2.7. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is convergent or divergent.

Proof. Consider the limit

$$\lim_{n\to\infty}\frac{n^2}{5n^2+4}=\lim_{n\to\infty}\frac{1}{5+\frac{4}{n^2}}=\frac{1}{5}\neq 0.$$

By the test for divergence, the series $\sum_{n=1}^{\infty}\frac{n^2}{5n^2+4}$ is divergent. □

Remark. (1) The series $\sum_{n=1}^{\infty}a_n$ diverges cannot imply $\lim_{n\to\infty}a_n\neq 0$. That is,

$$\lim_{n\to\infty}a_n\neq 0 \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \sum_{n=1}^{\infty}a_n \;\; (\text{or } \lim_{n\to\infty}s_n) \;\; \text{diverges}$$

For example, $a_n=\frac{1}{n}$.

(2) If $\lim_{n\to\infty}a_n\neq 0$, then $\sum_{n=1}^{\infty}a_n$ diverges. But if $\lim_{n\to\infty}a_n=0$, then $\sum_{n=1}^{\infty}a_n$ could be convergent or divergent. For example, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent but $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

❑ Laws of Series

Theorem 11.2.8. *If $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are convergent series and c is a constant. Then*

(1) $\sum_{n=1}^{\infty}(a_n\pm b_n)$ converges and $\sum_{n=1}^{\infty}(a_n\pm b_n)=\sum_{n=1}^{\infty}a_n\pm\sum_{n=1}^{\infty}b_n$.

(2) $\sum_{n=1}^{\infty}(ca_n)$ converges and $\sum_{n=1}^{\infty}(ca_n)=c\sum_{n=1}^{\infty}a_n$.

Example 11.2.9. Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\frac{1}{2^n}\right]$.

Proof. Since $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$ (converges), we have $\sum_{n=1}^{\infty}\frac{3}{n(n+1)}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=3$. For the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$, it is a geometric series with the first term $a=\frac{1}{2}$ and the ratio $r=\frac{1}{2}$. Then it converges and $\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$. Hence,

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\frac{1}{2^n}\right]=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n}=3+1=4.$$

□

Remark. The result of Theorem 11.2.8 is false if one of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is divergent.

Remark. For a sequence $\{a_n\}$, any finite terms of $\{a_n\}$ doesn't affect the convergence or divergence of the sequence. A series has similar results. If we only concern whether a series $\sum a_n$ is convergent or divergent (but not the exact value of the sereis), the sum of a finite number terms does not change its convergence or divergence. That is, for any number $n_0 \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=n_0}^{\infty} a_n$ both converge or both diverge.

Homework 11.2. 15, 17, 20, 23, 26, 29, 32, 37, 42, 45, 50, 52, 55, 59, 62, 65, 72, 77, 88, 91

11.3 The Integral Test and Estimates for Sums

In general, it is difficult to find the exact sum of a series. We can compute the sums of some special series. For example, geometric series, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Even for a simple series (like $\sum_{n=1}^{\infty} \frac{1}{n^2}$), we cannot find its sum easily. It is not easy to discover the formula of partial sum. Hence, we usually only discuss the convergence of a series. Observe two examples

Example 11.3.1. For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the partial sum is

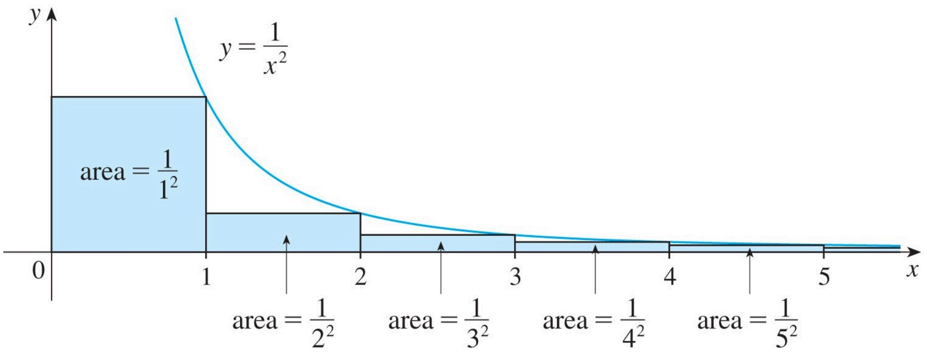
$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

To determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges. We observe that the sequence $\{s_n\}_{n=1}^{\infty}$ is increasing in n. In order to prove that the series is convergent, it suffices to show that the series is bounded above.

Consider the function $f(x) = \frac{1}{x^2}$ on $[1, \infty)$. We have

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2}\, dx < 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2.$$

Hence, $\{s_n\}$ is bounded above (by 2). By the bounded criterion, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.



n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

Example 11.3.2. For the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$, the partial sum

$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$$

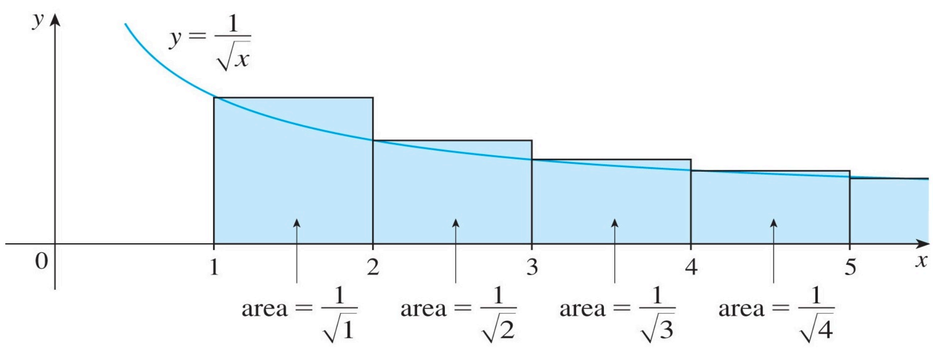
The sequence $\{s_n\}_{n=1}^{\infty}$ is increasing in n. Consider the function $f(x) = \frac{1}{\sqrt{x}}$ on $[1, \infty)$. We have

$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{n+1} - 1.$$

Then

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} (2\sqrt{n+1} - 1) = \infty.$$

and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.



n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

Theorem 11.3.3. *(Integral Test) Suppose that f is a continuous, positive and decreasing function on $[1, \infty)$ and $f(n) = a_n$. Then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge. That is,

$$\sum_{n=1}^{\infty} a_n \text{ converges (diverges)} \iff \int_1^{\infty} f(x)\,dx \text{ converges (diverges)}$$

Proof. Since f is decreasing, for every $k \in \mathbb{N}$,

$$f(k+1)\cdot 1 \le \int_k^{k+1} f(x)\,dx \le f(k)\cdot 1.$$

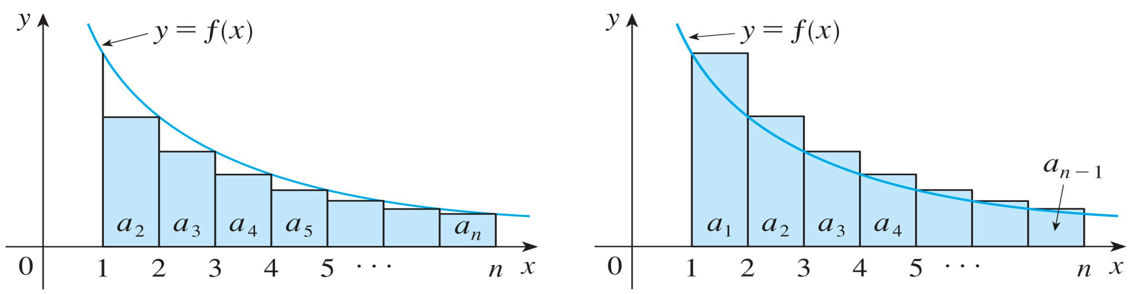
Since f is positive, for every $n \in \mathbb{N}$,

$$0 \le \underbrace{\sum_{k=1}^{n-1} a_{k+1}}_{s_n - a_1} = \sum_{k=1}^{n-1} f(k+1) \le \underbrace{\sum_{k=1}^{n-1}\int_k^{k+1} f(x)\,dx}_{\int_1^n f(x)\,dx} \le \sum_{k=1}^{n-1} f(k) = \underbrace{\sum_{k=1}^{n-1} a_k}_{s_{n-1}}.$$

Hence,

$$\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n.$$

This inequality implies that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.



□

Remark.

(1) To determine whether a series is convergent or divergent, it is not necessary to start with the first term. That is, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=n_0}^{\infty} a_n$ either both converge or both diverge. Hence, to use the integral test, it suffices to compute the integral with lower limit at $x = n_0$ instead of $x = 1$. That is,

$$\int_{n_0}^{\infty} f(x)\,dx \quad \text{converges (diverges)} \iff \sum_{n=n_0}^{\infty} a_n \quad \text{converges (diverges)}$$

$$\iff \sum_{n=1}^{\infty} a_n \quad \text{converges (diverges).}$$

(2) It is not necessary that f is "always" decreasing. We can use the integral test as long as the function f is positive and decreasing on (n_0, ∞) and $f(n) = a_n$ for some large number n_0 and $n \geq n_0$.

Example 11.3.4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent or divergent.

Proof. The function $f(x) = \dfrac{1}{x^2+1}$ is positive and decreasing on $[1, \infty)$. Also, $f(n) = \dfrac{1}{n^2+1}$ for all $n \in \mathbb{N}$. Since the improper integral

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1} x\Big|_1^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

by the integral test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges. □

Example 11.3.5. (p-series) For what values of p is the series $\dfrac{1}{n^p}$ convergent?

Proof. If $p \leq 0$, $\dfrac{1}{n^p} = n^{-p} \geq 1$ for all $n \in \mathbb{N}$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Consider the cases $0 < p < \infty$. The function $f(x) = \dfrac{1}{x^p}$ is positive and decreasing on $[1, \infty)$, and $f(n) = \dfrac{1}{n^p}$. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \infty & \text{when } 0 < p < 1 \quad \text{(divergent)} \\ \dfrac{1}{p-1} & \text{when } p > 1 \quad \text{(convergent).} \end{cases}$$

By the integral test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$. □

Example 11.3.6.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series with $p = 3 > 1$)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges (p-series with $p = \frac{1}{3} < 1$)

Note. The integral test can only determine whether a series is convergent (or divergent). But it cannot give the sum of the series.

Example 11.3.7. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Proof. Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e$. Hence, $f(x)$ is positive and decreasing on (e, ∞). Since the integral

$$\int_e^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_e^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\frac{(\ln x)^2}{2}\Big|_e^t = \lim_{t\to\infty}\frac{(\ln t)^2-1}{2} = \infty,$$

by the integarl test, the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges. □

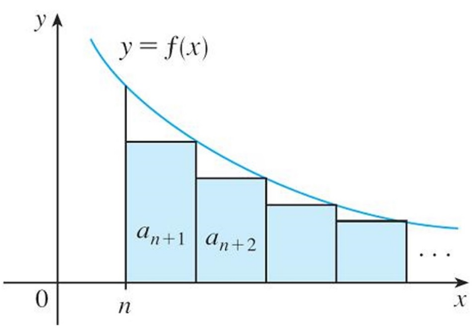
❑ Estimating the Sum of a Series

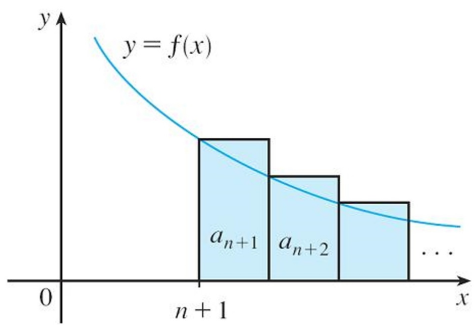
Although it is difficult to use the integral test to find the limit of a series $\sum a_n$, it can still help us to approximate the sum of the series. Recall that "$s = \sum_{n=1}^{\infty} a_n$ converges" means that the partial sum $s_n = \sum_{k=1}^{n} a_k \to s$ as $n \to \infty$. Hence, in order to evaluate the sum s, we want to estimate the difference between s_n and s. Define

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots = \sum_{k=n+1}^{\infty} a_k \qquad \text{as the "}\textit{remainder}\text{".}$$

Theorem 11.3.8. *(Remainder Estimate for the Integral Test) Let f be a continuous, positive and decreasing function for every $x \geq n_0$, and $f(n) = a_n$ for every $n \in \mathbb{N}$ and $n \geq n_0$. Then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq \underbrace{\sum_{k=n+1}^{\infty} a_k}_{=s-s_n} = R_n \leq \int_n^{\infty} f(x)\,dx$$





Note.

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_n^{\infty} f(x)\,dx.$$

Example 11.3.9.

(a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in the approximation.

Proof. Let $f(x) = \frac{1}{x^3}$. Then $\int_n^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$ and

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{200}.$$

□

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Proof. Consider

$$R_n \leq \int_n^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2} \leq 0.0005.$$

Then $n^2 \geq 1000$ and hence $n \geq 31.6$. We need 32 terms to ensure accuracy to within 0.0005. □

(c) Use $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Proof.

$$s_{10} + \frac{1}{2(11)^2} = \sum_{n=1}^{10} \frac{1}{n^3} + \int_{11}^{\infty} \frac{1}{x^3}\, dx \leq s \leq \sum_{n=1}^{10} \frac{1}{n^3} + \int_{10}^{\infty} \frac{1}{x^3}\, dx = s_{10} + \frac{1}{2(10)^2}.$$

Since $s_{10} \approx 1.197532$, we have $1.201664 \leq s \leq 1.202532$. □

Note. In fact, to make the error smaller than 0.0005, it only needs 10 terms by part(c) instead of 32 terms by part(b).

Homework 11.3. 7, 13, 19, 23, 27, 31, 34, 39, 42, 45

11.4 The Comparison Tests

In Section 11.3, we know that the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

Question: Does it say the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$?

Observe that the sequence of the partial sum $s_n = \sum_{k=1}^{n} \frac{1}{2^k + 1}$ is an increasing sequence. Since $0 < \frac{1}{2^k + 1} < \frac{1}{2^k}$ for every $k \in \mathbb{N}$, we have

$$s_n = \sum_{k=1}^{n} \frac{1}{2^k + 1} \leq \sum_{k=1}^{n} \frac{1}{2^k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Hence, $\{s_n\}$ is bounded above. By the bounded criterion, the series $\sum_{n=1}^{\infty} \frac{1}{2^k+1}$ converges. Moreover, $\sum_{n=1}^{\infty} \frac{1}{2^k+1} < 1$.

Heuristically, we may have the insight of two nonnegative series.

(i) If every term of one series is smaller than the corresponding term of another convergent series, then the former series is also convergent.

(ii) If every term of one series is larger than the corresponding term of another divergent series, then the former series is also divergent.

❑ The Comparison Test

Theorem 11.4.1. *(The Direct Comparision Test) Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with nonnegative terms and* $0 \le b_n \le a_n$ *for all* $n \in \mathbb{N}$.

(1) If $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\sum_{n=1}^{\infty} b_n$ *is convergent.*

(2) If $\sum_{n=1}^{\infty} b_n$ *is divergent, then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = b_1 + b_2 + \cdots + b_n$. Then the sequences $\{s_n\}$ and $\{t_n\}$ are increasing and $0 \le t_n \le s_n$ for every $n \in \mathbb{N}$.

(1) If $\sum_{n=1}^{\infty} a_n$ is convergent, $\{s_n\}$ is convergent. Since $\{t_n\}$ is increasing and bounded above, it is convergent and thus $\sum_{n=1}^{\infty} b_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} b_n$ is divergent, then $\lim_{n\to\infty} t_n = \infty$. Therefore, $\lim_{n\to\infty} s_n = \infty$ and thus $\sum_{n=1}^{\infty} a_n$ is divergent.

□

Remark.

(i) In order to use the Comparison Test, the "nonnegative" condition is necessary. For example, $b_n = -1$ and $a_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $b_n < a_n$. But the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(-1) = -\infty$ is divergent and the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(ii) In the use of the Comparsion Test, we need to know some convergent or divergent series. Some important series are:

- p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{converges when } p>1\\ \text{diverges when } p\le 1\end{cases}$
- geometric series $\displaystyle\sum_{n=1}^{\infty} ar^n\begin{cases}\text{converges when } |r|<1\\ \text{diverges when } |r|\ge 1\end{cases}$

Example 11.4.2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2n^2+4n+3}$ is convergent or divergent.

Proof. That the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (p-series, $p=2$) implies the series $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2n^2}$ is also convergent. Since $\dfrac{5}{2n^2+4n+3}<\dfrac{5}{2}\cdot\dfrac{1}{n^2}$ for every $n\in\mathbb{N}$, by the Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2n^2+4n+3}$ is convergent.

□

Remark. To determine whether a series is convergent, it suffices to consider the convergence of the "tail " ($\displaystyle\sum_{n=n_0}^{\infty} a_n$) of the series. Therefore, in the use of the Comparison Test, we can replace the condition $0\le b_n\le a_n$ "for every $n\ge 1$" by " for every $n\ge n_0$" and for some integer n_0, and the test still holds.

Example 11.4.3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ is convergent or divergent.

Proof. Since $\ln n>1$ for $n>e$, we have $\dfrac{\ln n}{n}>\dfrac{1}{n}$ when $n\ge 3$. Also, the series $\displaystyle\sum_{n=1}\frac{1}{n}$ diverges (p-series, $p=1$). By the Comparison Test, the series $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges and thus the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ also diverges. □

Example 11.4.4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3-5n-2}$ is convergent or divergent.

Proof.

Observe that

(i) Not all terms are positive

(ii) We guess the series is convergent and hope $\dfrac{1}{n^3 - 5n - 2} < \dfrac{2}{n^3}$ for all $n \geq n_0$. To find n_0, consider

$$2n^3 - 10n - 4 > n^3 \quad \Longleftrightarrow \quad n^3 > 10n + 4 \quad \Rightarrow n \geq 4.$$

When $n \geq 4$, the term $\dfrac{1}{n^3 - 5n - 2} > 0$ and $\dfrac{1}{n^3 - 5n - 2} < \dfrac{2}{n^3}$. Also, $\displaystyle\sum_{n=4}^{\infty} \frac{2}{n^3}$ converges (p-series, $p = 3 > 1$). By the Comparison Test, the series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^3 - 5n - 2}$ converges. Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 - 5n - 2}$ converges. □

Note. Recall that for $\sum a_n$ and $\sum b_n$ with $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$, the Comparison Test says that

(1) $\sum a_n$ converges $\Longrightarrow$ $\sum b_n$ converges;

(2) $\sum b_n$ diverges $\Longrightarrow$ $\sum a_n$ diverges.

But the converse is false. That is,

(1) $\sum b_n$ converges $\not\Longrightarrow$ $\sum a_n$ converges;

(2) $\sum a_n$ diverges $\not\Longrightarrow$ $\sum b_n$ diverges.

Example 11.4.5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$. In order to use the Comparison Test to show $\sum \dfrac{1}{2^n - 1}$ converges, we cannot choose the known convergent series $\sum \dfrac{1}{2^n}$ because $\dfrac{1}{2^n - 1} > \dfrac{1}{2^n}$. However, $\dfrac{1}{2^n - 1}$ looks very close to $\dfrac{1}{2^n}$. It is reasonable to guess that the series $\sum \dfrac{1}{2^n - 1}$ also converges.

■ The Limit Comparison Test

Theorem 11.4.6. *(The Limit Comparison Test) Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some $0 < c < \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges. That is, either both series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or both diverge.

Proof. (Exercise) □

Example 11.4.7. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n - 1}$ is convergent or divergent.

Proof. Consider the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with $r = \frac{1}{2} < 1$) and

$$\lim_{n\to\infty} \frac{\frac{3}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{3}{1 - \frac{1}{2^n}} = 3,$$

by the Limit Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n - 1}$ is convergent. □

Example 11.4.8. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ is convergent or divergent.

Proof. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (p-series, $p = \frac{1}{2} < 1$) and

$$\lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{\sqrt{\frac{5}{n^5} + 1}} = 2,$$

by the Limit Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ diverges. □

■ Estimating Sums

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are two convergent series with nonnegative terms and $0 \le b_n \le a_n$ for all $n \in \mathbb{N}$. Let

$$s = \sum_{n=1}^{\infty} a_n, \quad s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$
$$t = \sum_{n=1}^{\infty} b_n, \quad t_n = \sum_{k=1}^{n} b_k \quad \text{and} \quad T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

then $0 \le T_n \le R_n$ for all $n \in \mathbb{N}$. Hence, if we can estimate R_n, then we have an upper bound of T_n.

Example 11.4.9. Use the sum of the first 100 terms to approximate the sum of the series $\displaystyle\sum \frac{1}{n^3 + 1}$. Estimate the error involved in this approximation.

Proof. Since $\frac{1}{n^3+1} < \frac{1}{n^3}$ for all $n \in \mathbb{N}$, we have

$$T_{100} = \sum_{n=101}^{\infty} \frac{1}{n^3+1} \le \sum_{n=101}^{\infty} \frac{1}{n^3} < \int_{100}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2(100)^2}.$$

The error is less than $\frac{1}{2(100)^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538.$ □

Homework 11.4. 9, 12, 15, 18, 21, 24, 28, 34, 37, 41, 46, 48(b)(i), 49(b)(i)

11.5 Alternating Series and Absolute Convergence

In the previous section, we consider the convergence tests for the nonnegative series (because of the bounded criterion). In the present section, we want to relax the condition and discuss the convergence for some special series which includes positive and negative terms alternatively.

❏ Alternating Series

Definition 11.5.1. An alternating series $\sum_{n=1}^{\infty} a_n$ is a series whose terms are alternatively positive and negative.

Let $b_n = |a_n|$. The general form of an alternating series is

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \sum_{n=1}^{\infty} (-1)^n b_n & \text{if } a_1 < 0 \\ \sum_{n=1}^{\infty} (-1)^{n-1} b_n & \text{if } a_1 \ge 0. \end{cases}$$

Example 11.5.2. The series $\sum_{n=1}^{\infty} (-1)^n$ is an alternating series.

■ Alternating Series Test

Theorem 11.5.3. *If the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \quad \text{where } b_n > 0$$

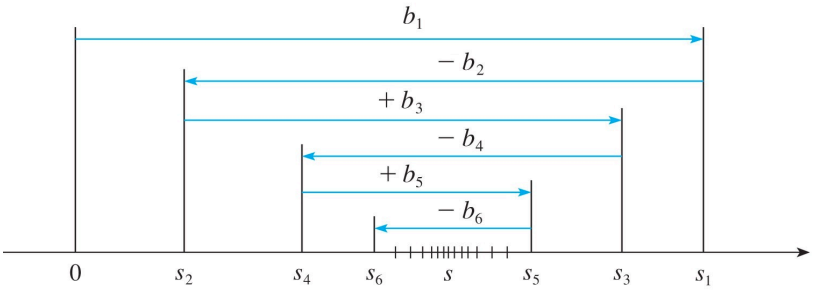
satisfies

(i) $b_{n+1} \le b_n$ *for all* $n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} b_n = 0,$

then the series is convergent.

Proof.



Let $\{s_n\}$ be the sequence of the partial sums of the alternating series. The condition (i) implies that, for every $n \in \mathbb{N}$,

$$s_{2n+2} = s_{2n} + \underbrace{(b_{2n+1} - b_{2n+2})}_{\geq 0} \geq s_{2n}$$

and

$$s_{2n} = b_1 - \underbrace{(b_2 - b_3)}_{\geq 0} - \cdots - \underbrace{(b_{2n-1} - b_{2n})}_{\geq 0} \leq b_1.$$

We have

$$0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots \leq b_1$$

which is increasing and bounded above by b_1. By the bounded criterion, $\lim\limits_{n\to\infty} s_{2n} = s$ is convergent. Since $s_{2n+1} = s_{2n} + b_{2n+1}$, by condition (ii),

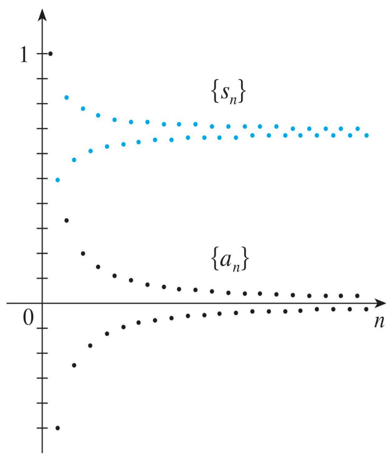
$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s.$$

Hence $\lim\limits_{n\to\infty} s_n = s$ and the alternating series is convergent. □

Example 11.5.4. (alternating harmonic series) Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent or divergent.

Proof.

Let $b_n = \frac{1}{n}$. Then $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$. Since $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, by the alternating series test, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.



□

Example 11.5.5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is convergent or divergent.

Proof. Let $b_n = \frac{3n}{4n-1}$ and $a_n = \frac{(-1)^n 3n}{4n-1} = (-1)^n b_n$. Then $|a_n| = b_n$ for every $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$, the limit $\lim_{n\to\infty} a_n$ is not equal to 0 (in fact, the limit does not exist). By the Test for Divergent, the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} = \lim_{n\to\infty} a_n$ is divergent. □

Example 11.5.6. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$ is convergent or divergent.

Proof. Let $b_n = \frac{n^2}{n^3+1}$ Then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1} = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$. Since

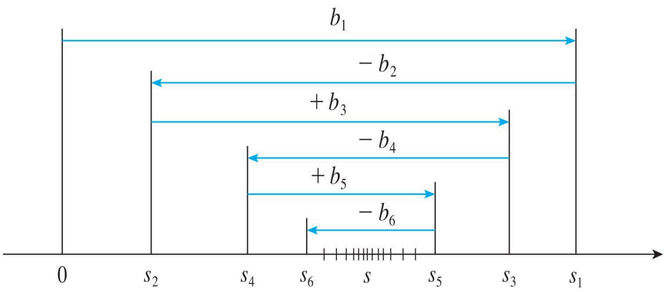
$$b_{n+1} - b_n = \frac{(n+1)^2}{(n+1)^3+1} - \frac{n^2}{n^3+1} = \frac{-n^4 - 2n^3 - n^2 + 2n + 1}{[(n+1)^3+1](n^3+1)} < 0 \quad \text{for all } n \in \mathbb{N},$$

we have $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Also, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$. By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$ is convergent. □

Note. In this example, we can compute $\frac{d}{dx}\left(\frac{x^2}{x^3+1}\right) = \frac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \geq 2$ to obtain $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$.

Remark. As the similar discussion as before, in the use of the alternating series test, it only needs that the series satisfies conditions (i) in Theorem 11.5.3 for every $n \geq n_0$ for some fixed integer n_0.

■ Estimating Sums



Observe the structure of an alternating series satisfying the two conditions (i) and (ii) in Theorem 11.5.3. Let $R_n = s - s_n$ be the remainder of the series, then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Theorem 11.5.7. *(Alternating Series Estimation Theorem) If* $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ *is the sum of an alternating series that satisfies*

$$(i)\ 0 \leq b_{n+1} \leq b_n \quad \textit{for every } n \in \mathbb{N} \qquad \textit{and} \qquad (ii)\ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 11.5.8. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Proof. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ is an alternating series. Let $b_n = \frac{1}{n!}$. Then $b_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = b_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$. To find n such that $b_n = \frac{1}{n!} < 0.001$, we have $n \geq 7$. Hence, by the alternating series estimation,

$$|R_6| = |s - s_6| \leq b_7 < 0.001 \qquad (\text{in fact, } b_7 < 0.0002).$$

Then $s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$. In fact $s = \frac{1}{e} \approx 0.36787944$. □

Remark. The rules does not apply to other type of series.

❑ Absolute Convergence and Conditional Convergence

From now on, we will continue to discuss the convergence of general series (without alternating patterns). Intuitively, it is difficult to give a nice test for every series because they may have too many varieties. Therefore, we hope to use some known results (discussed in the previous sections) to deal with the convergence of certain general series.

For a general series $\sum_{n=1}^{\infty} a_n$, we consider the correspondg series

$$\sum_{n=1}^{\infty} \left|a_n\right| = \left|a_1\right| + \left|a_2\right| + \cdots + \left|a_n\right| + \cdots .$$

Definition 11.5.9. (a) A series $\sum_{n=1}^{\infty} a_n$ is called "*absolutely convergent*" if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(b) A series $\sum_{n=1}^{\infty} a_n$ is called "*conditionally convergent*" if it is convergent but not absolutely convergent.

Example 11.5.10.

(1) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by the alternating series test. But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ is divergent (harmonic series, p-series with $p = 1$). Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a conditionally convergent series.

(2) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent by the alternating series test and $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent (p-series with $p = 2$). Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Question: For the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$, can the convergence of one series imply the convergence of the other one?

Theorem 11.5.11. *If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. That is, if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Observe that $0 \leq a_n + |a_n| \leq 2|a_n|$. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} 2|a_n|$ converges. By the Comparison Test, the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges. Hence, the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

converges.

□

Note.

(1) The converse of Theorem 11.5.11 is false. That is, the convergence of $\sum_{n=1}^{\infty} a_n$ cannot imply the convergence of $\sum_{n=1}^{\infty} |a_n|$.

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \not\Longrightarrow \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges.}$$

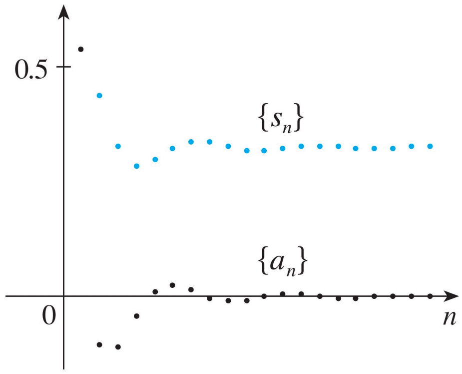
For example, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ is divergent.

(2) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} |a_n|$ must be divergent.

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} |a_n| \quad \text{diverges.}$$

Example 11.5.12. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

Proof. The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is not an alternating series. Consider $\left|\dfrac{\cos n}{n^2}\right| \le \dfrac{1}{n^2}$ for every $n \in \mathbb{N}$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2$), by the Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos n}{n}\right|$ converges. Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and this implies that it is convergent. □



Example 11.5.13. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \qquad (b)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \qquad (c)\ \sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$$

Exercise. Let $\{a_n\}$ be a sequence and define

$$a_n^+ = \begin{cases} a_n, & \text{if } a_n \ge 0 \\ 0, & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad a_n^- = \begin{cases} 0, & \text{if } a_n \ge 0 \\ a_n, & \text{if } a_n < 0 \end{cases}$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges if and only if both of the series $\displaystyle\sum_{n=1}^{\infty} a_n^+$ and $\displaystyle\sum_{n=1}^{\infty} a_n^-$ converge and moreover,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.$$

Hint: ($\Longrightarrow$) Using the Comarison Test with the fact $0 \le |a_n^{\pm}| \le |a_n|$ for every $n \in \mathbb{N}$ and moreover, the equality holds from the laws for series.
($\Longleftarrow$) Using the laws for series with the fact $|a_n| = a_n^+ - a_n^-$ for every $n \in \mathbb{N}$.

❑ Rearrangement

Question: What is the difference between absolutely convegent or conditionally convergent series? Whether the behaviors of infinite sums are like the ones of finite sums?

- For a finite sum, we can rearrange the order of the terms and the value of the sum remains unchanged.
- For an infinite sum, the rearrangement may change the sum.

Consider an example of a paradox. Let

$$\begin{aligned} x &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots \\ &\stackrel{?}{=} (1-\frac{1}{2})+\frac{1}{4}+(\frac{1}{3}-\frac{1}{6})-\frac{1}{8}+(\frac{1}{5}-\frac{1}{10})-\frac{1}{12}+(\frac{1}{7}-\frac{1}{14})-\frac{1}{16}+(\frac{1}{9}-\frac{1}{18})-\frac{1}{20}+\cdots \\ &= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots \\ &= \frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots) \\ &= \frac{1}{2}x. \end{aligned}$$

Hence $x = \frac{1}{2}x$ and we obtain a contradiction that $x = 0$.

Question: What's wrong with this?

For a sum of finitely many numbers, we obtain the same value if arbitrarily rearraneging the order of those numbers.

Question: Can we get the same value of the sum of infinitely many numbers if we arbitrarily rearrange the order of these numbers?

Definition 11.5.14. Let $\{a_n\}$ and $\{b_n\}$ be two sequences. We say that $\{b_n\}$ is a "*rearrangement*" of $\{a_n\}$ if there exists a one-to-one and onto function f on $\mathbb{N}$ such that $b_n = a_{f(n)}$ for every $n \in \mathbb{N}$.

Note. In general, $\sum_{n=1}^{\infty} a_n \neq \sum_{n=1}^{\infty} b_n$ if $\{b_n\}$ is a rearrangement of $\{a_n\}$.

Theorem 11.5.15. *If* $\sum_{n=1}^{\infty} a_n$ *is conditionally convergent then, for any number* $L \in \mathbb{R}$*, there exists a rearrangement* $\{b_n\}$ *of* $\{a_n\}$ *such that* $\sum_{n=1}^{\infty} a_n = L$.

Proof. We only sketch the proof by the following steps.

(I) Let $\{p_n\}$ be the nonnegative subsequence of $\{a_n\}$ and $\{q_n\}$ be the negative subsequence of $\{a_n\}$. Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, we have $\sum_{n=1}^{\infty} |a_n|$ diverges. Hence, at least one of the series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ is divergent. Moreover, the fact that $\sum_{n=1}^{\infty} a_n$ converges implies both series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ are divergent. We have that

$$\sum_{n=1}^{\infty} p_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} q_n = -\infty.$$

(II) W.L.O.G, say $L > 0$. We construct a sequence $\{b_n\}$ from $\{p_n\}$ and $\{q_n\}$ by the following process. Since $\sum_{n=1}^{\infty} p_n = \infty$, there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_1-1} p_n < L \leq \sum_{n=1}^{n_1} p_n.$$

Let $S_1 = \sum_{n=1}^{n_1} p_n$. Then $S_1 \geq L$ and $S_1 - p_{n_1} < L$. Hence, $|S_1 - L| < p_{n_1}$.

Since $\sum_{n=1}^{\infty} q_n = -\infty$, there exists $m_1 \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1-1} q_n > L \geq \sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n.$$

Let $T_1 = \sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n = S_1 + \sum_{n=1}^{m_1} q_n$. Then $T_1 \leq L$ and $T_1 - q_{m_1} > L$. Hence, $|T_1 - L| < q_{m_1}$.

Continue this process, we have $1 \leq n_1 < n_2 < \cdots$ and $1 \leq m_1 < m_2 < \cdots$ and $\{S_k\}$ and $\{T_k\}$ such that for every $k \in \mathbb{N}$,

$$S_k = T_{k-1} + \sum_{n=n_{k-1}+1}^{n_k} p_n, \quad S_k \geq L, \quad S_k - p_{n_k} < L \quad \Longrightarrow |S_k - L| < p_{n_k}$$

and

$$T_k = S_k + \sum_{n=m_{k-1}+1}^{m_k} q_n, \quad T_k \leq L, \quad T_k - q_{m_k} \geq L \quad \Longrightarrow |T_k - L| < q_{m_k}.$$

Define $\{b_n\} = \left\{p_1, p_2, \cdots, p_{n_1}, q_1, q_2, \cdots, q_{m_1}, p_{n_1+1}, \cdots, p_{n_2}, q_{m_1+1}, \cdots q_{m_2}, \cdots\right\}$

(III) To check that $\{b_n\}$ is a rearrangement of $\{a_n\}$, we have to show that

(i) To show that each a_n appears at most once in $\{b_n\}$. Since each a_n is either in $\{p_n\}$ or in $\{q_n\}$, and each p_n or each q_n appears in $\{b_n\}$ at most once by the construction of $\{b_n\}$, we have each a_n appears in $\{b_n\}$ at most once.

(ii) To show that each a_n appears at least once in $\{b_n\}$. For $K \in \mathbb{N}$, a_K must appear in $\{p_n\}_{n=1}^{K}$ or in $\{q_n\}_{n=1}^{K}$. Hence, a_K appears in $\{b_n\}$ at least once.

(IV) Check that $S_k \to L$ and $T_k \to L$ as $k \to \infty$. Since the series $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ as $n \to \infty$. Then $p_n \to 0$ and $q_n \to 0$ as $n \to \infty$. Hence, by part (II), $S_k \to L$ and $T_k \to L$ as $k \to \infty$.

By the above argument $\{b_n\}$ is a rearrangement of $\{a_n\}$ and $\sum_{n=1}^{\infty} b_n = L$.

□

Theorem 11.5.16. *If* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent and* $\{b_n\}$ *is a rearrangement of* $\{a_n\}$*, then*

(a) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ *and*

(b) $\sum_{n=1}^{\infty} b_n$ *is absolutely convergent.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_m = \sum_{k=1}^{m} b_k$.

(a) Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence it is convergent, the series $\sum_{n=1}^{\infty} a_n$ is a finite number. Given $\varepsilon > 0$, we want to prove $\left|t_m - \sum_{n=1}^{\infty} a_n\right| < \varepsilon$ as m is sufficiently large.

Since $\sum_{n=1}^{\infty} |a_n|$ converges, there exists $N \in \mathbb{N}$ such that

$$|a_{N+1}| + |a_{N+2}| + \cdots < \frac{\varepsilon}{2}.$$

Since $\{b_n\}$ is a rearrangement of $\{a_n\}$, there exists $M \in \mathbb{N}$ such that $\{a_1, a_2, \cdots, a_N\} \subseteq \{b_1, b_2, \cdots, b_M\}$. For $m > M$

$$|t_m - s_N| \leq |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\varepsilon}{2}.$$

Then

$$|t_m - \sum_{n=1}^{\infty} a_n| \leq |t_m - s_N| + |s_N - \sum_{n=1}^{\infty} a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\{t_m\}$ converges to $\sum_{n=1}^{\infty} a_n$ and we have $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

(b) Consider the sequence $\{|a_n|\}$. Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, $\sum_{n=1}^{\infty} |a_n|$ is also absolutely convergent. On the other hand, since $\{b_n\}$ is a rearrangement of $\{a_n\}$, $\{|b_n|\}$ is a rearrangement of $\{|a_n|\}$. By part(a),

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |b_n|.$$

Hence, $\sum_{n=1}^{\infty} |b_n|$ converges; that is, $\sum_{n=1}^{\infty} b_n$ is absutely convergent.

□

■ **Product of two sequences**

Suppose that $\{a_n\}$ and $\{b_n\}$ are summable sequences. We recall that

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n \quad \text{where } c \text{ is a constant.}$$

Question: Can we express $(\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n)$ as a form of series? If yes, what is the expression?

Heuristically, we observe the product of two finite series.

$$\Big(\sum_{n=1}^{N} a_n\Big)\Big(\sum_{m=1}^{M} b_m\Big) = \sum_{k=1}^{L} c_k.$$

where $\{c_k\}$ contains all products of $a_n b_m$.

Question: Is the formula still true for the product of two arbitrary infinite series?
Anserer: In general, it is not true for two summable sequences.

Exercise. Find two summable sequences $\{a_n\}$ and $\{b_n\}$ such that there is no summable sequence $\{c_n\}$ satisfying

$$\Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{n=1}^{\infty} b_n\Big) = \sum_{n=1}^{\infty} c_n.$$

Theorem 11.5.17. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely and $\{c_n\}$ is any sequence containing all products $a_i b_j$ for each pair (i, j), then*

$$\sum_{n=1}^{\infty} c_n = \Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{n=1}^{\infty} b_n\Big).$$

Proof. (Exercise) □

Homework 11.5. 3, 6, 9, 11, 13, 17, 20, 22, 25, 28, 31, 34, 37, 41, 46, 48

11.6 The Ratio and Root Tests

In the previous section, we study that an absolutely convergent series is also convergent. However, it is not easy to check whether a general series is absolutely convergent. In the present section, we will introduce two methods which can determine whether certain series are convergent or divergent. The spirit of these two methods is from the comparison with geometric series.

❑ The Ratio Test

Theorem 11.6.1. *(Ratio Test) For the series* $\sum_{n=1}^{\infty} a_n$*, suppose that* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$*.*

(a) If $L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore it is convergent).*

(b) If $L > 1$ (*or* $L = \infty$)*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) If $L = 1$ *the Ratio Test is inconclusive. (For example,* $\sum \frac{1}{n}$ *diverges and* $\sum \frac{1}{n^2}$ *converges).*

Proof. (Postponed) □

Example 11.6.2. Determine whether the following series are convergent or divergent.

(1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Proof. Let $a_n = \frac{1}{n!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. □

(2) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Proof. Let $a_n = \frac{1}{n!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. □

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{(n+1)!}$ for some $r \in \mathbb{R}$.

Proof. Let $a_n = \dfrac{r^n}{(n+1)!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{r^{n+1}}{(n+2)!}}{\frac{r^n}{(n+1)!}} = \lim_{n\to\infty} \frac{r}{n+2} = 0 < 1.$$

By the ratio test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{(n+1)!}$ is convergent. □

(4) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Proof. Let $a_n = (-1)^n \dfrac{n^3}{3^n}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}\frac{(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}\right| = \lim_{n\to\infty} \frac{1}{3}\Big(\frac{n+1}{n}\Big)^3 = \frac{1}{3} < 1.$$

By the ratio test, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is convergent. □

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Proof. Let $a_n = \dfrac{n^n}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \lim_{n\to\infty} \Big(\frac{n+1}{n}\Big)^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e > 1.$$

By the ratio test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent. □

Note. Consider $\dfrac{n^n}{n!} = \dfrac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} \geq n \to \infty$ as $n \to$. By the Test for Divergence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

<u>Proof of Ratio Test</u>

(a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, choosing a number s such that $L < s < 1$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$

$$\frac{|a_{n+1}|}{|a_n|} < s < 1.$$

Hence, $|a_{n+1}| < |a_n|s$ for every $n > N$. We have

$$\begin{aligned}
|a_{N+2}| &< |a_{N+1}|s \\
|a_{N+3}| &< |a_{N+2}|s < |a_{N+1}|s^2 \\
&\vdots \\
|a_{N+k}| &< |a_{N+k-1}|s < \cdots < |a_{N+1}|s^{k-1} \qquad \text{for } k = 1, 2, 3, \cdots
\end{aligned}$$

For every $n > N$, the partial sum s_n of $\sum_{n=1}^{\infty} |a_n|$ satisfies

$$\begin{aligned}
s_n &= |a_1| + |a_2| + \cdots + |a_N| + |a_{N+1}| + \cdots + |a_n| \\
&= \sum_{k=1}^{N} |a_k| + |a_{N+1}| + \cdots + |a_n| \\
&< \sum_{k=1}^{N} |a_k| + |a_{N+1}| + |a_{N+1}|s + |a_{N+1}|s^2 + \cdots + |a_{N+1}|s^{n-(N+1)} \\
&= \sum_{k=1}^{N} |a_k| + \frac{|a_{N+1}|(1 - s^{n-N})}{1 - s} \\
&< \sum_{k=1}^{N} |a_k| + \frac{|a_{N+1}|}{1 - s} \qquad \text{since } 0 < s < 1.
\end{aligned}$$

Since $\{s_n\}$ is an increasing sequence and bounded above, by the bounded criterion, $\{s_n\}$ converges and hence $\sum_{n=1}^{\infty} a_n$ is absolutely convegent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, choosing a number s such that $1 < s < L$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$

$$\frac{|a_{n+1}|}{|a_n|} > s > 1.$$

Hence, $|a_{n+1}| > |a_n|s$ for every $n > N$. We have

$$\begin{aligned}
|a_{N+2}| &> |a_{N+1}|s \\
|a_{N+3}| &> |a_{N+2}|s < |a_{N+1}|s^2 \\
&\vdots \\
|a_{N+k}| &> |a_{N+k-1}|s > \cdots < |a_{N+1}|s^{k-1} \qquad \text{for } k = 1, 2, 3, \cdots
\end{aligned}$$

W.L.O.G, we may assume that $|a_{N+1}| > 0$. Then

$$\lim_{n\to\infty} |a_n| \geq \lim_{n\to\infty} |a_{N+1}| s^{n-(N+1)} = \infty \qquad \text{(since } s > 1\text{)}.$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

❑ The Root Test

Theorem 11.6.3. *(Root Test) For the series* $\sum_{n=1}^{\infty} a_n$*, suppose that* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$*.*

(a) If $L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore it is convergent).*

(b) If $L > 1$ *(or* $L = \infty$*), then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) If $L = 1$ *the Ratio Test is inconclusive. (For example,* $\sum \frac{1}{n}$ *diverges and* $\sum \frac{1}{n^2}$ *converges).*

Proof. (Postponed) □

Example 11.6.4. Determine whether the following series are convergent or divergent.

(1) $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$.

Proof. Let $a_n = \frac{1}{(\ln n)^n}$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(\ln n)}\right|^n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1.$$

By the root test, the series $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$ is convergent. □

(2) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$.

Proof. Let $a_n = \frac{2^n}{n^3}$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^3}\right|} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^3}} = 2 > 1.$$

By the root test, the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ is divergent. □

(3) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Proof. Let $a_n = \left(\frac{2n+3}{3n+2}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2n+3}{3n+2}\right|^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1.$$

By the root test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent. □

(4) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$.

Proof. Let $a_n = \left(\frac{n}{n+1}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{n+1}\right|^n} = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1.$$

The Root Test is inconclusive. However,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0.$$

By the Test for Divergence, the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$ diverges. □

Proof of Ratio Test

(a) Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, choosing a number s such that $L < s < 1$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$

$$\sqrt[n]{|a_n|} < s < 1.$$

Hence, $|a_n| < s^n$ for every $n \geq N$. The partial sum s_n of $\sum_{n=1}^{\infty} |a_n|$ satisfies

$$\begin{aligned} s_n &= |a_1| + |a_2| + \cdots + |a_N| + |a_{N+1}| + \cdots + |a_n| \\ &< \sum_{k=1}^{N} |a_k| + s^{N+1} + s^{N+2} + \cdots + s^n \\ &= \sum_{k=1}^{N} |a_k| + \frac{s^{N+1}(1 - s^{n-N})}{1 - s} \\ &< \sum_{k=1}^{N} |a_k| + \frac{s^{N+1}}{1 - s} \qquad \text{since } 0 < s < 1. \end{aligned}$$

Since $\{s_n\}$ is an increasing sequence and bounded above, by the bounded criterion, $\{s_n\}$ converges and hence $\sum_{n=1}^{\infty} a_n$ is absolutely convegent.

(b) Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, choosing a number s such that $1 < s < L$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$

$$\sqrt[n]{|a_n|} > s > 1.$$

Hence, $|a_n| > s^n$ for every $n > N$. We have

$$\lim_{n\to\infty} |a_n| \geq \lim_{n\to\infty} s^n = \infty \qquad (\text{since } s > 1).$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Homework 11.6. 3, 6, 9, 13, 19, 22, 25, 28, 32, 35, 39, 41

11.7 Strategy for Testing Series

In the present section, we will organize all tests introduced in previous sections. The following steps are some strategies for convergence or divergence for series.

$$\sum_{n=1}^{\infty} a_n$$

1. ***p*-series:**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{when } p > 1 \\ \text{divergent} & \text{when } p \leq 1. \end{cases}$$

2. **geometric series:**

$$\sum_{n=1}^{\infty} ar^n \ (a \neq 0) \text{ is } \begin{cases} \text{convergent} & \text{when } |r| < 1 \\ \text{divergent} & \text{when } |r| \geq 1. \end{cases}$$

3. When the form of the series is similar to a p-series or a geometric series (for example, $\sum \frac{2}{n^2 + 3n + 1}$ or $\sum \frac{2^{n+1} - 5}{3^n + 2}$), we could determine the convergence or divergence by using the comparison test (or limit comparison test).

4. **Test for Divergence:**

$$\lim_{n \to \infty} a_n \neq 0 \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{is divergent.}$$

5. **Alternating Series Test:** If the series has the form $\sum_{n=1}^{\infty} (-1)^n b_n$ for $b_n > 0$ satisfying

$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n \in \mathbb{N} \qquad \text{and} \qquad \text{(ii) } \lim_{n \to \infty} b_n = 0$$

then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

6. **Ratio Test:** Suppose that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

$$\sum_{n=1}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1 \\ \text{divergent} & \text{if } L > 1 \\ \text{inconclusive} & \text{if } L = 1 \end{cases}$$

7. **Root Test:** Suppose that $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$.

$$\sum_{n=1}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1 \\ \text{divergent} & \text{if } L > 1 \\ \text{inconclusive} & \text{if } L = 1 \end{cases}$$

8. **Integral Test:** Suppose that f is positive and nonincreasing on $[1, \infty)$, and $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{is convergent (divergent)} \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\, dx \quad \text{is convergent (divergent).}$$

Homework 11.7. 9, 12, 15, 18, 20, 22, 26, 30, 33, 36, 38, 40, 43, 46, 68

11.8 Power Series

So far, we have studied series of numbers: $\sum a_n$. For a number $s \in \mathbb{R}$, it can be expressed as a series (sum of infinite numbers). For a function f, we want to ask whether a (smooth) function can be expressed as a sum of infinite function. Here we consider series, called "*power series*", in which each term includes a power of the variable x: $\sum c_n x^n$.

❑ Power Series

Definition 11.8.1. A "*power series*" is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n are constants called the "*coefficients*" of the series.

For given $x = x_0$, we should determine whether the series $\sum_{n=0}^{\infty} c_n x_0^n$ converges or diverges. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be as a function. Then the domain of $f(x)$ is the set of all x for which the series converges.

Example 11.8.2. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a power series. (We regard the power function as a geometric series with ratio x.) The series converges when $|x| < 1$ and diverges when $|x| \geq 1$. Therefore, the domain of $\sum_{n=0}^{\infty} x^n$ is $(-1, 1)$.

Definition 11.8.3. In general, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a "*power series in* $(x-a)$" or a "*power series centered at* a" or a "*power series about* a".

Note. For a power series, it is important to determine for what values of x the series converges.

Example 11.8.4. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Proof. (Idea: using the ratio test or root test)

Let $a_n = n! x^n$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = (n+1)|x|$. If $x = 0$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ and if $x \neq 0$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.

By the Ratio Test, the series converges when $x = 0$. □

Example 11.8.5. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Proof. Let $a_n = \frac{(x-3)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \frac{n}{n+1}|x-3| \longrightarrow |x-3| \quad as\ n \to \infty.$$

By the Ratio Test, if $|x-3| < 1$ (i.e. $2 < x < 4$), the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges and if $|x-3| > 1$ (i.e. $x < 2$ or $x > 4$) the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ diverges.

For $|x-3| = 1$,

(i) When $x - 3 = 1$ (i.e. $x = 4$), $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1$).

(ii) When $x-3 = -1$ (i.e. $x = 2$), $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test.

Hence, the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges on $[2, 4)$ and diverges on $(-\infty, 2) \cup [4, \infty)$. □

Example 11.8.6. For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ converge?

Proof. Let $a_n = \frac{x^n}{(2n)!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{[2(n+1)]!}}{\frac{x^n}{(2n)!}}\right| = \frac{|x|}{(2n+1)(2n+2)} \longrightarrow 0 < 1 \quad \text{as } n \to \infty$$

for all x. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ converges for all x. □

Example 11.8.7. (Bessel function of order 0) Find the domain of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

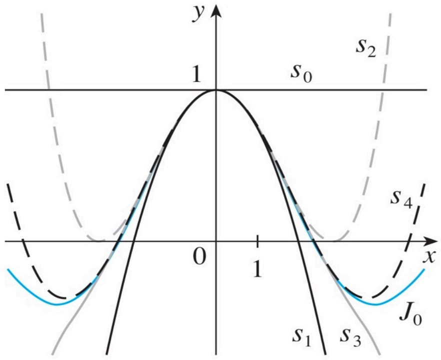
Proof. Let $a_n = \frac{(-1)x^{2n}}{2^{2n}(n!)^2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}}{\frac{(-1)x^{2n}}{2^{2n}(n!)^2}}\right| = \frac{1}{2^2(n+1)^2}|x|^2.$$

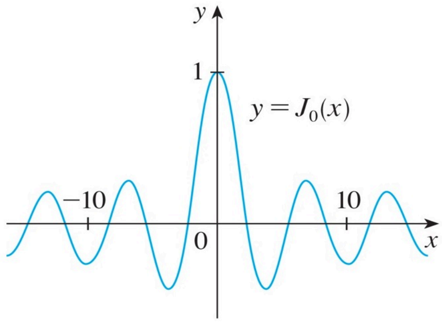
For every $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2^2(n+1)^2}|x|^2 = 0 < 1.$$

By the Ratio Test, the series converges for every x and the domain of $J_0(x)$ is $\mathbb{R}$. □



Partial sums of the Bessel function J_0



■ Interval of Convergence

Definition 11.8.8. (a) We say that a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges

(i) at x_0 if $\sum_{n=0}^{\infty} c_n(x_0-a)^n$ converges;

(ii) on the set S if $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at each $x \in S$.

(b) If we regard a series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ as a function, then the domain of $f(x)$ is the set of all x for which the series converges.

Remark. A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges at its center a. In fact, when putting $x = a$, the series converges to the constant term c_0.

Example 11.8.9. Consider the series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ as a geometric series with ratio x. Then the series converges when $|x| < 1$ and diverges when $|x| \geq 1$. Therefore, the domain of $\sum_{n=0}^{\infty} x^n$ is $(-1, 1)$.

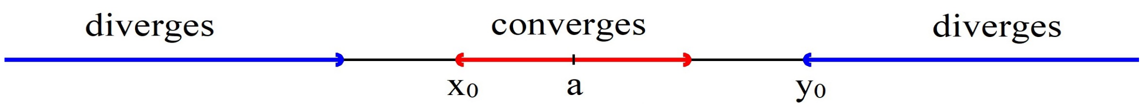
From the above examples, we observe that the region where the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is convergent has always turned out to be an interval (e.g. $\{a\}$, finite interval, $(-\infty, \infty)$ etc).

Question: Is the set where a power series converges always an interval (including the case that converges at a single point)?

Theorem 11.8.10. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$,

(a) if the series converges at $x_0 \neq a$, then it converges absolutely at every x with $|x-a| < |x_0-a|$.

(b) if the series diverges at y_0, then it diverges at every x with $|x-a| > |y_0-a|$.



Proof.

(a) Since $\sum_{n=0}^{\infty} c_n(x_0-a)^n$ converges, we have $\lim_{n\to\infty} |c_n(x_0-a)^n| = 0$. Thus, there exists $N \in \mathbb{N}$ such that for every $n > N$ such that $|c_n(x_0-a)^n| < 1$.

Let x satisfy $|x-a| < |x_0-a|$. Since $\left|\frac{x-a}{x_0-a}\right| < 1$, the series $\sum_{n=N+1}^{\infty} \left|\frac{x-a}{x_0-a}\right|^n$ converges. Also,

$$|c_n(x-a)^n| = |c_n(x_0-a)^n|\left|\frac{x-a}{x_0-a}\right|^n < \left|\frac{x-a}{x_0-a}\right|^n \qquad \text{for } n > N.$$

By the comparison test, the series $\sum_{n=N+1}^{\infty} |c_n(x-a)^n|$ conveges and hence $\sum_{n=1}^{\infty} |c_n(x-a)^n|$ also converges.

(b) Let z_0 be a number such that $|y_0-a| < |z_0-a|$. Assume that the series $\sum_{n=0}^{\infty} c_n(z_0-a)^n$ converges. By part(a), for every x with $|x-a| < |z_0-a|$, the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges. Hence the series $\sum_{n=0}^{\infty} c_n(y_0-a)^n$ converges. It contradicts the hypothesis that the series $\sum_{n=0}^{\infty} c_n(y_0-a)^n$ diverges. Therefore, $\sum_{n=0}^{\infty} c_n(z_0-a)^n$ must diverges.

Since z_0 is an arbitrary number with $|y_0-a| < |z_0-a|$, part(b) is proved.

□

Theorem 11.8.11. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilities:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R *such that the series conveges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Proof. (Exercise) □

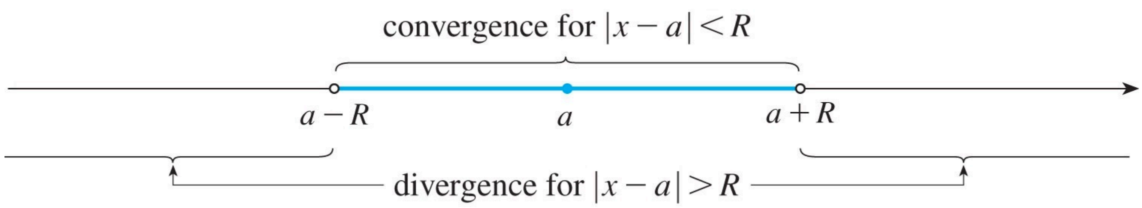
Note.

(a) The number R in part(c) of Theorem 11.8.11 is called the "*radius of convergence*".

(b) By convention, we define the radius of convergence as $R = 0$ in part(a), and as $R = \infty$ in part(b).

(c) The interval which consists of all values of x for which the series converges is called the "*interval of convergence*" of the power series.

(d) In order to find the interval of convergence in part(c) if the radius of convergence is obtained, we still need to consider the endpoints of the interval. That is, to consider whether the series converges at the endpoints $x = a - R$ and $x = a + R$. All situations would occur. Hence, the interval of convergence could be $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$ or $[a-R, a+R]$.



Example 11.8.12.

Question: How to find the radius of convergece for a given power series? What is the connection between the coefficients and the radius of convergence?

Suppose that $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$. Let $a_n = c_n(x-a)^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}}{c_n}\right||x-a| \longrightarrow L|x-a| \qquad \text{as } n \to \infty.$$

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

By the ratio test,

$$\text{if } L|x-a| < 1 \iff |x-a| < \frac{1}{L}, \text{ then the series } \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is convergent;}$$

$$\text{if } L|x-a| > 1 \iff |x-a| > \frac{1}{L}, \text{ then the series } \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is divergent.}$$

Hence, the radius of convergence of the series is $R = \frac{1}{L}$ where $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$.

Example 11.8.13. Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Proof. Let $a_n = \frac{x^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{|x|}{n+1}.$$

Hence, for every $x \in \mathbb{R}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$. The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every $x \in \mathbb{R}$. The radius of convergence is ∞ and the interval of convergence is $\mathbb{R}$. □

Example 11.8.14. Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} n^n x^n$.

Proof. For every $x \neq 0$, if $n \in \mathbb{N}$ and $n > \frac{2}{|x|}$, then $|nx| > 2$. Hence,

$$\lim_{n\to\infty} |n^n x^n| = \lim_{n\to\infty} |nx|^n \geq \lim_{n\to\infty} 2^n = \infty.$$

By the test for divergence, the series $\sum_{n=0}^{\infty} n^n x^n$ diverges at every $x \in 0$. The radius of convergence is 0 and the interval of convergence is $\{0\}$.

□

Example 11.8.15. Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Proof. Let $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n x^n}{\sqrt{n+1}}}\right| = 3\sqrt{\frac{n+1}{n+2}} \longrightarrow 3|x| \quad \text{as } n \to \infty.$$

By the Ratio Test,

(1) When $3|x| < 1 \iff |x| < \frac{1}{3}$, the power series is convergent.

(2) When $3|x| > 1 \iff |x| > \frac{1}{3}$, the power series is divergent.

(3) At the endpoints,

(i) if $x = \frac{1}{3}$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergnet by the alternating series test.

(ii) if $x = -\frac{1}{3}$, the series is $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent (p-series, $p = \frac{1}{2} < 1$).

Hence, the radius of convergence is $\frac{1}{3}$ and the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$. □

Example 11.8.16. Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Proof. Let $a_n = \frac{n(x+2)^n}{3^{n+1}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\frac{n(x+2)^n}{3^{n+1}}}\right| = \frac{n}{3(n+1)}|x+2| \longrightarrow \frac{1}{3}|x+2| \quad \text{as } n \to \infty.$$

By the Ratio Test,

(1) When $\frac{1}{3}|x+2 < 1 \iff |x+2| < 3$, the power series is convergent.

(2) When $\frac{1}{3}|x+2 > 1 \iff |x+2| > 3$, the power series is divergent.

(3) At the endpoints, consider $\frac{1}{3}|x+2| = 1 \iff |x+2| = 3$.

(i) If $x = 1$, the series is $\frac{1}{3}\sum_{n=0}^{\infty} n = \infty$ is divergent.

(ii) If $x = -5$, the series is $\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$ is divergent by the test for divergence.

Hence, the radius of convergence is 3 and the interval of convergence is $(-5, 1)$. □

Remark.

(i) The Ratio Test (or Root Test) do not apply for the endpoints of the interval of convergence.

(ii) Theorem 11.8.11 is false for general series $\sum_{n=0}^{\infty} f_n(x)$.

Homework 11.8. 7, 10, 13, 17, 21, 24, 26, 31, 35, 37, 39

11.9 Representations of Functions as Power Series

Motivation: Many functions have no elementary antiderivatives or it is difficult to solve differential equations, or the approximation of them are difficult to find. We hope to express those functions as sums of power series and do the differentiation or integration on the power series rather than dealing with the original functions.

Difficulties:

(1) What kinds of functions can be expressed as a powe series?

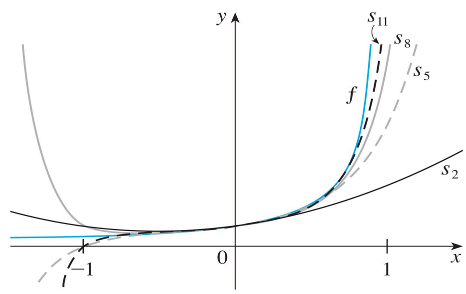
(2) If a function can be expressed as a power series, can we do the differentiation or integration on the power series "term by term"?

Example 11.9.1.

Consider the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$. If we regard the series as a geometric series with ratio x, then the series diverges when $|x| > 1$ and converges when $|x| < 1$. Moreover,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for } |x| < 1. \tag{11.1}$$

Hence, the power series is regarded as expressing the function $f(x) = \frac{1}{1-x}$.



Note. Observe that the domain of $f(x) = \frac{1}{1-x}$ is $\mathbb{R}\backslash\{1\}$ but the domain of the series $\sum_{n=0}^{\infty} x^n$ is $(-1, 1)$. This says that a power series representation of a function may equal this function only on a proper subset of its domain rather than the whole domain.

Question: For a given function,

(1) does it have a power series representation?

(2) If yes, for what values of x does $f(x)$ equal $\sum_{n=0}^{\infty} c_n x^n$?

(3) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, can we take differentation or integration on the power series term-by-term?

Example 11.9.2. Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Proof. Consider $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$. Replacing x by $-x^2$ in Equation (11.1), we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

The geometric series converes when $|-x^2| < 1$. Thus, the interval of convergence is $(-1, 1)$. □

Example 11.9.3. Find the power series representation of $\frac{1}{x+2}$.

Proof. Consider $\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})}$. Replacing x by $-\frac{x}{2}$ in Equation (11.1), we have

$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{x}{2})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n.$$

The power series converges when $|-\frac{x}{2}| < 1$. The interval of convergence is $(-2, 2)$. □

Example 11.9.4. Find a power series representation of $\frac{x^3}{x+2}$.

Proof. The power series representation is

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}.$$

The interval of convergence is $(-2, 2)$.

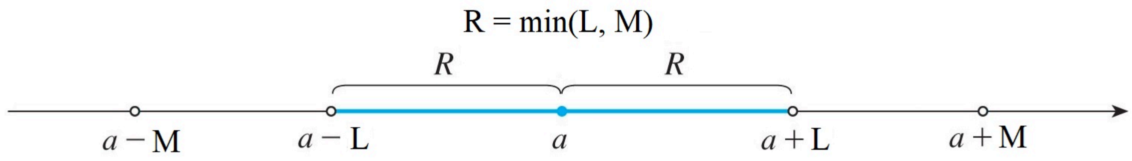
□

❑ Operations on Power Series

Theorem 11.9.5. *Let* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ *with the intervals of convergence* $(a-L, a+L)$ *and* $(a-M, a+M)$ *respectively. Let* $R = \min(L, M)$. *Then*

(1) $\big(f \pm g\big)(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n$ *on* $(a-R, a+R)$.

(2) $\big(f \cdot g\big)(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *on* $(a-R, a+R)$ *where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.



❑ Differentiation and Integration of Power Series

Theorem 11.9.6. *Let* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *with the radius of convergence* $R > 0$. *Then* f *is differentiable (and therefore continuous) on* $(a-R, a+R)$ *and*

(1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(2)

$$\begin{aligned}\int f(x)\,dx &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + \frac{c_n}{n+1}(x-a)^{n+1} + \cdots \\ &= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.\end{aligned}$$

The radii of convergence of the power series of above equations are both R.

Remark. (1)

$$\begin{aligned}\frac{d}{dx}\Big[\sum_{n=0}^{\infty} c_n(x-a)^n\Big] &= \sum_{n=0}^{\infty}\frac{d}{dx}\Big[c(x-a)^n\Big] \\ \int \sum_{n=0}^{\infty} c_n(x-a0^n\,dx &= \sin_{n=0}^{\infty}\int c_n(x-a)^n\,dx\end{aligned}$$

(2) The radius of convergence remains the same when a power series is differentiated or integrated. But it does NOT mean that the interval of convergence remains the same. (For example, $f(x) = \tan^{-1} x$)

(3) A powerful method to solve differential equations.

Example 11.9.7. (Bessel function) The function

$$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \text{is defined for all } x \in \mathbb{R}$$

Then

$$J_0'(x) = \frac{d}{dx}\Big[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\Big] = \sum_{n=0}^{\infty}\frac{d}{dx}\Big[\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\Big] = \sum_{n=1}^{\infty}\frac{(-1)^n 2n x^{2n-1}}{2^n(n!)^2} \quad \text{on } \mathbb{R}.$$

Example 11.9.8. Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$. What is the radius of convergence?

Proof. Since $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$,

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\Big[\frac{1}{1-x}\Big] = \frac{d}{dx}\Big[\sum_{n=0}^{\infty} x^n\Big] = \sum_{n=1}^{\infty}\frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n x^{n-1} \Big(= \sum_{n=0}^{\infty}(n+1)x^n\Big)\\ &= 1 + 2x + 3x^2 + \cdots .\end{aligned}$$

The radius of convergence of the power series of $\frac{1}{(1-x)^2}$ is 1 which is the same as the radius of convergence of the power series of $\frac{1}{1-x}$. □

Example 11.9.9. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Proof. Since $\ln(1+x) = \int \frac{1}{1+x}\,dx$ and $\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$,

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

To determine C, taking $x = 0 \in (-1, 1)$, we have $0 = \ln(1+0) = C$ and hence

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}.$$

Since the radius of convergence of the series for $\frac{1}{1+x}$ is 1, the radius of convergence of the series for $\ln(1+x)$ is also 1. □

Example 11.9.10. Find a power series representation for $f(x) = \tan^{-1} x$.

Proof. Since $f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ on $|x| < 1$, we have

$$f(x) = \tan^{-1} x = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}.$$

To determine C, taking $x = 0$, we have $0 = \tan^{-1} 0 = C$ and hence

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

Since the radius of convergence of the series for $\dfrac{1}{1+x^2}$ is 1, the radius of convergence of the series for $\tan^{-1} x$ is also 1. □

Note. In fact, the power series representation is also true when $x = \pm 1$. But this result is not given by the above theorem.

Example 11.9.11. Express $\dfrac{\pi}{4}$ as a series.

Proof. From Example 11.9.10,

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

In fact, $\dfrac{\pi}{4}$ has several different series representations. For example,

$$\begin{aligned} \frac{\pi}{4} &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \\ &= \left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{7}\left(\frac{1}{2}\right)^7 + \cdots\right] + \left[\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \frac{1}{7}\left(\frac{1}{3}\right)^7 + \cdots\right] \end{aligned}$$

□

Note. If we use the idnetity $\pi = 48\tan^{-1}\frac{1}{18} + 32\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239}$ to approximate π, it will give more rapid rate of convergence than the above series representation since $\frac{1}{18}$, $\frac{1}{57}$ and $\frac{1}{239}$ are much smaller than $\frac{1}{2}$ and $\frac{1}{3}$. This implies that the reminder of the former decays to zero much more rapidly than the one of latter.

Example 11.9.12. (a) Evaluate $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series

(b) Approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}.

Proof. (a) Since $\dfrac{1}{1+x^7} = \dfrac{1}{1-(-x^7)} = \displaystyle\sum_{n=0}^{\infty}(-x^7)^n = \sum_{n=0}^{\infty}(-1)^n x^{7n}$ for $|x < 1$, we have

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1} \quad \text{for } |x| < 1.$$

(b)

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} x^{7n+1}\Big|_0^{0.5} = \sum_{n=0}^{\infty}(-1)^n \frac{(0.5)^{7n+1}}{7n+1}.$$

By the alternating series estimation, for $\displaystyle\sum_{n=0}^{\infty}(-1)b_n$ with $b_n > 0$, the estimate of remainder $|R_n| < b_{n+1}$. Hence, for $b_n = \dfrac{(0.5)^{7n+1}}{7n+1} < 10^{-7}$, we have $n \geq 4$.

Therefore,

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374.$$

□

Remark. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$. Then $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^n$ also converges for $|x-a| < R$. Hence $f'(x)$ has a power series representation on $(x-R, x+R)$. We can also take term-by-term differentiation and obtain

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)(x-a)^{n-2} \qquad \text{converges on } (a-R, a+R)$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)(x-a)^{n-k} \qquad \text{converges on } (a-R, a+R).$$

Homework 11.9. 7, 10, 13, 15, 19, 22, 27, 30, 31, 38, 40(a), 49

11.10 Taylor and Maclaurin Series

So far, we can find power series representations for a centain restricted class of functions.

Question: Which functions do have power series representations?

Suppose that f has a power series representation

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{for } |x-a| < R$$

Question: what are the coefficients c_n?

By the term-by-term differentiation, we can take $\frac{d^k}{dx^k}$ on f and obtain

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n(x-a)^{n-k}.$$

Plugging $x = a$ into the equation, we have

$$c_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, \cdots.$$

Note. For the sake of conventions, we denote $0! = 1$ and $f^{(0)} = f$.

Definition 11.10.1. (a) Let f be a function with infinitely many times derivatives at a, that is, $f'(a), f''(a), \cdots, f^{(k)}(a), \cdots$ exist for $k = 1, 2, \cdots$. Then the series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the "*Taylor series for f at a*" (or "*Taylor series for f about a*" or "*Taylor series for f centered at a*").

(b) For the special case $a = 0$, the Taylor series at 0, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is also called the "*Maclaurin series for f*".

Note. If f can be represented as a power series about a with radius of convergence $R > 0$, then f is equal to the sum of its Taylor series about a.

Example 11.10.2. Find the Taylor series for the following functions at the given points.

(1) $f(x) = e^x$ at $x = 0$.

Proof. Since $f^{(k)}(x) = e^x$, we have $f^{(k)}(0) = 1$ for $k = 0, 1, 2, \cdots$. Hence, the Taylor series for f at 0 (Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

Moreover, let $a_n = \frac{x^n}{n!}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \longrightarrow 0 < 1$ as $n \to \infty$ for every x. By the Ratio Test, the Taylor series converges for all x and the radius of convergence is ∞. □

(2) $f(x) = \sin x$ at $x = 0$.

Proof. For $k \in \mathbb{N}$,

$$\begin{array}{llll} f^{(4n)}(x) = \sin x, & f^{(4n+1)}(x) = \cos x, & f^{(4n+2)}(x) = -\sin x, & f^{(4n+3)}(x) = -\cos x \\ f^{(4n)}(0) = 0, & f^{(4n+1)}(0) = 1, & f^{(4n+2)}(0) = 0, & f^{(4n+3)}(0) = -1 \end{array}$$

The Taylor series for f at 0 (Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Let $a_n = \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}}\right| = \left|\frac{x^2}{(2n+1)(2n+2)}\right| \longrightarrow 0 \quad \text{for all } x.$$

Therefore, the Taylor series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ converges for all $x \in \mathbb{R}$ and the radius of convergence is ∞. □

■ When is a Function Represented by Its Taylor Series?

Note. Suppose that the functions $f(x) = e^x$ or $f(x) = \sin x$ has power series representation. Then we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{or} \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

By the definition of Taylor series, as long as a function f has infinitely many derivatives at a, the Taylor series for f about a is defined. It is natural to ask the following questions:

Remark. (1) As long as f has derivatives of all orders at a, then its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ exists.

(2) If f can be represented as a power series, then its power series representation mush be its Taylor series.

(3) There are examples that a function is not equal to its Taylor series at all points except the center. We usually determine whether and where a Taylor series converges by using the Ratio test or Root test. Even if the Taylor series for f about a converges at some number $x \neq a$, it may not converge to $f(x)$. For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We can evaluate that $f(0) = f'(0) = f''(0) = \cdots = f^{(k)}(0) = \cdots = 0$. Hence, the Taylor series for f at 0 is the zero function which does not converge to f except at the center 0.

Question:

(i) What values of x for which the Taylor sereis is convergent or divergent?

(ii) If the Taylor series converges at x, does it converge to $f(x)$? That is, $f(x) \overset{??}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Consider

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \Big(= \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{T_n(x)} \Big)$$

means that $f(x)$ is equal to the limit of the partial sum. Define

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

We call T_n "*nth-degree Taylor polynomial of f at a*". Let

$$R_n(x) = f(x) - T_n(x)$$

be the "*remainder of the Taylor series*". To check whether the Taylor series converges to f, we have

$$f(x) = \lim_{n\to\infty} T_n(x) \quad \text{if and only if} \quad \lim_{n\to\infty} R_n(x) = \lim_{N\to\infty} \left[f(x) - T_n(x)\right] = 0.$$

Theorem 11.10.3. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

■ Taylor Theorem

Theorem 11.10.4. *(Taylor Theorem) Let $f(t)$ be a $n+1$ times differentiable function on $[a, x]$ and $R_{n,a}(x)$ be defined by*

$$f(x) = P_{n,a}(x) + R_{n,a}(x).$$

Then

(a) (Cauchy form)

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a) \qquad \text{for some } \xi \in (a, x).$$

(b) (Lagrange form)

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } \xi \in (a, x).$$

(c) (Integral form)

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$$

By using the part(b) of Taylor Theorem, we can derive the Taylor inequality

Lemma 11.10.5. *(Taylor Inequality) Let $f(x)$ be a $(n+1)$ times differentiable function and $|f^{(n+1)}(x)| \le M$ for all $|x-a| \le d$. Then the remainder $R_n(x)$ of the Taylor series satisfies the inequaltiy*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d.$$

Example 11.10.6. Determine whether the equality $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ holds. If yes, for what values of x does the equality hold?

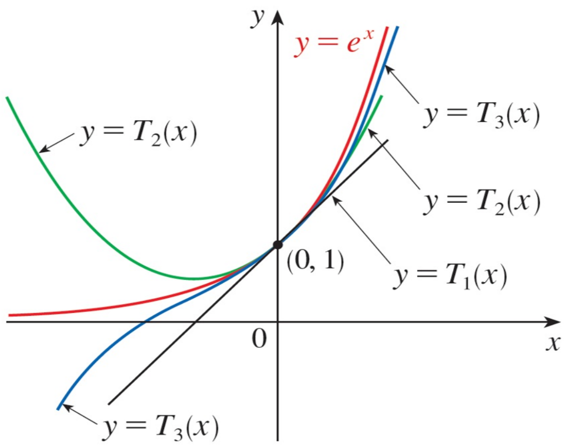
Proof. Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$ and

$$f(x) = e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x).$$

Fix a number x_0 and choose a number $d \geq |x_0|$. Then $|f^{(n+1)}(z)| \leq e^{|z|} \leq e^d$ for all $0 \leq |z| \leq |x_0| \leq d$. By the Taylor inequality,

$$0 \leq |R_n(x_0)| \leq \frac{e^d}{(n+1)!}|x_0 - 0|^{n+1} \leq e^d \frac{d^{n+1}}{(n+1)!}.$$

By the Squeeze Theorem, $\lim_{n\to\infty} |R_n(x_0)| = 0$. Hence, the Taylor series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x at x_0. Since x_0 is an arbitrary number in $\mathbb{R}$, the Taylor series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for every number in $\mathbb{R}$.



□

Remark. $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for every $x \in \mathbb{R}$. Taking $x = 1$, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Example 11.10.7. Find the Taylor series for $f(x) = e^x$ at $a = 2$, and determine whether and for what values of x, $f(x)$ equals its Taylor series about $a = 2$.

Proof. Since $f^{(n)}(x) = e^x$, $f^{(n)}(2) = e^2$. The Taylor series for f at $a = 2$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

- To determine for which values of x the Taylor series conveges.

  Let $a_n = \frac{e^2}{n!}(x-2)^n$. Then for every $x \in \mathbb{R}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{e^2}{(n+1)!}(x-2)^{n+1}}{\frac{e^2}{n!}(x-2)^n}\right| = \frac{1}{n+1}|x-2| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

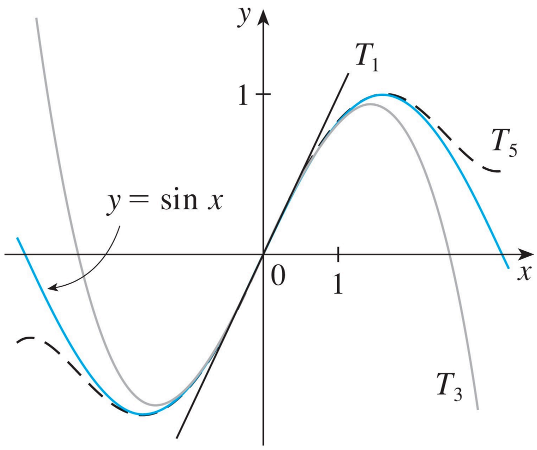
- To determine whether $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$.

  Fix a number $d > 0$. By the Taylor theorem, for x with $|x-2| < d$, there exists z_x between 2 and x such that
  $$R_{n,2}(x) = \frac{f^{(n+1)}(z_x)}{(n+1)!}|x-2|^{n+1} = \frac{e^{z_x}}{(n+1)!}|x-2|^{n+1}.$$
  Hence, for $|x-2| < d$,
  $$0 \le |R_{n,2}(x)| \le \frac{e^{2+d}}{(n+1)!}|x-2|^{n+1} \le e^{2+d}\frac{d^{n+1}}{(n+1)!}.$$
  By the Squeeze Theorem, $\lim_{n\to\infty} R_{n,2}(x) = 0$ for every $|x-2| < d$ and this imiplies that $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$ for every $|x-2| < d$. Sicne d is arbitrary number, we have
  $$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for every } x \in \mathbb{R}.$$
  The radius of convergence of the series is ∞.

□

Example 11.10.8. Find the Maclaurin series for $f(x) = \sin x$ and prove that it represents $\sin x$ for all x.

Proof. The derivatives of f are
$$f^{(4k)}(x) = \sin x,\ f^{(4k+1)}(x) = \cos x,\ f^{(4k+2)}(x) = -\sin x,\ f^{(4k+3)}(x) = -\cos x.$$
Then
$$f^{(4k)}(0) = 0,\ f^{(4k+1)}(0) = 1,\ f^{(4k+2)}(0) = 0,\ f^{(4k+3)}(0) = -1.$$
The Maclaurin series for $\sin x$ is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$
Since $|f^{(n+1)(x)}| = |\pm\sin x|$ or $|\pm\cos x| \le 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have
$$|R_{n,0}(x) \le \frac{1}{(n+1)!}|x|^{n+1}.$$
Hence, for every fixed x, $R_{n,0}(x) \to 0$ as $n \to \infty$. That is,
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad \text{for all } x \in \mathbb{R}.$$



□

Example 11.10.9. Prove that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$.

Proof.

$$\begin{aligned} \cos x &= \frac{d}{dx}(\sin x) = \frac{d}{dx}\Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\Big) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\Big(\frac{d}{dx} x^{2n+1}\Big) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

□

Exercise. Find the Taylor series for $f(x) = \ln(1 + x)$ and for what values of x the Taylor series converges to $f(x)$.
Answer: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ for $-1 < x \leq 1$.

■ **Binomial Series**

Example 11.10.10. (Binomial Series) Use the Maclaurin series for $f(x) = (1 + x)^k$ to deduce the formula of the binomial series where k is any real number.

Proof. The derivatives of f is

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n} \quad \text{for } n = 1, 2, \cdots.$$

Then

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1) \quad \text{for } n = 1, 2, \cdots.$$

The Maclaurin series for $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n \qquad \text{(binomial series)}$$

□

Note. (1) (Convergence)

(i) For $k \in \mathbb{N}$, $k - n + 1 = 0$ when $n = k + 1$. Then the binomial series is a finite sum and a k degree polynomial. Therefore, the series converges for all x.

(ii) For $k \in \mathbb{R} \setminus \mathbb{N}$, let $a_n = \dfrac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$. Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|k-n|}{n+1}|x| = \frac{|1-\frac{k}{n}|}{1+\frac{1}{n}}|x| \longrightarrow |x| \quad \text{as } n \to \infty.$$

By the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Question: How about $x = \pm 1$?
Answer: depending on k.

- If $-1 < k \leq 0$, the series converges at 1.
- If $k \geq 0$, the series converges at ± 1.

(2) Denote the coefficients in the binomial series

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad \text{(binomial coefficients)}$$

If $k \in \mathbb{N}$ and $k \geq n$, then $\dbinom{k}{n} = \dfrac{k!}{n!(k-n)!}$.

(3) **The binomial series:** if $k \in \mathbb{R}$ and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty}\binom{k}{n}x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n.\end{aligned}$$

Example 11.10.11. Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Proof. The function $f(x) = \dfrac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}}$. By the binomial series with $k = -\frac{1}{2}$ and replacing x by $-\dfrac{x}{4}$, we have

$$\begin{aligned}f(x) &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\ &= = \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!\,8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!\,8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\,8^n}x^n + \cdots\right]\end{aligned}$$

The series converges when $|-\dfrac{x}{4}| < 1$, that is, on $(-4, 4)$. □

Exercise. Evaluate the sum of the series $\sum_{n=0}^{\infty}(-1)^n\frac{2n+2}{(2n+1)!}$.

Answer: $\sum_{n=0}^{\infty}(-1)^n\frac{2n+2}{(2n+1)!} = \sin 1 + \cos 1$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

■ New Taylor Series from Old

Example 11.10.12. Find the Maclaurin series for the function $f(x) = x\cos x$

Proof. Since $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ for all x, we have

$$x\cos x = x\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n+1} \quad \text{for all } x.$$

□

Example 11.10.13. Find the Maclaurin series for $f(x) = \ln(1+3x^2)$.

Proof. We know that $\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ for every $|x| < 1$. Replacing x by $3x^2$, we have

$$\ln(1+3x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n x^{2n}}{n}.$$

for every $|3x^2| < 1$. That is the representation valid for $|x| < \dfrac{1}{\sqrt{3}}$. □

Example 11.10.14. Find the function represented by the power series $\sum_{n=0}^{\infty}(-1)^n \dfrac{2^n x^n}{n!}$.

Proof. By writing

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

and comparing with the Taylor series of e^x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

we have

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = e^{-2x}.$$

□

Example 11.10.15. Find the sum of the series

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots + \frac{(-1)^{n-1}}{n\cdot 2^n} + \cdots.$$

Proof. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n} = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}.$$

Using the Maclarin series for $\ln(1 + x)$ by taking $x = \frac{1}{2}$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} = \ln(1 + \frac{1}{2}) = \ln \frac{3}{2}.$$

□

■ Multiplication and Divison of Power Series

Recall that if $f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then

$$f(x)g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n \quad \text{where } d_n = \sum_{k=0}^{n} b_k c_{n-k}$$

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} e_n(x-a)^n \quad \text{where } e_n \text{ satisfying } b_n = \sum_{k=0}^{n} c_k e_{n-k}.$$

Example 11.10.16.

(1) Find the first three nonzero terms in the Maclaurin series for $e^x \sin x$.

Proof. Since

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{and} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)^n} x^{2n+1} + \cdots, \end{aligned}$$

we have

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= x + x^2 + \frac{x^3}{3} + \cdots . \end{aligned}$$

□

(2) Find the first three nonzero terms in the Maclaurin series for $\tan x$.

Proof. Since

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \qquad \text{and} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} + \cdots, \end{aligned}$$

we have

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots . \end{aligned}$$

□

Note. One reason that Taylor series are important is that they enable us to integrate functions which we cannot find and express their antiderivatives as elementary functions.

Example 11.10.17.

(1) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

Proof. Since $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$ for any x, we obtain

$$\begin{aligned} \int e^{-x^2}\, dx &= \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} x^{2n}\, dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} x^{2n+1}. \end{aligned}$$

□

(2) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Proof. Consider

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} x^{2n+1}\Big|_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \qquad \text{(alternating series)}\end{aligned}$$

By the alternating series estimation, $|s - \sum_{k=0}^{n} b_n| \leq b_{n+1}$. Consider

$$\left| \frac{(-1)^n}{n!\,(2n+1)} \cdot 1^{2n+1} \right| < 0.001.$$

Then $n \geq 5$ and $\int_0^1 e^{-x^2}\,dx \approx 0.7475$. □

(3) Evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Proof.

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - 1 - x}{x^2} \\ &= \lim_{x\to 0} \left(\frac{1}{2!} + \frac{x}{3!} + \cdots + \frac{x^{n-2}}{n!} + \cdots \right) \\ &= \frac{1}{2}.\end{aligned}$$

Note: we can also obtain the above limit by the L'Hospital's Rule. □

Exercise.

(1) Find the Taylor series for the function $f(x) = \sin^{-1} x$ and find its interval of convergence. (Hint: $\sin^{-1}(x) = \int \frac{1}{\sqrt{1-t^2}\,dt}$ and using the binomial series.)

(2) Express the following functions as their Taylor series and find the limits

(i) $\lim_{x\to 1} \frac{\ln x}{x-1}$.

(ii) $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$.

(iii) $\lim_{x\to 0} \frac{(e^{2x} - 1)\ln(1 + x^3)}{(1 - \cos 3x)^2}$.

Homework 11.10. 4, 6, 10, 11, 16, 18, 23, 28, 30, 35, 37, 39, 43, 47, 56, 59, 62, 65, 69, 72, 73, 74, 83, 90, 96(a)

11.11 Applications of Taylor Polynomials

Notivation:

- computer scientists use Taylor polynomials to approximate functions
- physicists and engineers use Taylor polynomials on the problems of relativity, optics, electric dipoles, the velocity of water waves etc.

■ Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For $n \in \mathbb{N}$, the polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the "*n-degree Taylor polynomials of f at a*".

Recall: Since $T_n(x) \longrightarrow f(x)$ as $n \to \infty$, $T_n(x)$ can be used as an approximation to $f(x) \approx T_n(x)$.

Note.

(1) Consider the 1st-degree Taylor polynomial $T_1(x)$ of f at a.

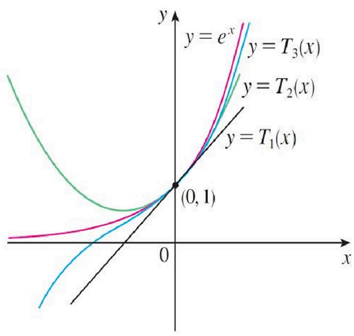
$$T_1(x) = f(a) + f'(a)(x-a)$$
$$T_1(a) = f(a),\ T_1'(a) = f'(a)$$

(2) Consider the nth degree Taylor polynomial $T_n(x)$ of f at a.

$$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$T_n(a) = f(a),\ T_n'(a) = f'(a), \cdots, T_n^{(n)}(a) = f^{(n)}(a)$$

For example, $f(x) = e^x$ and $T_n(x) \to e^x$ as $n \to \infty$.



	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

Question: When using a Taylor polynomial $T_n(x)$ to approximate a function f,

(1) how good approximation is it?

(2) how large should we take n to be in order to achieve a desired accuracy?

Consider the remainder

$$|R_n(x)| = |f(x) - T_n(x)|.$$

There are three possible methods for estimating the size of the remainder:

(1) using the graphing device

(2) using the Alternating Series Estimation Theorem (if it happens)

(3) using Taylor's Inbequality: if $|f^{(n+1)}(x)| \leq M$ for every $|x-a| < d$ then

$$|R_n(x)| < \frac{M}{(n+1)!}d^{n+1}.$$

Example 11.11.1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

Proof. Compute

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, \; f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}, \; f'''(x) = \frac{10}{57}x^{-\frac{8}{3}}$$

Then

$$f(8) = 2, \; f'(8) = \frac{1}{12}, \; f''(8) = -\frac{1}{144}$$

We have

$$T_2(x) = 2 + \frac{1}{12}(x-2) - \frac{1}{144}(x-8)^2.$$

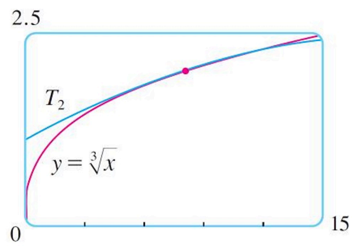
Therefore,

$$\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8) - \frac{1}{144}(x-8)^2.$$

□

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Proof.

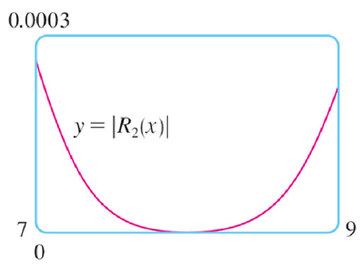


To find a bound M such that $|f'''(x)| \leq M$ for $7 \leq x \leq 9$. Consider

$$|f'''(x)| = \frac{10}{27}|x|^{-\frac{8}{3}} \leq \frac{10}{27} \cdot 7^{-\frac{8}{3}} \qquad \text{for } 7 \leq x \leq 9.$$

Hence,

$$\begin{aligned} |R_2(x)| &\leq \frac{1}{3!} \cdot \frac{10}{27} \cdot 7^{-\frac{8}{3}} |x-8| \quad \text{for } 7 \leq x \leq 9 \\ &\leq \frac{0.0021}{3!} \cdot 1 < 0.0004 \end{aligned}$$



Remark. In fact, $|R_2(x)| = |f(x) - T_2(x)| < 0.0003$ when $7 \leq x \leq 9$.

□

Example 11.11.2. (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? use this approximation to find 12° correct to six decimal places.

Proof. Two methods:

(i) (**Alternating Series**)
When $-0.3 \leq x \leq 0.3$, the series is an alternating series and

$$\underbrace{\frac{|x|^{2k+1}}{(2k+1)!} \leq \frac{|x|^{2k-1}}{(2k-1)!}}_{(b_{k+1} \leq b_k)} \quad \text{and} \quad \underbrace{\frac{|x|^{2k+1}}{(2k+1)!} \to 0}_{(b_k \to 0)} \text{ as } k \to \infty$$

By the alternating series test, for $-0.3 \leq x \leq 0.3$,

$$\left| \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right| \leq \frac{|x|^7}{7!} \leq \frac{(0.3)^7}{7!} \approx 4.3^{-8}.$$

$$\sin 12^\circ = \sin\left(\frac{\pi}{15}\right) \approx \frac{\pi}{15} - \frac{1}{3!}\left(\frac{\pi}{15}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{15}\right)^5 \approx 0.20791169.$$

(ii) (**Taylor's Inequality**)
Let $f(x) = \sin x$. Then $T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the 6th degree Taylor's polynomial for f at 0. The remainder

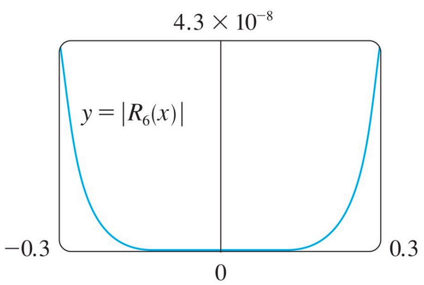
$$\left|R_6(x)\right| \leq \frac{M}{7!}|x|^7$$

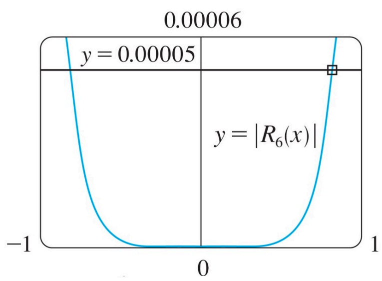
where M is a number such that $\left|f^{(7)}(x)\right| \leq M$ for $-0.3 \leq x \leq 0.3$.
To find M, consider $f^{(7)}(x) = -\cos x$. Thus, when $-0.3 \leq x \leq 0.3$,

$$\left| -\cos x\right| \leq \left|\cos 0\right| = 1 = M.$$

We have

$$\left|R_6(x)\right| \leq \frac{M}{7!}|x|^7 \leq \frac{1}{7!}(0.3)^7 \leq 4.3 \times 10^{-8}.$$





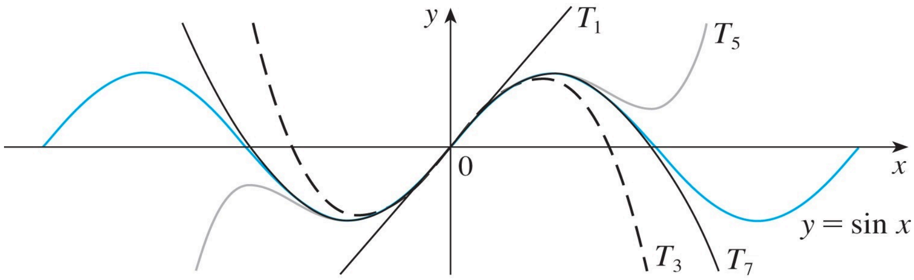
□

(b) For what values of x is this approximation accurate to within 0.00005?

Proof. Consider

$$\left|R_6(x)\right| \le \frac{1}{7!}|x|^7 < 0.00005.$$

We have $|x| \le (0.252)^{1/7} \approx 0.821$.



□

■ Applications to Physics

(Skip)

Homework 11.11. 13(a)(b), 15(a)(b), 18(a)(b), 21(a)(b), 25, 28, 30, 37

Chapter 13

Vector Functions

In this chapter, we will use the vector-valued functions to describe curves, surfaces and the motion of objects through space.

13.1 Vector Functions and Space Curves

As we know, we can regard $\mathbb{R}^n$ as a n-dimensional vector space. Every element in $\mathbb{R}^n$ can be expressed as a vector $\mathbf{a} = \langle a_1, \cdots, a_n\rangle$. In this chapter, we consider the function whose range consisting of vectors in 3-dimensional vector space $\mathbb{R}^3$.

Definition 13.1.1. A vector-valued function (vector function) is a function whose domain is a set of real numbers and whose range is a set of vectors

$$\mathbf{r}(t) : \{\text{subset in } \mathbb{R}\} \longrightarrow \{\text{set of vectors}\}.$$

Note. In the present chapter, we will focus the vector function $\mathbf{r}(t)$ whose values are three-dimensional vectors.

We recall the expressions of vectors and vector-valued functions.

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{r}(t) &= \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \end{aligned}$$

where $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$, and $f, g, h : \mathbb{R} \to \mathbb{R}$ are component functions

Example 13.1.2. Let $\mathbf{r}(t) = \langle 2t^2, 3t - 4, \sqrt{t}\rangle$ be a vector-valued function. The domain of $\mathbf{r}(t)$ is $[0, \infty)$.

❑ Limits and Continuity

■ Limits of Vector-valued Functions

To study the calculus of vector-valued functions, motivated by the concepts of real-valued functions, we will discuss the limits and continuity of vector-valued functions. We heuristically consider that

(i) a limit of a vector valued function is supposed to be a vector; and

(ii) if $\mathbf{L}$ is the limit of a vector valued function $\mathbf{r}(t)$ as $t \to a$, we expect that $\mathbf{r}(t)$ arbitrarily approaches to $\mathbf{L}$ by taking t arbitrarily close to a.

Definition 13.1.3. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector valued function defined on an open interval I and $a \in I$. We say that the limit of $\mathbf{r}(t)$ exists, as t approaches a if there exists a vector $\mathbf{L} = \langle L_1, L_2, L_3\rangle$ such that

$$\lim_{t\to a} f(t) = L_1, \ \lim_{t\to a} g(t) = L_2 \text{ and } \lim_{t\to a} h(t) = L_3.$$

The vector $\mathbf{L}$ is called the "*limit of* $\mathbf{r}(t)$ *as t arpproaches a*" and we write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}.$$

Note. Suppose that $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Then the limit $\lim_{t\to a} \mathbf{r}(t)$ exists if and only if all the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ and $\lim_{t\to a} h(t)$ exist. Moreover,

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a}\langle f(t), g(t), h(t)\rangle = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle.$$

Example 13.1.4. Suppose that $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \dfrac{\sin t}{t}\mathbf{k}$. Then

$$\lim_{t\to 0} \mathbf{r}(t) = \left[\lim_{t\to 0}(1 + t^3)\right] \mathbf{i} + \left[\lim_{t\to 0} te^{-t}\right] \mathbf{j} + \left[\lim_{t\to 0} \frac{\sin t}{t}\right] \mathbf{k} = \mathbf{i} + \mathbf{k}.$$

■ **Laws of Limts**

Theorem 13.1.5. *Let* $\mathbf{r}(t)$ *and* $\mathbf{s}(t)$ *be vector valued functions defined on I, u be a real-valued function defined on I and α be a number. Suppose that*

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}, \ \lim_{t\to a} \mathbf{s}(t) = \mathbf{M} \quad \text{and} \quad \lim_{t\to a} u(t) = c.$$

Then

(a) $\lim_{t\to a} \left(\mathbf{r} \pm \mathbf{s}\right)(t) = \mathbf{L} \pm \mathbf{M}$.

(b) $\lim_{t\to a} \alpha\mathbf{r}(t) = \alpha\mathbf{L}$.

(c) $\lim_{t\to a} \mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{L} \cdot \mathbf{M}$.

(d) $\lim_{t\to a} u(t)\mathbf{r}(t) = c\mathbf{L}$.

(e) $\lim_{t\to a} \mathbf{r}(t) \times \mathbf{s}(t) = \mathbf{L} \times \mathbf{M}$.

Proof. (Exercise) □

■ Continuity of Vector-valued Functions

Definition 13.1.6. Let $\mathbf{r}(t)$ be a vector-valued function defined on $I \subseteq \mathbb{R}$ and $a \in I$. We say that

(1) $\mathbf{r}$ is continuous at a if
$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

(2) $\mathbf{r}$ is continuous on I if $\mathbf{r}$ is continuous at every point of I.

Note. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at a, then
$$\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t)\rangle = \lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a) = \langle f(a),\ g(a),\ h(a)\rangle.$$

We have
$$\lim_{t\to a} f(t) = f(a), \quad \lim_{t\to a} g(t) = g(a) \quad \text{and} \quad \lim_{t\to a} h(t) = h(a)$$
Thus, $\mathbf{r}(t)$ is continuous at a if and only if all its component functions f, g and h are continuous at a.

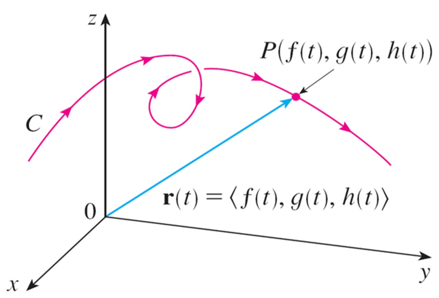
Theorem 13.1.7. *Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be vector valued functions defined on I, u be a real-valued function defined on I and α be a number. Suppose that $\mathbf{r}$, $\mathbf{s}$ and u are continuous at a. Then $\mathbf{r} \pm \mathbf{s}$, $\alpha\mathbf{r}$, $u\mathbf{r}$, $\mathbf{r}\cdot\mathbf{s}$ and $\mathbf{r}\times\mathbf{s}$ are continuous at a.*

Proof. Exercise. □

❏ Space Curves

Consider the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. The tip of $\mathbf{r}(t)$ is the point $P\big(f(t), g(t), h(t)\big)$ and $\mathbf{r}(t)$ is the position vector of the point $P\big(f(t), g(t), h(t)\big)$.

As t ranges over an interval I, the point P traces out some path C in $\mathbb{R}^3$. That is,
$$C = Range\big(\mathbf{r}(t)\big), \quad t \in I.$$



C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Definition 13.1.8. Let $f(t)$, $g(t)$ and $h(t)$ be three functions defined on an interval I. The set C of all points (x, y, z) in space, where
$$x = f(t), \quad y = g(t), \quad z = h(t) \quad \text{for } t \in I \tag{13.1}$$
is called a "*space curve*".

Note.

(1) The equation (13.1) is called the "*parametric equation of C*" and t is called a "*parameter*".

(2) The space curve C is "*oriented*" in the direction as t increases.

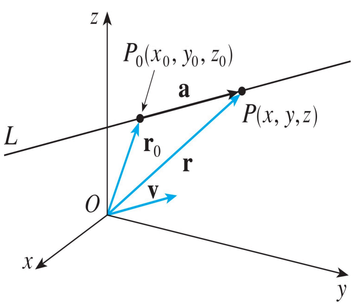
Example 13.1.9. Describe the curve defined by the vector function

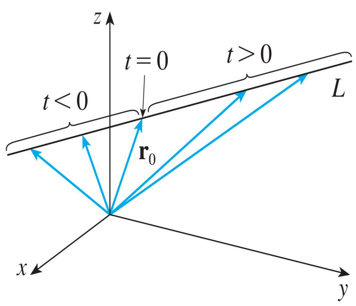
$$\mathbf{r}(t) = \langle 5 + t, 1 + 4t, 3 - 2t \rangle$$

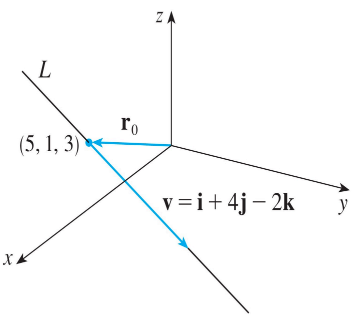
Proof. From the parametric equation, the coordinates are

$$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t.$$

The curve represents a line passing through $(5, 1, 3)$ and parallel to the vector $\langle 1, 4, -2 \rangle$. Let $\mathbf{r_0} = \langle 5, 1, -3 \rangle$ and $\mathbf{v} = \langle 1, 4, -2 \rangle$. Then $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}$. □







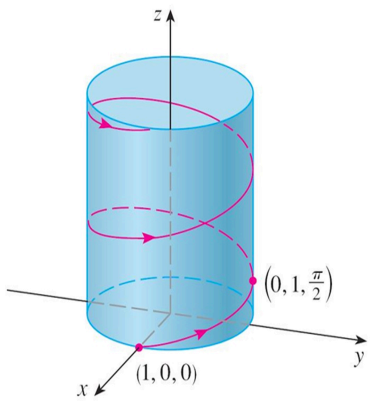
Example 13.1.10. Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j} + t \, \mathbf{k}$$

Proof. The parametric equation represents the curve with coordinates

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

The curve is called a "*helix*". □



Example 13.1.11. Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

Proof.

The line segment joining the tip of $\mathbf{r_0} = \langle 1, 3, -2\rangle$ to the tip of $\mathbf{r_1} = \langle 2, -1, 3\rangle$ is

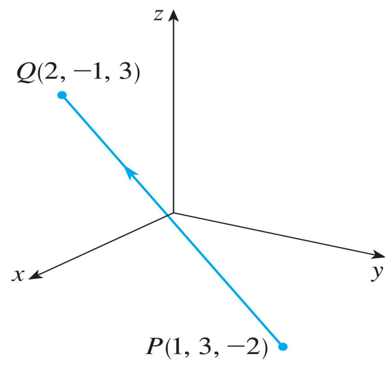
$$\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1}, \quad 0 \le t \le 1.$$

The vector equation of the line segment is

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\langle 1, 3, -2\rangle + t\langle 2, -1, 3\rangle \\ &= \langle 1+t, 3-4t, -2+5t\rangle, \quad 0 \le t \le 1.\end{aligned}$$

The parametric equation of the line segment is

$$x = 1 + t, \quad y = 3 - 4t, \quad z = -2 + 5t \qquad 0 \le t \le 1.$$



□

Example 13.1.12. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Proof.

For (x, y, z) on the cylinder $x^2 + y^2 = 1$,

$$x = \cos t, \quad y = \sin t \quad 0 \le t \le 2\pi.$$

Also, for (x, y, z) on the plane $y + z = 2$, $z = 2 - y$. Then for (x, y, z) on the intersection of $x^2 + y^2 = 1$ and $y + z = 2$,

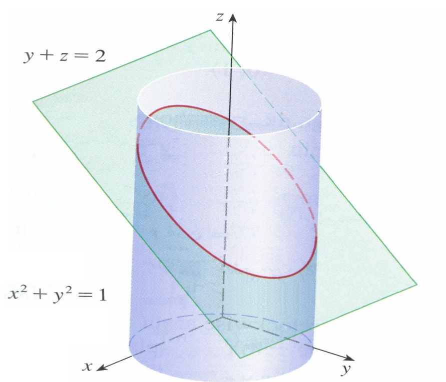
$$z = 2 - y = 2 - \sin t, \quad 0 \le t \le 2\pi.$$

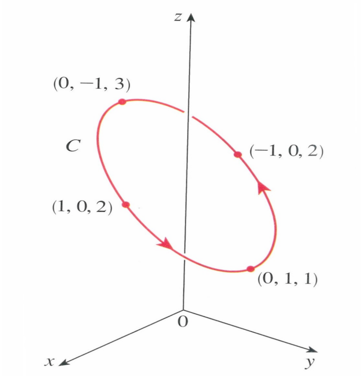
Hnece, the parametric equation for the curve is

$$x = \cos t, \quad y = \sin t, \quad z = 2 - \sin t \quad 0 \le t \le 2\pi.$$

The parametrization of the curve is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2 - \sin t)\,\mathbf{k} \quad 0 \le t \le 2\pi.$$



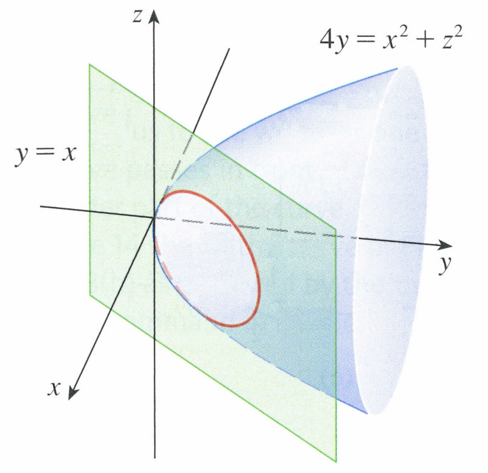


□

Example 13.1.13. Find parametric equations for the curve of intersection of the paraboloid $4y = x^2 + z^2$ and the plane $y = x$.

Proof.

A point on the curve C satisfies the equations of both surfaces. Thus, substituting $y = x$ into the equation of the paraboloid, $4y = x^2 + z^2$, we have $4x = x^2 + z^2$ which is equivalent to $(x-2)^2 + z^2 = 4$. Then the equation of C must contain $(x-2)^2 + z^2 = 4$.



Consider the projection of C on the xz-plane is the curve $(x-2)^2 + z^2 = 4$, $y = 0$ which is a circle with center $(2, 0, 0)$ and radius 2. Therefore, we can write $x = 2 + 2\cos t$, $z = 2\sin t$, $0 \le t \le 2\pi$. Furthermore, since $y = x$ on the curve C, the parametric equation for C is

$$x = 2 + 2\cos t, \quad y = 2 + 2\cot t, \quad z = 2\sin t \qquad 0 \le t \le 2\pi.$$

□

■ **Using Technology to Draw Space Curves**

(Skip)

Homework 13.1. 6, 14, 21, 25-30, 31, 35, 39, 40, 51, 54, 58

13.2 Derivatives and Integrals of Vector Functions

❑ Derivatives

Recall that the derivative of a real-valued function f is defined by

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Let $\mathbf{r}(t)$ be a vector-valued function. Consider

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

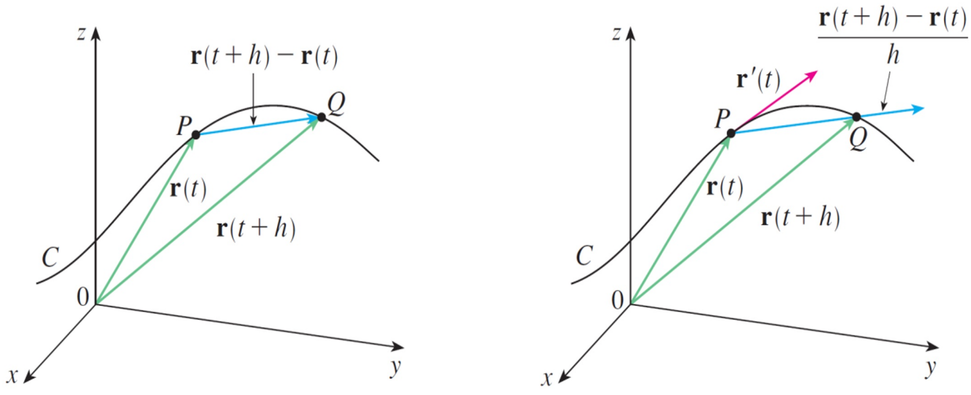
Note. (1) The numerator $\mathbf{r}(t+h) - \mathbf{r}(t) = \overrightarrow{PQ}$ means a secant vector.

(2) The term $\dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$ represents the vector $\dfrac{1}{h}\big(\mathbf{r}(t+h) - \mathbf{r}(t)\big)$ which has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$.

(3) As $h \to 0$, the vector $\dfrac{1}{h}\big(\mathbf{r}(t+h) - \mathbf{r}(t)\big)$ approaches a vector which lies on the tangent line.

Definition 13.2.1. Let $\mathbf{r}(t)$ be a vector function defined on $I \subseteq \mathbb{R}$, C be the curve consisting of the graph of $\mathbf{r}(t)$ and $P = \mathbf{r}(a)$ be a point on C.

(a) We say that $\mathbf{r}(t)$ is differentiable at a if the limit $\displaystyle\lim_{h\to 0} \frac{\mathbf{r}(a+h) - \mathbf{r}(a)}{h}$ exists. The limit is called the "*derivative of* $\mathbf{r}$ *at* a" and denoted by $\mathbf{r}'(a)$. Moreover, we say $\mathbf{r}(t)$ is differentiable on I if it is differentiable at every point in I.



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

(b) If the derivative $\mathbf{r}'(a)$ exists, it is the "*tangent vector*" to the curve C at the point P provided $\mathbf{r}'(a) \neq \mathbf{0}$.

(c) The "tangent line" to C at P is defined to be the line through P parallel to the tangnet vector $\mathbf{r}'(a)$.

(d) The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Note. From the definition of part(c), the parametric equation of the tangent line to C at P is

$$\mathbf{r}(a) + t\mathbf{r}'(a), \quad t \in \mathbb{R}.$$

Theorem 13.2.2. *If* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, *where* f, g *and* h *are differentiable functions, then*

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Proof. (Exercise) □

Example 13.2.3. Suppose that $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(a) The tangent vector function is $\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$.

(b) To find the unit tangent vector at the point where $t = 0$, consider the position vector $\mathbf{r}(0) = \mathbf{i}$ and the tangnet vector $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$. Therefore, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{5}}(\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}.$$

Example 13.2.4. For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

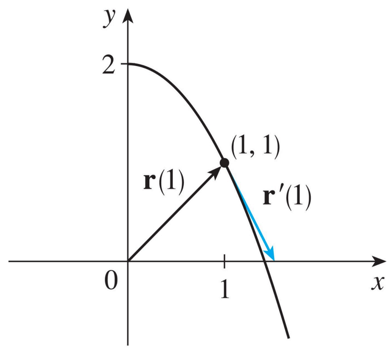
Proof. The tangent vector is $\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\,\mathbf{i} - \mathbf{j}$. Then $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}'(1) = \dfrac{1}{2}\,\mathbf{i} - \mathbf{j}$.

To sketch the position vector and the tangent vector, consider the parametric equation

$$x = \sqrt{t}, \quad y = 2 - t \qquad \Rightarrow \qquad y = 2 - x^2,\ x \geq 0.$$

Then parametric equation of the tangent line to the plane curve at $(1, 1)$ is

$$\ell(t) = \mathbf{r}(1) + t\mathbf{r}'(1) = (\mathbf{i}+\mathbf{j}) + t(\frac{1}{2}\mathbf{i}-\mathbf{j}) = (1+\frac{1}{2}t)\,\mathbf{i} + (1-t)\,\mathbf{j}$$



□

Example 13.2.5. Find parametric equations for the tangent line to the helix with parametric equation

$$x = 2\cos t, \quad y = \sin t, \quad z = t.$$

at the point $(0, 1, \frac{\pi}{2})$.

Proof. The vector function is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle$. Then the tangent vector function is

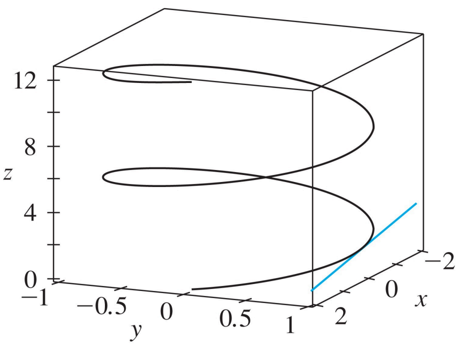
$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle.$$

At the point $(0, 1, \frac{\pi}{2})$, $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle = \langle 0, 1, \frac{\pi}{2}\rangle$. Thus, $t = \frac{\pi}{2}$. The tangent vector is

$$\mathbf{r}'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle.$$

Hence, the parametric equation of the tangent line through $(0, 1, \frac{\pi}{2})$ is

$$x = 0 + (-2)t = -2t, \quad y = 1 + 0t = 1, \quad z = \frac{\pi}{2} + t.$$



□

Theorem 13.2.6. *Suppose that* $\mathbf{r}(t)$ *is differentiable at* a*. Then it is continuous at* a*.*

Proof. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Since $\mathbf{r}(t)$ is differentiable at a, f, g and h are also differentiable at a and hence they are continuous at a. This implies that $\mathbf{r}(t)$ is continuous at a. □

■ Second Derivatives

$$\mathbf{r}(t) \xrightarrow{\frac{d}{dt}} \mathbf{r}'(t) \xrightarrow{\frac{d}{dt}} (\mathbf{r}')' = \mathbf{r}''(t)$$

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle \overset{\frac{d}{dt}}{\Longrightarrow} \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

$$\overset{\frac{d}{dt}}{\Longrightarrow} \mathbf{r}''(t) = \langle f''(t), g''(t), h''(t)\rangle.$$

Similarly, if $\mathbf{r}^{(k)}(t)$ exists, then

$$\mathbf{r}^{(k)}(t) = \langle f^{(k)}(t), g^{(k)}(t), h^{(k)}(t)\rangle.$$

■ **Differentiation Rules**

Theorem 13.2.7. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two differentiable vector functions, c be a number and f be a real-valued function. Then*

(1) $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t).$

(2) $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t).$

(3) $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).$

(4) $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).$ *(real-valued function)*

(5) $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t).$

(6) $\dfrac{d}{dt}[\mathbf{u}\big(f(t)\big)] = f'(t)\mathbf{u}'\big(f(t)\big).$ *(Chain Rule)*

Exercise. Let $\mathbf{r}(t) = \langle e^{3t},\ \sin(t^2),\ 2t^2-t\rangle$, $\mathbf{s}(t) = \langle \dfrac{t^2}{t+1},\ \sec(2t),\ \ln(t^2+1)\rangle$ and $\mathbf{u}(t) = \langle 1,\ t,\ t^2\rangle$. Find $\dfrac{d}{dt}\Big((\mathbf{r}\times\mathbf{s})\cdot\mathbf{u}\Big)$.

Proposition 13.2.8. *Let* $\mathbf{r}(t)$ *be a differentiable vector function on I and* $\mathbf{r}'(t) \neq \mathbf{0}$ *for every* $t \in I$. *Then*

(a) $\dfrac{d}{dt}|\mathbf{r}(t)| = \dfrac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}.$

(b) $\dfrac{d}{dt}\Big(\dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}\Big) = \dfrac{-|\mathbf{r}|'}{|\mathbf{r}|^2}\mathbf{r} + \dfrac{1}{|\mathbf{r}|}\mathbf{r}' \overset{(n=3)}{=} \dfrac{1}{|\mathbf{r}|^2}\big[(\mathbf{r}\times\mathbf{r}')\times\mathbf{r}\big].$

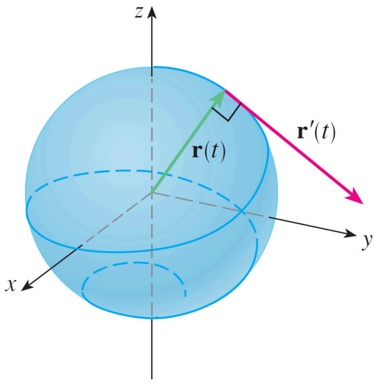
Proof. (Direct computation! We left the proof to the readers as exercise)

□

Remark. The results of Proposition 13.2.8 are true for all n-dimensional vector valued functions except for the last equality of part(b) which is true for 3-dimensional vector valued functions.

Example 13.2.9. Show that if $\big|\mathbf{r}(t)\big| = C$, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Proof.



Since $\mathbf{r}(t)\cdot\mathbf{r}(t) = \left|\mathbf{r}(t)\right|^2 = C^2$ (constant), we have

$$2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \frac{d}{dt}(C^2) = 0$$

Hence, $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t.

For example, $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. □

❏ Integrals

Recall that the integral of a real-valued function $f(t)$ over $[a,b]$ is defined by the limit of Riemann sums.

$$\int_a^b f(t)\,dt = \lim_{|P|\to 0}\sum_{i=1}^n f(t_i^*)\triangle t_i$$

We try to use the same strategy to define the definite integral of vector-valued functions. Let $\mathbf{r}(t)$ be a continuous vector-valued function defined on $[a,b]$. Let $P = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[a,b]$ and $\triangle t_i = |t_i - t_{i-1}|$. Define

$$\begin{aligned}
\int_a^b \mathbf{r}(t)\,dt &= \lim_{|P|\to 0}\sum_{i=1}^n \mathbf{r}(t_i^*)\triangle t_i \\
&= \lim_{|P|\to 0}\left[\sum_{i=1}^n \langle f(t_i^*), g(t_i^*), h(t_i^*)\rangle \triangle t_i\right] \\
&= \lim_{|P|\to 0}\left\langle \sum_{i=1}^n f(t_i^*)\triangle t_i, \sum_{i=1}^n g(t_i^*)\triangle t_i, \sum_{i=1}^n h(t_i^*)\triangle t_i\right\rangle \\
&= \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right\rangle \\
&= \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}
\end{aligned}$$

Definition 13.2.10. Let $\mathbf{r}(t)$ be a vector valued function defined on $[a,b]$ where $\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle$. We say that $\mathbf{r}$ is integrable on $[a,b]$ if f, g,and h are integrable on $[a,b]$ and

$$\begin{aligned}
\int_a^b \mathbf{r}(t)\,dt &= \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right\rangle \\
&= \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.
\end{aligned}$$

Remark. If $\mathbf{r}(t)$ is continuous on $[a,b]$, then $\mathbf{r}(t)$ is integrable on $[a,b]$.

Theorem 13.2.11. *(Integral Rule) Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be integrable vector-valued functions on $[a,b]$, $\mathbf{c}$ be a vector, and α and β be two numbers. Then* *(a) The vector valued function* $\big(\alpha\mathbf{r}+\beta\mathbf{s}\big)(t)$ *is also integrable on* $[a,b]$ *and*

$$\int_a^b \big(\alpha\mathbf{r}+\beta\mathbf{s}\big)(t)\,dt = \alpha\int_a^b \mathbf{r}(t)\,dt + \beta\int_a^b \mathbf{s}(t)\,dt.$$

(b) $\displaystyle\int_a^b \mathbf{c}\cdot\mathbf{r}(t)\,dt = \mathbf{c}\cdot\int_a^b \mathbf{r}(t)\,dt.$

(c) $\displaystyle\left|\int_a^b \mathbf{r}(t)\,dt\right| \le \int_a^b |\mathbf{r}(t)|\,dt.$

Proof. The proofs of part(a) and (b) are easy and left to the readers. We will prove part(c) here.

Let $\mathbf{R} = \displaystyle\int_a^b \mathbf{r}(t)\,dt$. Then

$$\begin{aligned}
|\mathbf{R}|\left|\int_a^b \mathbf{r}(t)\,dt\right| &= |\mathbf{R}|^2 = \mathbf{R}\cdot\mathbf{R} \\
&= \mathbf{R}\cdot\int_a^b \mathbf{r}(t)\,dt = \int_a^b \mathbf{R}\cdot\mathbf{r}(t)\,dt \\
&\le \int_a^b \big|\mathbf{R}\cdot\mathbf{r}(t)\big|\,dt \le \int_a^b |\mathbf{R}||\mathbf{r}(t)|\,dt \\
&= \big|\mathbf{R}\big|\int_a^b \big|\mathbf{r}(t)\big|\,dt.
\end{aligned}$$

Hence,

$$\left|\int_a^b \mathbf{r}(t)\,dt\right| \le \int_a^b |\mathbf{r}(t)|\,dt.$$

□

■ **Fundamental Theorem of Caluclus**

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is $\mathbf{R}'(t) = \mathbf{r}(t)$. Denote

$$\mathbf{R}(t) = \int \mathbf{r}(t)\,dt.$$

Example 13.2.12. Let $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$. Then

$$\int \mathbf{r}(t)\,dt = 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$$

and

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\,dt = 2\sin t\Big|_0^{\frac{\pi}{2}}\mathbf{i} - \cos t\Big|_0^{\frac{\pi}{2}}\mathbf{j} + t^2\Big|_0^{\frac{\pi}{2}}\mathbf{k} = 2\,\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\,\mathbf{k}.$$

Homework 13.2. 3, 9, 12, 15, 19, 21, 24, 27, 30, 36, 39, 44, 51, 57

13.3 Arc Length and Curvature

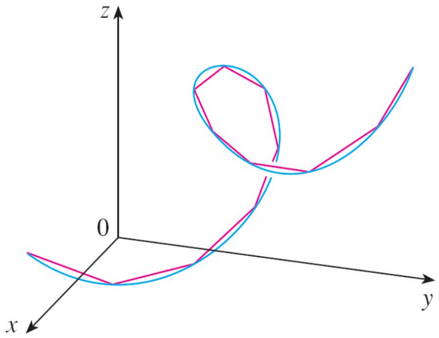
❑ Length of a Curve

In Section **??**, we have learned how to evaluate the arc length of a parametric curve. Let

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

The arc length of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$



The length of a space curve is the limit of lengths of inscribed polygons.

Consider the space curve with the vector equations

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle, \quad a \le t \le b$$

If the curve is traversed exactly once as t increases from a to b, the arc length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Note. (1) If $\mathbf{r}(t)$ is the position vector of an object at time t, then $\mathbf{r}'(t)$ is the velocity vector and $|\mathbf{r}'(t)|$ is the speed.

(2) Since $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$, we have $|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$. The arc length is

$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$

We give a precise proof of formula of arc length here.

Theorem 13.3.1. *Let $\mathbf{r}(t)$ be a continuously differentiable vector function on $[a, b]$. Let C be the curve parametrized by $\mathbf{r}$. The arc length of C is*

$$L(C) = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Proof. Let $P = \{t_0, t_1, \cdots, t_n\}$ be a partitition of $[a, b]$. By the Fundamental Theorem of Calculus,

$$|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| = \left|\int_{t_{i-1}}^{t_i} \mathbf{r}'(t)\, dt\right| \le \int_{t_{i-1}}^{t_i} |\mathbf{r}'(t)|\, dt.$$

Then

$$\sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\mathbf{r}'(t)|\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Since P is an arbitrary partition of $[a, b]$, we have

$$L(C) = \sup_P \sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \le \int_a^b |\mathbf{r}'(t)|\, dt. \tag{13.2}$$

On the other hand, define $s(t)$ as arc length of the curve from $\mathbf{r}(a)$ to $\mathbf{r}(t)$. Then $s(t+h)-s(t)$ is the arc length from $\mathbf{r}(t)$ to $\mathbf{r}(t+h)$. □

By (13.2),

$$|\mathbf{r}(t+h)-\mathbf{r}(t)| \le s(t+h)-s(t) \le \int_t^{t+h} |\mathbf{r}'(u)|\,du.$$

Then, for $h\rangle 0$,

$$\left|\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}\right| = \frac{|\mathbf{r}(t+h)-\mathbf{r}(t)|}{h} \le \frac{s(t+h)-s(t)}{h} \le \frac{1}{h}\int_t^{t+h} |\mathbf{r}'(u)|\,du.$$

By the Fundamental Theorem of Calculus, as $h \to 0$,

$$|\mathbf{r}'(t)| \le \underbrace{\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}}_{=s'(t)} \le |\mathbf{r}'(t)|.$$

Therefore, the arc length of C *is*

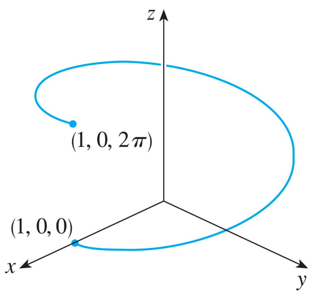
$$s(b) = \int_a^b s'(t)\,dt = \int_a^b |\mathbf{r}'(t)|\,dt.$$

Example 13.3.2. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, from the point $(1,0,0)$ to the point $(1,0,2\pi)$.

Proof.

Compute $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$ and then $|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$. The length of the arc is

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi.$$

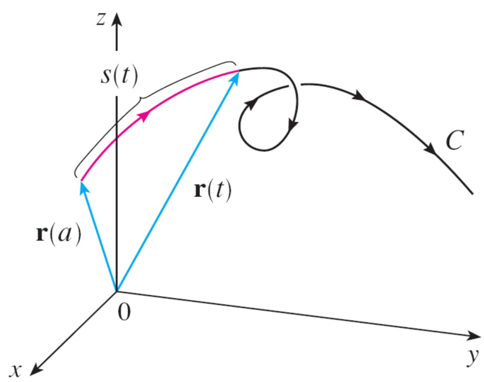


□

■ The Arc Length Function

Let C be a curve with vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, $a \le t \le b$. Suppose that $\mathbf{r}'(t)$ is continuous and C is traversed exactly once as t increases from a to b. The "*arc length function*" is

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$



Note. The value of $s(t)$ is the arc length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

Observe that the arc length function $s(t)$ is one-to-one. Hence, we may also regard t as a function of s, say $t = t(s)$. Then we can "parametrize a curve with respect to are length.

$$\mathbf{r} = \mathbf{r}\big(t(s)\big).$$

For example, when $s = 3$, $\mathbf{r}\big(t(3)\big)$ is the position vector of the point 3 unit of length along the curve from its starting point.

Example 13.3.3. Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t

Proof. Find the arc length function from the starting time $t = 0$.

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}t.$$

Hence, $t = t(s) = \frac{1}{\sqrt{2}}s$. We have

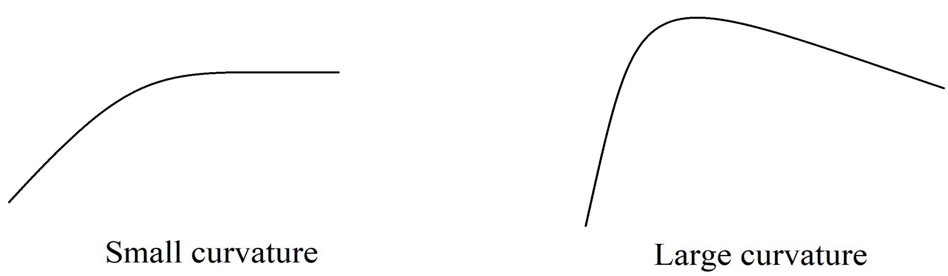
$$\mathbf{r}\big(t(s)\big) = \cos(\frac{1}{\sqrt{2}}s)\,\mathbf{i} + \sin(\frac{1}{\sqrt{2}}s)\,\mathbf{j} + \frac{1}{\sqrt{2}}s\,\mathbf{k}.$$

□

❑ Curvature

Question: How do we feel the "curvature" of a curve?

From our expericence, when we ride a bike at a constant speed, it is more difficult to turn the direction along a path with "larger curvature" than the one with a smaller curvature.*†

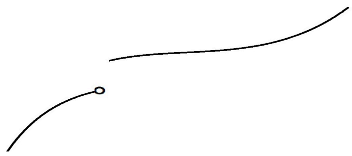


To discuss the curvature of a curve, we should discard some cases:

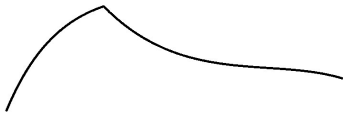
*Heuristically speaking, along the larger curvature path, we need to change directions more at the same time. The constant speed says that the same period is corresponding to the same travelling distance. Thus, we can also explain the larger curvature path as, when travelling the same distance, the direction changes more.

†The "curvature" is a geometric word. It is supposed to only depend on distance and direction but not "time". Hence, to define "curvature", we usually parametrize in s.

(i) Discontinuous curve



(ii) The curve has sharp corners or cusps



(iii) Imagine a particle moves along a curve, we don't expect that it "stays" at a point for a period since it cannot decide whether the direction changes there. Thus, we assume $|\mathbf{r}'(t)| \neq 0$. We parametrize the curve with respect to arc length parameter "s" rather than time parameter "t".

Definition 13.3.4.

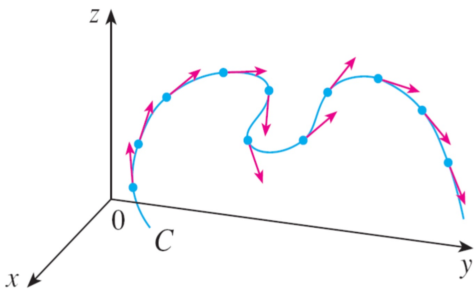
(a) A parametrization $\mathbf{r}(t)$ is called "*smooth* " on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I.

(b) A curve is called "smooth" if it has a smooth parametrization.

Suppose that C is a smooth curve defined by the vector function $\mathbf{r}$. The unit tangnet vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

indicates the direction of the curve.



Unit tangent vectors at equally spaced points on C

Heuristically, the curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Definition 13.3.5. The curvature of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

Note. (1) The unit tangent vector $\mathbf{T}$ is usually expressed as a vector function in "t". By the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Then

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|.$$

(2) Since the arc length function $s(t) = \int_0^t |\mathbf{r}'(u)|\, du$, by the Fundamental Theorem of Calculus, $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$. Hence,

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Example 13.3.6. Show that the curvature of a circle of radius a is $\dfrac{1}{a}$.

Proof. A parametrization of a circle of radius a is $\mathbf{r}(t) = a\cos t\, \mathbf{i} + a\cos t\, \mathbf{j}$. Then $\mathbf{r}'(t) = -a\sin t\, \mathbf{i} + a\cos t\, \mathbf{j}$ and $|\mathbf{r}'(t)| = a$. The unit tangent vector function is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j}.$$

Then

$$\mathbf{T}'(t) = -\cos t\, \mathbf{i} - \sin t\, \mathbf{j} \quad \text{and} \quad \left|\frac{d\mathbf{T}}{dt}\right| = 1.$$

The curvature is

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}.$$

□

Note. Small circles have large curvature and large circles have small curvature.

Theorem 13.3.7. *The curvature of the curve given by the vector function* $\mathbf{r}$ *is*

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Proof. Since $\mathbf{T} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$ and $|\mathbf{r}'| = \dfrac{ds}{dt}$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}.$$

By the product rule,

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

Consider

$$\mathbf{r}' \times \mathbf{r}'' = \frac{ds}{dt}\frac{d^2s}{dt^2}\underbrace{\mathbf{T} \times \mathbf{T}}_{=\mathbf{0}} + \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'.$$

Since $|\mathbf{T}| = 1$, we have $\mathbf{T}(t) \perp \mathbf{T}'(t)$. Then $|\mathbf{T} \times \mathbf{T}'| = \underbrace{|\mathbf{T}|}_{=1}|\mathbf{T}'| = |\mathbf{T}'|$. Also,

$$|\mathbf{r} \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 \underbrace{|\mathbf{T}|}_{=1}|\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|.$$

Hence,

$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{\left(\frac{ds}{dt}\right)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}.$$

The curvature is

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

□

Example 13.3.8. Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at general point and at $(0, 0, 0)$.

Proof. Since $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$, we have

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle.$$

Then $|\mathbf{r}' \times \mathbf{r}''| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$ and $|\mathbf{r}'| = \sqrt{1 + 4t^2 + 9t^4}$. The curvature is

$$\kappa = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

At $t = 0$, $\kappa(0) = 2$. □

• <u>**Special Case** $y = f(x)$</u>

Suppose that the curve C is the graph of $f(x)$. We can express it as vector-valued function.

$$\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}\,\Big(+ 0\,\mathbf{k}\Big).$$

Then

$$\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j} \quad \text{and} \quad \mathbf{r}''(x) = f''(x)\,\mathbf{j}.$$

The cross product is

$$\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}.$$

We have

$$|\mathbf{r}' \times \mathbf{r}''| = |f''(x)| \quad \text{and} \quad |\mathbf{r}'| = \sqrt{1 + [f'(x)]^2}.$$

Hence, the curvature is

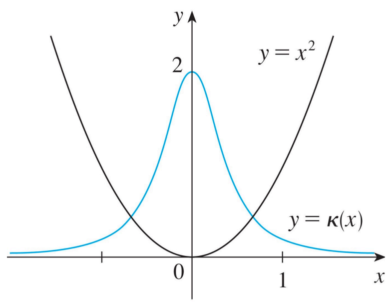
$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}.$$

Example 13.3.9. Find the curvature of the parabola $y = x^2$ at the point $(0, 0)$, $(1, 1)$ and $(2, 4)$.

Proof.
Compute that $y' = 2x$ and $y'' = 2$. The curvature of the curve is

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

At $(0, 0)$, $\kappa(0) = 2$.
At $(1, 1)$, $\kappa(1) = \dfrac{2}{5^{3/2}} \approx 0.18$.
At $(2, 4)$, $\kappa(2) = \dfrac{2}{17^{3/2}} \approx 0.03$.
We can observe that $\kappa(x) \to 0$ as $x \to \pm\infty$.

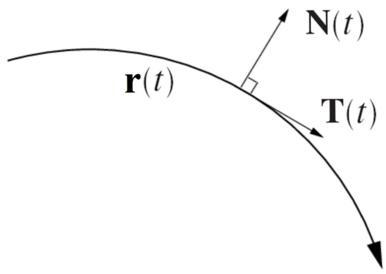


The parabola $y = x^2$ and its curvature function $y = \kappa(x)$

□

❑ The Normal and Binormal Vectors

Let $\mathbf{r}(t)$ be smooth space curve and $\mathbf{T}(t)$ be the unit tangent vector. Then



$$\left|\mathbf{T}(t)\right| = 1 \quad \overset{\frac{d}{dt}}{\Longrightarrow} \quad \mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$$
$$\Longrightarrow \quad \mathbf{T}(t) \perp \mathbf{T}'(t) \quad \text{for all } t.$$

Note. (1) $\mathbf{T}'(t)$ may not be a unit vector.

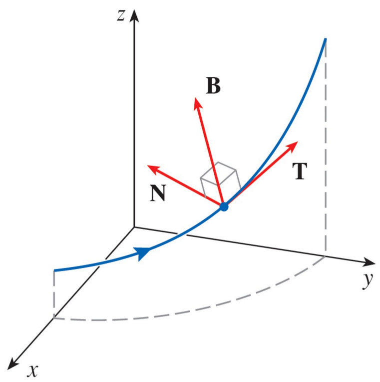
(2) If $\mathbf{T}'(t) \neq \mathbf{0}$ (hence $\kappa \neq 0$), $\mathbf{T}'(t)$ indicates the direction where the curve is turning.

Definition 13.3.10.

(1) We define the "*principal unit normal vector*" (or "*unit normal*") as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|}.$$

(2) The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the "*binormal vector*".



Remark. $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are unit vectors and they are orthogonal each other.

Example 13.3.11. Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}.$$

Proof. Compute that

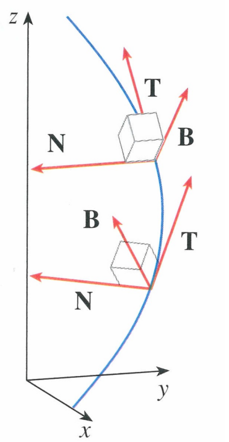
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}.$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left|\mathbf{r}'(t)\right|} = \frac{1}{\sqrt{2}}\big(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}\big)$$

and

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\big(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}\big).$$



The normal and the binormal vectors are

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cot t\,\mathbf{i} - \sin t\,\mathbf{j}$$

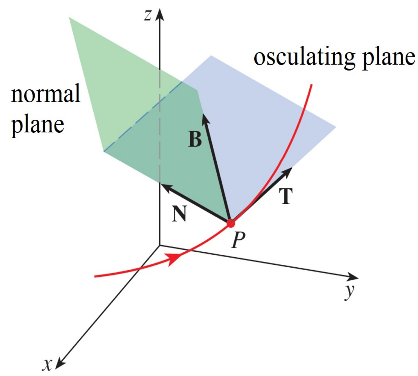
and

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}\right).$$

□

■ The Normal Plane

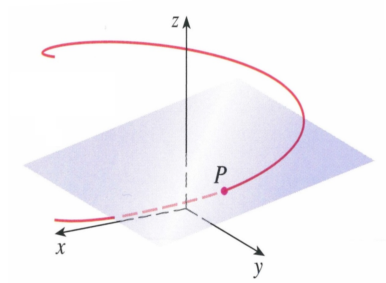
Definition 13.3.12.



(1) The plane determined by the normal and binormal vectors $\mathbf{N}(t)$ and $\mathbf{B}(t)$ at a point P on a curve C is called the "*normal plane*" of C at P.

(2) The plane determined by the vector $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the "*osculating plane*" of C at P.

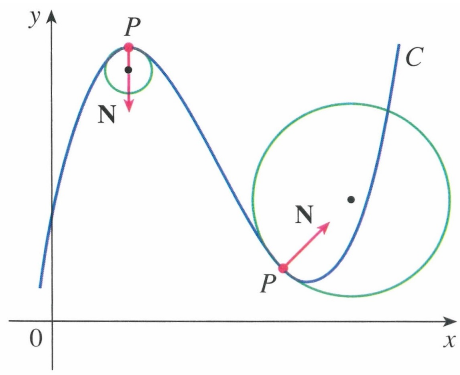
Note. (1) The normal plane consists of all ines that are orthogonal to the tangent vector $\mathbf{T}$.



(2) The osculating plane comes closest to containing the part of the curve near P.

(3) For a plane curve, the osculating plane is the plane that contains the curve.

Definition 13.3.13.



Let C be a smooth space curve and O be the circle lies in the osculating plane of C at P and has the same tangnet as C at P and lies on the concave side of C (toward which $\mathbf{N}$ points) with radius $\rho = \dfrac{1}{\kappa}$.

The circle is called the "*osculating circle*" (or "*circle of curvature*") of C at P.

Note. The osculating circle nicely describes how C behaves near P. It shares the same tangent, normal and curvature at P.

Example 13.3.14. Find the equation of the normal plane and the osculating plane of the helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}.$$

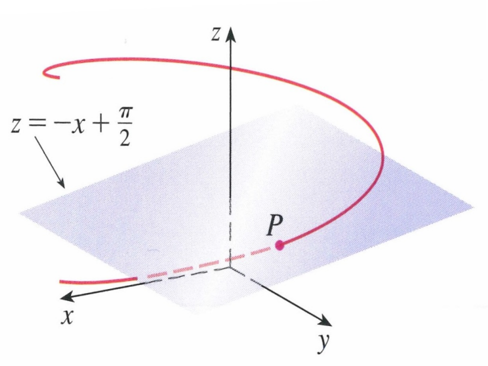
at the point $P(0, 1, \frac{\pi}{2})$.

Proof. The tangent vector is $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$ and hence $\mathbf{r}'(\frac{\pi}{2}) = -\mathbf{i} + \mathbf{k}$. Then the equation of the normal plane is

$$-1 \cdot (x - 0) + 0 \cdot (y - 1) + 1 \cdot (z - \frac{\pi}{2}) \qquad \text{or} \qquad z = x + \frac{\pi}{2}.$$

Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, we have

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}}\langle \sin t, -\cot t, 1\rangle \quad \text{and} \quad \mathbf{B}(\frac{\pi}{2}) = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle.$$



The equation of the osculating plane is

$$\frac{1}{\sqrt{2}}(x - 0) + 0(y - 0) + \frac{1}{\sqrt{2}}(x - \frac{\pi}{2}) = 0.$$

That is,

$$x + z - \frac{\pi}{2} \qquad \text{or} \qquad z = -x + \frac{\pi}{2}.$$

□

Example 13.3.15. Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

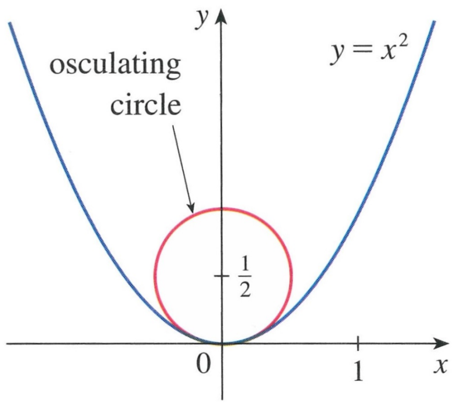
Proof.

Let $f(x) = x^2$. Then

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{3/2}} = \frac{2}{\left(1 + 4x^2\right)^{3/2}}.$$

Then the radius of the osculating circle is $\frac{1}{\kappa(0)} = \frac{1}{2}$ and its center is $\left(0, \frac{1}{2}\right)$. The equation of the osculating circle is

$$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}.$$



□

■ Summary

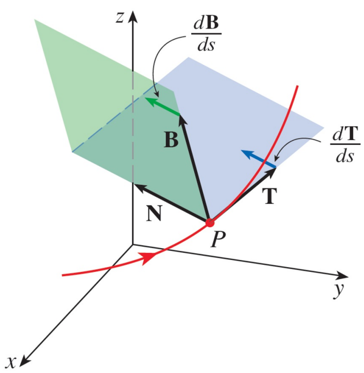
- $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

- $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$

■ **Torsion**

Remark. Curvature $\kappa = \left|\dfrac{d\mathbf{T}}{dx}\right|$ at a point P on a curve C indicates how tightly the curve "bends." Since $\mathbf{T}$ is a normal vector for the normal plane, $\dfrac{d\mathbf{T}}{ds}$ tells us how the normal plane changes as P moves along C. (The tangent vector at P rotates in the direction of $\mathbf{N}$.)

A space curve can also lift or "twist" out of the osculating plane at P. Since $\mathbf{B}$ is normal to the osculating plane, $\dfrac{d\mathbf{B}}{ds}$ gives us information about how the osculating plane changes as P moves along C.



Note. We can show that $\dfrac{d\mathbf{B}}{dx}$ is parallel to $\mathbf{N}$. Hence, the scalar τ such that

$$\frac{d\mathbf{B}}{dx} = -\tau\mathbf{N}$$

is called the "*torsion*" of C at P. Moreover, $\tau = -\tau\mathbf{N} \cdot (-\mathbf{N}) = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$.

Definition 13.3.16. The "*torsion*" of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

Remark. By using the Chain Rule,

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} \qquad \text{so} \qquad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|}.$$

We have

$$\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}.$$

Example 13.3.17. Find the torsion of the helix

$$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle.$$

Proof. For Example 13.3.14, we have

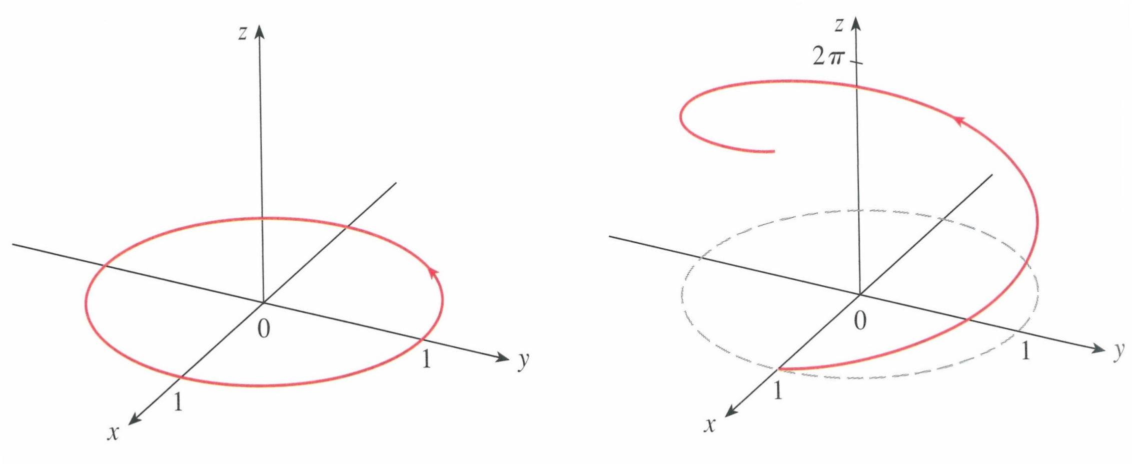
$$\frac{ds}{dt} = \left|\mathbf{r}'(t)\right| = \sqrt{2}, \quad \mathbf{N}(t) = \langle -\cos t, -\sin t, 0\rangle \quad \text{and} \quad \mathbf{B}(t) = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle.$$

Then $\mathbf{B}'(t) = \frac{1}{1}\sqrt{2}\langle \cot t, \sin t, 0\rangle$ and

$$\tau(t) = -\frac{\mathbf{B}(t)\cdot\mathbf{N}(t)}{\left|\mathbf{r}'(t)\right|} = -\frac{1}{2}\langle \cos t, \sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle = \frac{1}{2}.$$

□

Remark. Compare with the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ in the xy-plane and the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Both of them have constant curvatrue, but the circle has constant torsion 0 whereas the helix has constant torsion $\frac{1}{2}$. We can think of the circle as bending at each point but never twisting, while the helix both bends and twist (upward) at each point.



$\kappa = 1, \tau = 0$ $\kappa = \frac{1}{2}, \tau = \frac{1}{2}$

Under certain conditions, the shape of a space curve is completely determined by the values of curvature and torsion at each point on the curve.

Theorem 13.3.18. *The torsion of the curve given by the vector function* $\mathbf{r}$ *is*

$$\tau(t) = \frac{[\mathbf{r}'(t)\times\mathbf{r}''(t)]\cdot\mathbf{r}'''(t)}{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|^2}.$$

Homework 13.3. 5, 7, 11, 16, 20, 23, 27, 28, 33, 43, 49, 50

Chapter 14

Partial Derivatives and Differentiability

14.1 Functions of Several Variables

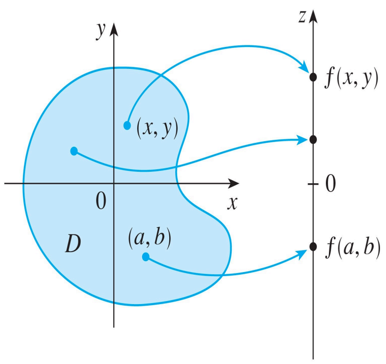
❏ Functions of Two Variables

Example 14.1.1.

(1) Let $T = f(x, y)$ represent the temperature at the position (x, y) where x and y indicate the longitude and latitude respectively.

(2) Let $V = V(r, h)$ represent the volume of a circular cylinder where r and h indicate the raidus and the height of the cylinder respectively.

Definition 14.1.2. A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the "*domain*" of f and its "*range*" is the set of values that f takes on. That is, $Range(f) = \{f(x, y) \mid (x, y) \in D\}$.



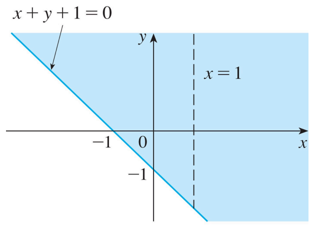
Sometimes, we express $z = f(x, y)$ where x and y are independent variables and z is a dependent variable.

Remark. If a function is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pair(x, y) for which the given expression is a well-defined real number.

Example 14.1.3.

(1) Let $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$. The domain of f is

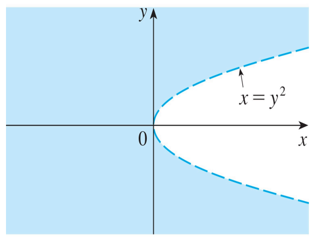
$$\begin{aligned} Dom(f) &= \left\{(x, y) \,\middle|\, x + y + 1 \geq 0,\ x - 1 \neq 0\right\} \\ &= \left\{(x, y) \,\middle|\, y \geq -x - 1,\ x \neq 1\right\}. \end{aligned}$$



Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

(2) Let $f(x, y) = x \ln(y^2 - x)$. The domain of f is

$$\begin{aligned} Dom(f) &= \left\{(x, y) \,\middle|\, y^2 - x > 0\right\} \\ &= \left\{(x, y) \,\middle|\, x < y^2\right\}. \end{aligned}$$



Domain of $f(x, y) = x \ln(y^2 - x)$

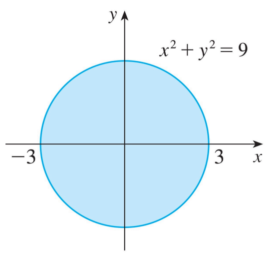
Example 14.1.4. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

The domain of g is

$$\begin{aligned} Dom(f) &= \left\{(x, y) \,\middle|\, 9 - x^2 - y^2 \geq 0\right\} \\ &= \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 9\right\}. \end{aligned}$$

The range of g is

$$\begin{aligned} Range(g) &= \left\{z \,\middle|\, z = \sqrt{9 - x^2 - y^2},\ (x, y) \in Dom(g)\right\} \\ &= \left\{z \,\middle|\, 0 \leq z \leq 3\right\}. \end{aligned}$$



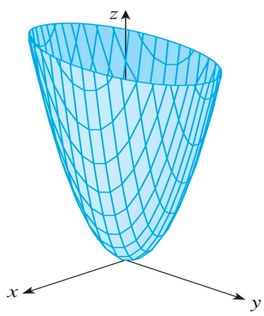
Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Example 14.1.5. Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

$Dom(h) = \mathbb{R}^2$ and $Range(f) = [0, \infty)$. The graph of h

$$Graph(h) = \{(x, y, z) \mid z = 4x^2 + y^2,\ (x, y) \in \mathbb{R}^2\}$$

is an elliptic paraboliod.



Graph of $h(x, y) = 4x^2 + y^2$

■ Some ways to figure out two variables functions

We introduce some visual methods to understand functions of two variables.

(I) Algebraically (by an explicit formula). Such as above examples.

(II) Verbally (by a description in words)

Example 14.1.6. In regions with severe winter weather, the wind-chill index is often used to describe the apparent severity of the cold. The index W is a subjective temperature that depends on the actual temperautre T and the wind speed v.

W is a function of T and v, and we write $W = f(T, v)$. For example, the value of W is record in a table

Wind speed (km/h)

Actual temperature (°C)

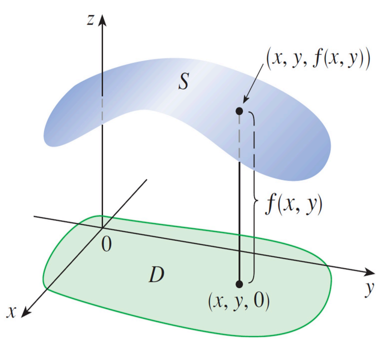
T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Wind-chill index as a function of air temperature and wind speed

■ Graph

Definition 14.1.7. If f is a function of two variables with domain D, then the "*graph*" of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D. That is,

$$Graph(f) = \{(x, y, z) \mid z = f(x, y),\ (x, y) \in D\}.$$



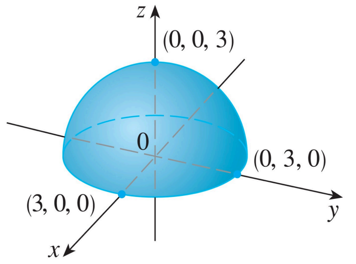
Example 14.1.8. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Proof.

Let $z = \sqrt{9 - x^2 - y^2}$. Then the graph of g is

$$\begin{aligned} Graph(g) &= \{(x, y, z) \mid z^2 = 9 - x^2 - y^2,\ z \geq 0\} \\ &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 9,\ z \geq 0\} \end{aligned}$$

Note. An entire sphere cannot be represented by a single function of x and y. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.



Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

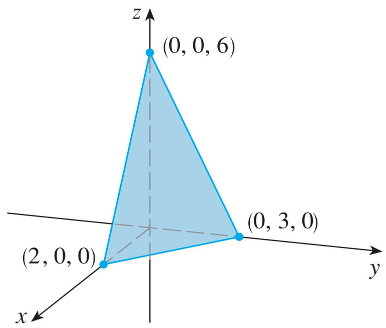
□

Example 14.1.9. Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y.$$

Proof.

Let $z = 6 - 3x - 2y$ or $3x + 2y + z = 6$. The intercepts of the function are $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$.



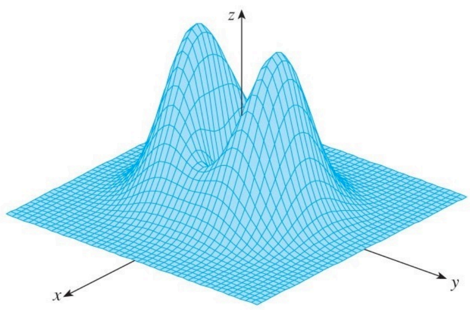
Graph of $f(x, y) = 6 - 3x - 2y$

□

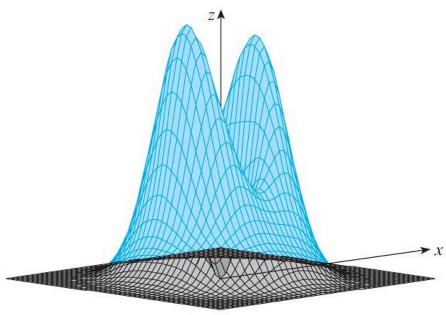
Note. The function $f(x, y) = ax + by + c$ is called a "*linear function*". The graph of such a function is a plane and has the equation $z = ax + by + c$ or $ax + by - z + c = 0$.

❑ Computer- generated Graphs

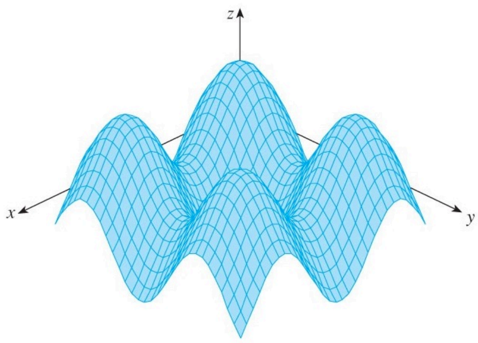
In general, it is difficult to sketch the graph of a two-variables function. A nice method to sketch the traces in the vertical plne $x = k$ and $y = h$. For example, fix $x = k$ and sketch the graph of a single variable function $z = f(k, y)$. It is a curve on the plane $x = k$. Draw all such curve as x ranges over all possible values in the x direction.



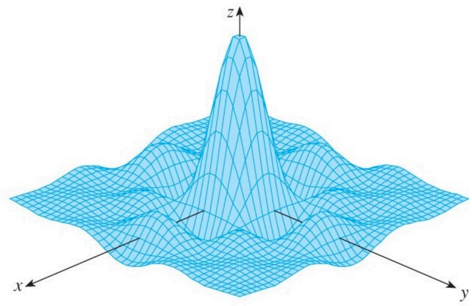
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(c) $f(x, y) = \sin x + \sin y$



(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

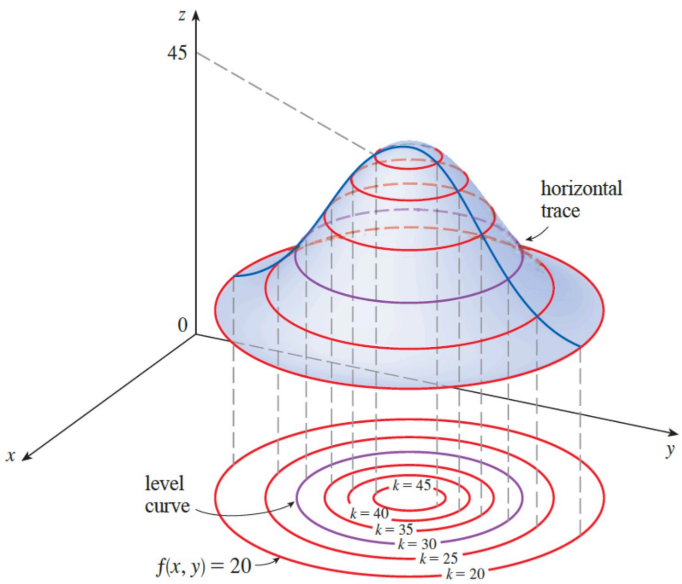
■ Level Curves and Contour Maps

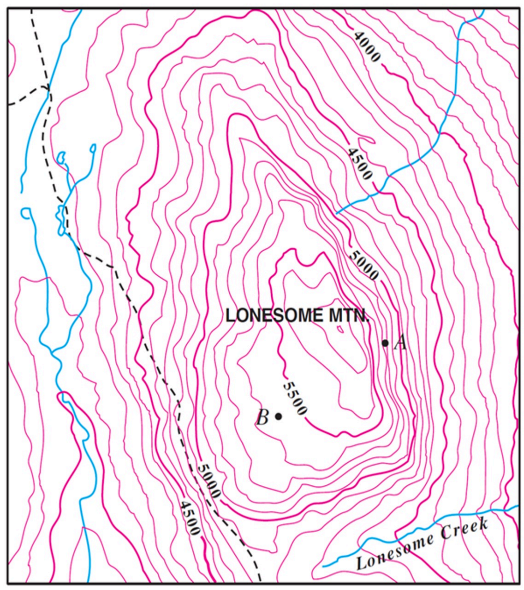
So far, we have two methods for visualizing functions: arrow diagrams and graphs. A third method is to consider a contour map on which points of constant elevation are joined to form "*contour curves*", or "*level curves*".

Definition 14.1.10. The "*level curves*" of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant (in the range of f). The level curve is the set $\{(x, y) \in D \mid f(x, y) = k\}$.

Note. (1) A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. (It shows where the graph of f has height k).

(2) Level curves are useful in the reality. For example, isothermals(等溫線), contour map, contour line.

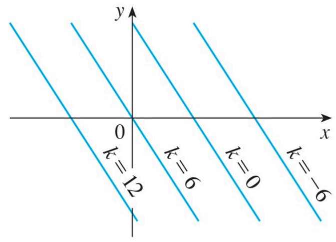




Example 14.1.11. Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Proof.

Consider the curves $6 - 3x - 2y = k$ in the domain. For $k = -6, 0, 6, 12$, the corresponding level curves are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$ and $3x + 2y + 6 = 0$.



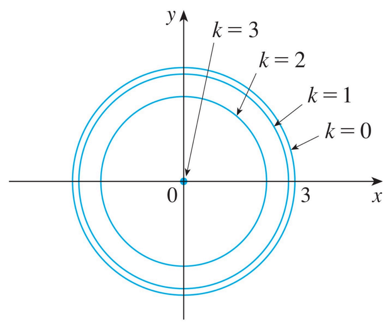
Contour map of $f(x, y) = 6 - 3x - 2y$

□

Example 14.1.12. Sketch the level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$ for the values $k = 0, 1, 2, 3$.

Proof.

Consider the curves $\sqrt{9 - x^2 - y^2} = k$ in the domain. For $k = 0, 1, 2, 3$, the corresponding level curves are $x^2 + y^2 = 9$, $x^2 + y^2 = 8$, $x^2 + y^2 = 5$ and $x^2 + y^2 = 0$.

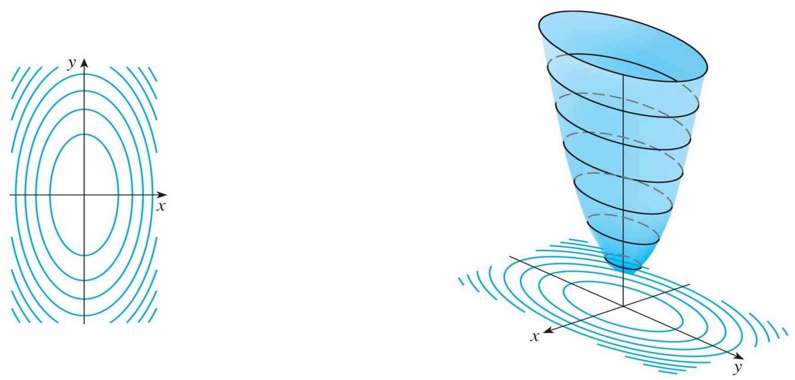


Contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$

□

Example 14.1.13. Sketch the level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Proof. Consider the curves $4x^2 + y^2 + 1 = k$ in the domain. We can rewrite the equation as $\dfrac{x^2}{\frac{1}{4}(k-1)} + \dfrac{y^2}{k-1} = 1$. For $k > 1$, the level curves are a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$.



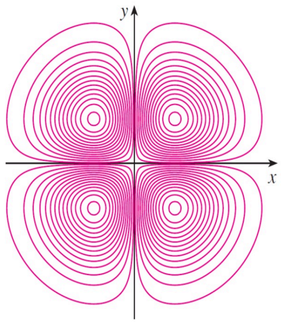
(a) Contour map (b) Horizontal traces are raised level curves

The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

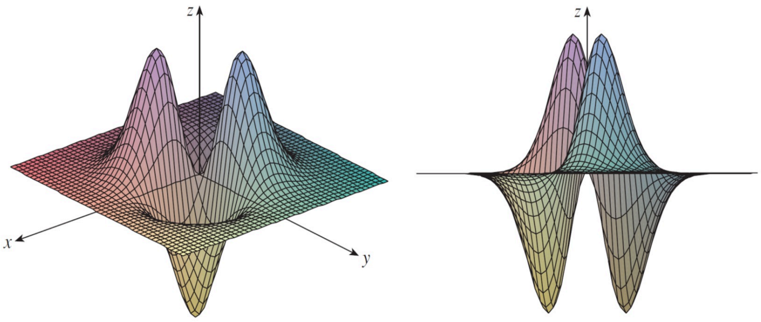
□

Note. The following two figures show different visualized concepts to figure out a two variables functions $f(x, y)$.

(1) $f(x, y) = -xye^{-x^2-y^2}$.

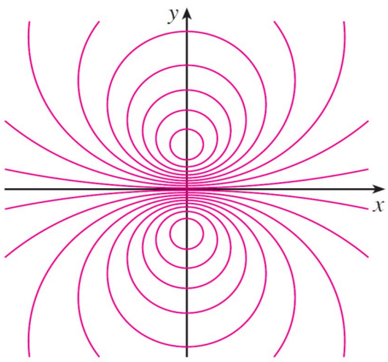


(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

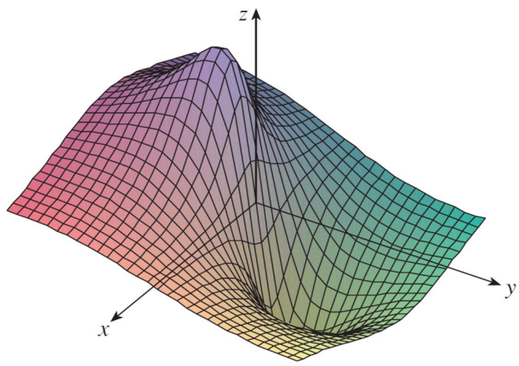


(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

(2) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$.



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

❑ Functions of Three or More Variables

■ Three variables functions

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Example 14.1.14. The function $f(x, y, z) = \ln(z - y) + xy \sin z$ has the domain

$$Dom(f) = \{(x, y, z) \mid z - y > 0\} = \{(x, y, z) \mid z > y\}.$$

Note. It is difficult to visualize a function f of three variables by its graph since that would lie in four-dimensional space.

We obtain some insight into f by examining its "level surfaces", which are surfaces with equation $f(x, y, z) = k$, where k is a constant in the range of f.

Example 14.1.15. Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z.$$

Proof.

$$Dom(f) = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; z > y\right\}$$

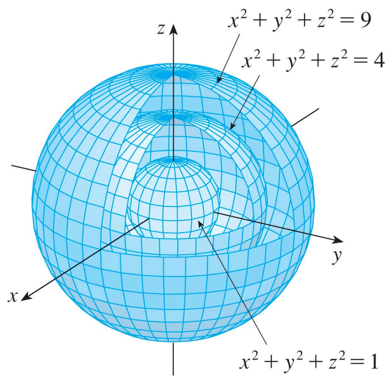
□

Example 14.1.16. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Proof.

Consider the surface with equation $x^2 + y^2 + z^2 = k$, $k \geq 0$. The corresponding level surfaces form a family of concentric spheres with radius $\sqrt{k}$.



□

■ n variables functions

A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \cdots, x_n)$ to an n-tuple $(x_1, x_2, \cdots, x_n)$ of real numbers.

Example 14.1.17. (Cost function) Let C_i be the cost per unit of the ith ingredient and x_i be the units of the ith ingredient are used. The total cost is

$$C = f(x_1, x_2, \cdots, x_n) = C_1x_1 + C_2x_2 + \cdots + C_nx_n.$$

which is a n-variable function.

Remark. Since the point $(x_1, x_2, \cdots, x_n)$ and the vector $\mathbf{x} = \langle x_1, x_2, \cdots, x_n\rangle$ are one-to-one correspondence, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$.

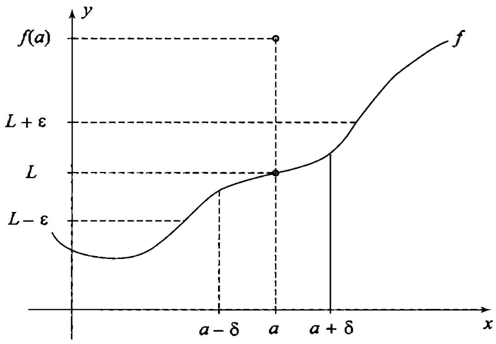
1. As a function of n real variables $x_1, x_2, \cdots, x_n$, denote $f(x_1, x_2, \cdots, x_n)$.
2. As a function of a single point variable $(x_1, x_2, \cdots, x_n)$, denote $f\big((x_1, x_2, \cdots, x_n)\big)$.
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \cdots, x_n\rangle$, denote $f(\mathbf{x}) = f\big(\langle x_1, x_2, \cdots, x_n\rangle\big)$.

Homework 14.1. 9, 12, 16, 25, 29, 31, 32, 36, 45, 49, 54, 61-66, 67

14.2 Limits and Continuity

❑ Limits

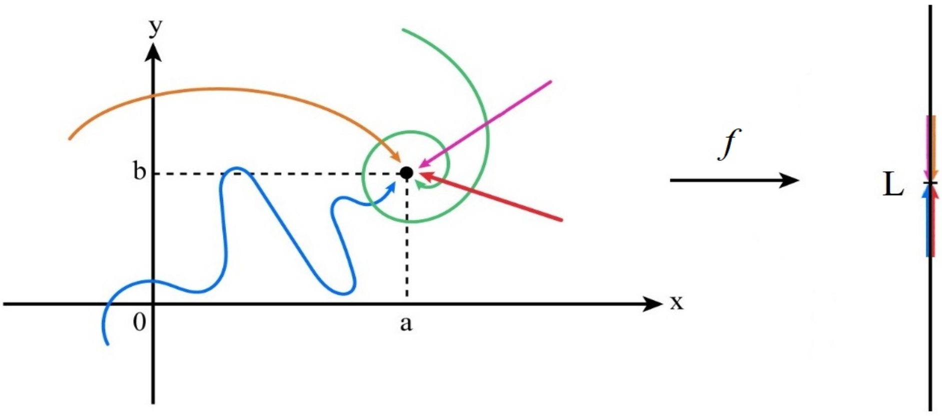
Recall that the limit of a single variable function $f(x)$ as x approaches a is followed by the concept that the value of $f(x)$ approaches L as x tends to a. The precise $\varepsilon - \delta$ definition is given in Chapter 3.



Question: How about the limit of a two variables function f as (x, y) approaches a point (a, b)?

Definition 14.2.1. (Heuristic definition) Let f be a function of two variables whose domain D containing a neighborhood of (a, b) (possibly except (a, b) itself). We say that the limit of $f(x, y)$ as (x, y) approaches (a, b) exists if there is a number L such that we can make $f(x, y)$ as close to L as we like by taking (x, y) sufficiently close to (a, b). Denote

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \qquad \text{or} \qquad f(x, y) \to L \quad \text{as } (x, y) \to (a, b).$$

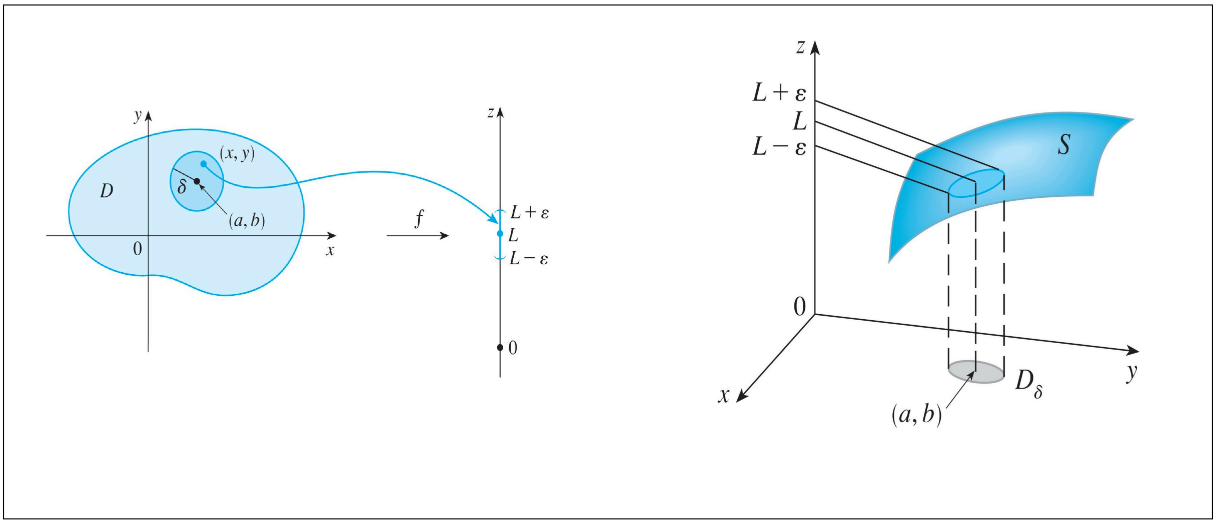


Definition 14.2.2. (Precise definition) Let f be a function of two variables whose domain D containing a neighborhood of (a, b) (possibly except (a, b) itself). We say that the limit of $f(x, y)$, as (x, y) approaches (a, b), exists if there is a number L such that for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - a)^2} < \delta$. Denote

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \qquad \text{or} \qquad f(x, y) \to L \quad \text{as } (x, y) \to (a, b).$$

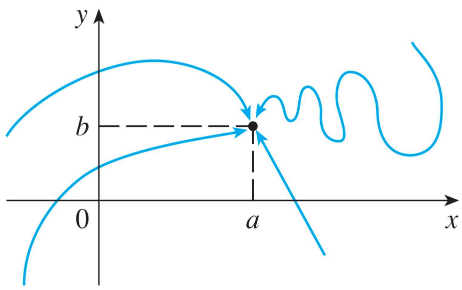


Remark. For functions of a single variable, we only need to consider two possible direction when x approaches a (from the left and from the right).

For functoins of two variables, we have to consider an infinite numbers of directions in any manner whatsover as long as (x, y) stays within the domain of f.

Hence, if the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, then $f(x, y)$ must approach the same limit no matter which direction and how (x, y) approaches (a, b).

Note. From the above remark, if $f(x, y) \to L_1$ and (x, y) approaches (a, b) along a path C_1 and $f(x, y) \to L_2$ when (x, y) approaches (a, b) along another path C_2 where $L_1 \neq L_2$, then the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.



Example 14.2.3. Let $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Consider the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$.

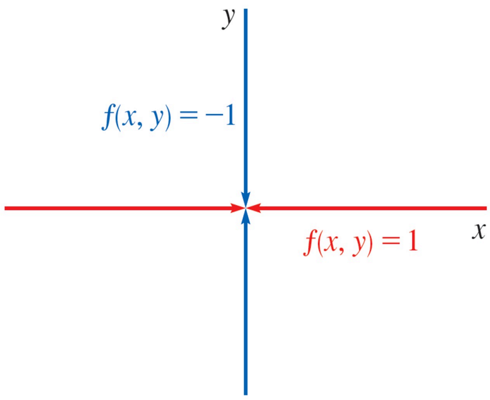
Proof. Along the x-axis $(y = 0)$,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

Along the y-axis $(x = 0)$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Hence, the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.



□

Example 14.2.4. If $f(x,y) = \dfrac{xy}{x^2+y^2}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Proof. Along the x-axis ($y = 0$),

$$\lim_{\substack{(x,y)\to(a,b)\\ y=0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

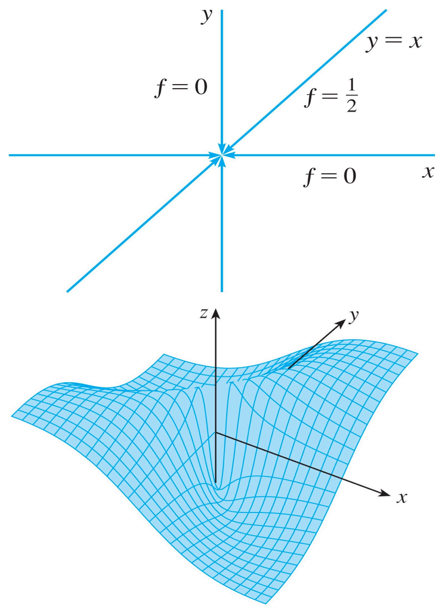
Along the y-axis ($x = 0$),

$$\lim_{\substack{(x,y)\to(a,b)\\ x=0}} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

But, along the line $y = x$,

$$\lim_{\substack{(x,y)\to(a,b)\\ x=y}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Hence, the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.



$f(x,y) = \dfrac{xy}{x^2+y^2}$

□

Example 14.2.5. If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

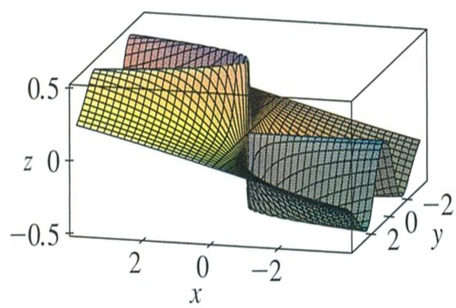
Proof. Along the the line $y = mx$ (not y-axis),

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{x^3(1+m^2)}{x^2(1+m^4x^2)} = 0.$$

Along the curve $x = y^2$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2\cdot y^2}{(y^2)^2+y^4} = \frac{1}{2}.$$

Hence, the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.



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■ Laws of Limits and Squeeze Theorem

Theorem 14.2.6. *(Laws of Limits) Let f and g be two variables functions defined on D containing a neighborhood of (a,b) (possibly except (a,b) itself) and c be a constant number. Suppose that the limits $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ exist. Then* *(a)* $\lim_{(x,y)\to(a,b)} [f \pm g](x,y)$ *exists and* $\lim_{(x,y)\to(a,b)} [f \pm g](x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$.

(b) $\lim_{(x,y)\to(a,b)} [cf](x,y)$ *exists and* $\lim_{(x,y)\to(a,b)} [cf](x,y) = c \lim_{(x,y)\to(a,b)} f(x,y)$.

(c) $\lim_{(x,y)\to(a,b)} [fg](x,y)$ *exists and* $\lim_{(x,y)\to(a,b)} [fg](x,y) = \Big(\lim_{(x,y)\to(a,b)} f(x,y)\Big)\Big(\lim_{(x,y)\to(a,b)} g(x,y)\Big)$.

(d) $\lim_{(x,y)\to(a,b)} \Big[\frac{f}{g}\Big](x,y)$ *exists if* $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$ *and*

$$\lim_{(x,y)\to(a,b)} \Big[\frac{f}{g}\Big](x,y) = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

provided $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$.

(e) In particular,

$$\lim_{(x,y)\to(a,b)} x = a, \qquad \lim_{(x,y)\to(a,b)} y = b, \qquad \lim_{(x,y)\to(a,b)} c = c$$

Theorem 14.2.7. *(Squeeze Theorem) Let $f(x,y)$, $g(x,y)$ and $h(x,y)$ be three functions defined near (a,b). Suppose that $f(x,y) \leq g(x,y) \leq h(x,y)$ for every (x,y) near (a,b). If*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y),$$

then the limit $\lim_{(x,y)\to(a,b)} g(x,y)$ *exists and*

$$\lim_{(x,y)\to(a,b)} g(x,y) = L.$$

Example 14.2.8. Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Proof. First of all, we may try the limits when (x,y) approaches $(0,0)$ along several paths. We observe that all the limits are 0. Therefore, we guess that the limit could exist and equal 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that if $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$, then $\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$. Consider

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \underbrace{\left|\frac{x^2}{x^2+y^2}\right|}_{<1} \cdot 3|y| < 3|y|.$$

Choose $\delta = \frac{1}{3}\varepsilon$. If $0 < \sqrt{x^2+y^2} < \delta = \frac{1}{3}\varepsilon$, then $|y| \leq \sqrt{x^2+y^2} < \frac{1}{3}\varepsilon$. Therefore,

$$|f(x,y) - 0| = \left|\frac{3x^2y}{x^2+y^2}\right| < 3|y| < 3 \cdot \frac{1}{3}\varepsilon = \varepsilon$$

whenever $0 < \sqrt{x^2+y^2} < \delta$ and this implies that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$. □

■ **Limt at Infinity**

In the previous chapter, we regard R^n as a vector space and every point $(x_1, \cdots, x_n)$ is identified as a vector $\mathbf{x} =< x_1, \cdots, x_n >$. The length of a vector is denoted by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Hence, if we want to describe a point (or a vector) $\mathbf{x} \in \mathbb{R}^n$ tending to infinity, we will use the notation "$\|\mathbf{x}\| \to \infty$" (or $\|(x_1, \cdots, x_n)\| \to \infty$ or $\| < x_1, \cdots, x_n > \| \to \infty$)

Remark. We usually use the words "*as* $\|\mathbf{x}\|$ *is sufficiently large*" which means that there exists a positive number M such that for every point $\mathbf{x}$ with $\|\mathbf{x}\| > M$ then $\cdots$. For example, "$f(x, y) > 1$ when $\|(x, y)\|$ is sufficiently large" means that there exists a number $M > 0$ such that $f(x, y) > 1$ for every $\|(x, y)\| > M$.

Definition 14.2.9. (Limit at infinity) Let f be a function of two variables whose domain D containing all points which are sufficiently large. We say that the limit of $f(x, y)$, as (x, y) approaches infinity, exists if there is a number L such that for every number $\varepsilon > 0$ there exists a corresponding number $M > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever $\sqrt{x^2 + y^2} > M$. Denote

$$\lim_{\|(x,y)\|\to\infty} f(x, y) = L \qquad \text{or} \qquad f(x, y) \to L \quad \text{as } \|(x, y)\| \to \infty.$$

Example 14.2.10. Let $f(x, y) = x$. Determine whether the limit $\lim\limits_{\|(x,y)\|\to\infty} f(x, y)$ exists.

Proof. Fix $x = 1$ and let $y \to \infty$, then $\|(x, y)\| \to \infty$ and $\lim\limits_{x=1,y\to\infty} f(x, y) = 1$.

Similarly, fix $x = 2$ and let $y \to \infty$, then $\|(x, y)\| \to \infty$ and $\lim\limits_{x=2,y\to\infty} f(x, y) = 2$. Hence, the limit $\lim\limits_{\|(x,y)\|\to\infty} f(x, y)$ does not exist. □

Example 14.2.11. Let $f(x, y) = \dfrac{1}{x^2 + y^2}$. Determine whether the limit $\lim\limits_{\|(x,y)\|\to\infty} f(x, y)$ exists.

Proof. Given $\varepsilon > 0$, choose $M = \dfrac{1}{\sqrt{\varepsilon}}$ and $L = 0$. For $\|(x, y)\| = \sqrt{x^2 + y^2} > M$,

$$|f(x, y) - L| = \left| \frac{1}{x^2 + y^2} - 0 \right| < \frac{1}{M^2} = \varepsilon.$$

Hence, $\lim\limits_{\|(x,y)\|\to\infty} f(x, y) = 0$. □

❑ **Continuity**

Recall that the continuity of a single variable function $f(x)$ at a is defined by

$$\lim_{x\to a} f(x) = f(a).$$

A slogan is that "the limit of f at a is equal to the value of f at a". We attempt to use the same idea to define the continuity of a multi-variables function.

Definition 14.2.12.

(a) A two variables function f is called "*continuous at* (a, b)" if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

(b) f is called continuous on D if f is continuous at every point in D.

Remark.

(1) A surface that is the graph of a continuous function has no hole or break.

(2) The sums, differeneces, products and quotients of continuous functions are continuous on their domains

(3) Every polynomial function or every rational function of two variables is continuous. For example, $f(x, y) = 3x^5 + 6y^4 + 10x^7y^6 + 5x - 7y + 6$ is continuous everywhere.

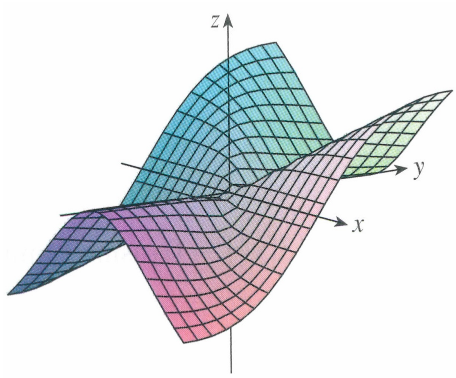
Example 14.2.13. Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Proof. Since f is a rational function, it is continuous on its domain. That is, f is continuous on $Dom(f) = \{(x, y) \mid x^2 + y^2 \neq 0\} = \{(x, y) \mid (x, y) \neq (0, 0)\} = \mathbb{R}\backslash\{(0, 0)\}$. □

Example 14.2.14. Let $g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$. Since the limit $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist, g is not continuous at $(0, 0)$.

Example 14.2.15. Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

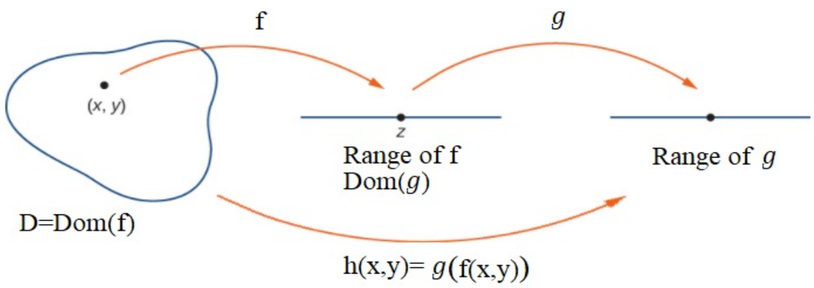


Since f is a rational function for $(x, y) \neq (0, 0)$, it is continuous on $\mathbb{R}^2\backslash\{(0, 0)\}$. Also, $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$. Thus, f is continuous at $(0, 0)$ and f is continuous on $\mathbb{R}^2$.

■ Composite Functions

We consider the composition of a two variables function and a single variable function.

Let $f(x, y)$ be a continuous function of two variables and $g(t)$ be a continuous function of a single variable that define on the range of f. Then $h = g \circ f$ defined by $h(x, y) = g\big(f(x, y)\big)$ is also a continuous function.



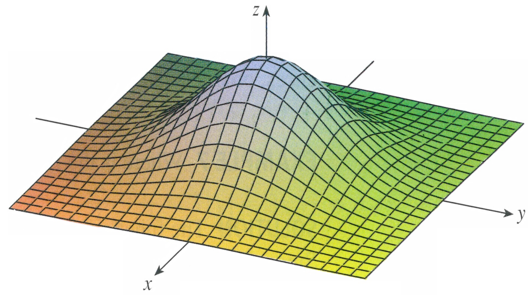
Example 14.2.16. Where is the function $h(x, y) = e^{-(x^2+y^2)}$ continuous?

Proof. Since the function $f(x, y) = x^2 + y^2$ is a polynomail and thus is continuous on $\mathbb{R}^2$.

Also, the function $g(t) = e^{-t}$ is continuous on $\mathbb{R}$.
Then the composite function

$$f(x, y) = g\big(f(x, y)\big) = e^{-(x^2+y^2)}$$

is continuous on $\mathbb{R}^2$

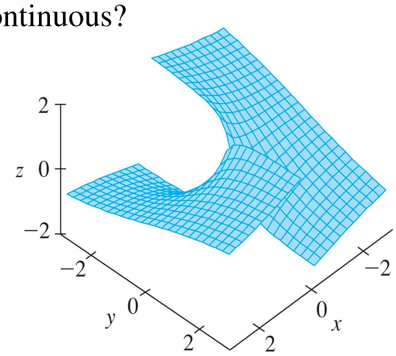


□

Example 14.2.17. Where is the function $h(x, y) = \arctan\left(\frac{y}{x}\right)$ continuous?

Proof.

Let $f(x, y) = \frac{y}{x}$ be continuous except on the line $x = 0$. Let $g(t) = \arctan t$ be continuous everywhere. Then the composite function $h(x, y) = \arctan\left(\frac{y}{x}\right) = g\big(f(x, y)\big)$ is continuous except the line $x = 0$.



The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

□

■ Functions of Three or more Variables

The definitions of limits and continuity of n-variables functions are similar as the ones of two variables functions. We ignore the details of their definitions here.

Homework 14.2. 7, 10, 12, 15, 18, 21, 26, 29, 31, 34, 38, 43, 46, 49, 51, 57

14.3 Partial Derivatives

Recall that for a single variable function $f(x)$, the derivative of f is defined by

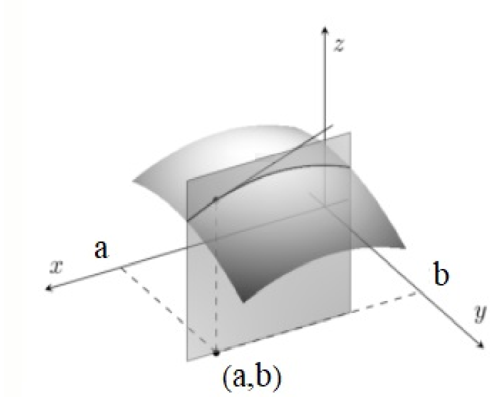
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

which represents the instantaneous rate of change of f with respect to x.

For a two variables function $f(x, y)$, let x vary while keeping y fixed, say $y = b$, where b is a constant. We can regard $f(x, b)$ as a single variable function.
Let $g(x) = f(x, b)$, then $g(a) = f(a, b)$. The derivative of $g(x)$ at $x = a$ is

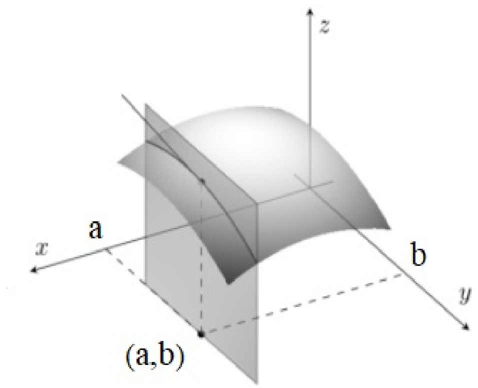
$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}..$$

We call it the "*partial derivative of f with respect to x at* (a, b)".



Similarly, let y vary while keeping x fixed, say $x = a$. Let $k(y) = f(a, y)$. The partial derivative of f with respect to y at (a, b) is

$$\lim_{h\to 0} \frac{k(b+h) - k(b)}{h} = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}$$



Definition 14.3.1. (Partial Derivatives) Let f be a function of two variables. The partial derivatives of f with respect to x and with respect to y are the functions f_x and f_y defined by setting

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided these limits exist.

Notation: Let $z = f(x, y)$. We write

$$\begin{aligned} f_x(x,y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f = D_1 f = f_1 \\ f_y(x,y) &= f_x = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f = D_2 f = f_2 \end{aligned}$$

■ **Find Partial Derivatives of** $z = f(x, y)$

- To find f_x, we regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, we regard x as a constant and differentiate $f(x, y)$ with respect to xy.

Example 14.3.2. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Proof. The partial derivatives of f are

$$f_{(}x, y) = 3x^2 + 2xy^3 \qquad \text{and} \qquad f_y(x, y) = 3x^2y^2 - 4y.$$

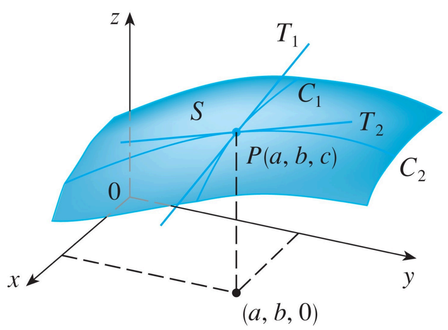
Then $f_x(2, 1) = 12 + 4 = 16$ and $f_y(2, 1) = 12 - 4 = 8$. □

Note. We should consider the single variable function $f(x, 1) = x^3 + x^2 - 4$ and $f(2, y) = 8 + 4y^3 - 2y^2$. Then

$$\begin{aligned} f_x(2,1) &= \left(\frac{d}{dx} f(x,1)\right)\Big|_{x=2} = 3x^2 + 2x\Big|_{x=2} = 12 + 4 = 16. \\ f_y(2,1) &= \left(\frac{d}{dy} f(2,1)\right)\Big|_{y=1} = 12y^2 - 4y\Big|_{y=1} = 12 - 4 = 8. \end{aligned}$$

■ Interpretation of Partial Derivatives

The equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.
Fix $y = b$, the curve C_1 is the intersection of the vertical plane and S. C_1 is also the graph of the function $g(x) = f(x, b)$, $y = b$. The slope of its tangent line T_1 at P is $g'(a) = f_x(a, b)$. Similar for the curve C_2, the tangnet line T_2 and its slope $f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Example 14.3.3. If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

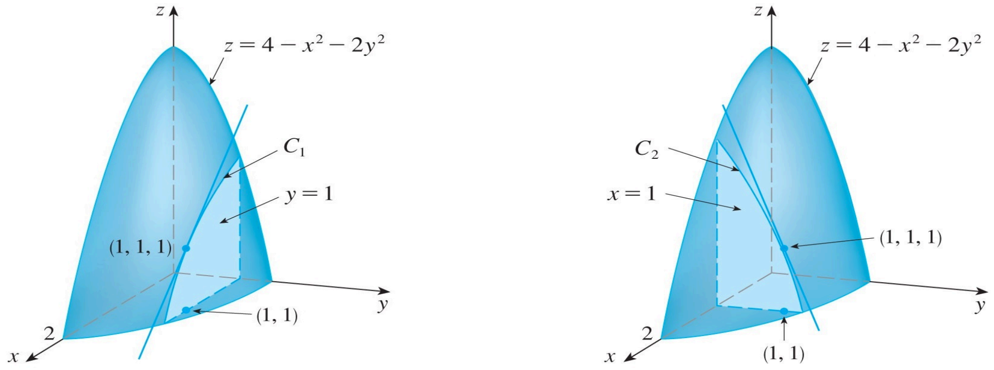
Proof. The partial derivatives of f are

$$f_x(x, y) = -2x \qquad \text{and} \qquad f_y(x, y) - 4y.$$

Then $f_x(1, 1) = -2$ and $f_y(1, 1) = -4$.

The equation $z = 4 - x^2 - 2y^2$ represents a paraboloid which is the graph of $f(x, y)$. Fix $y = 1$, $z = 2 - x^2$ is the equation of a parabola which is the intersection of the vertical plane $y = 1$ and the graph of $f(x, y)$. The value $f_x(1, 1) = -2$ is the slope of the tangent line to the parabola $C_1 : z = 2 - x^2, y = 1$ at $(1, 1, 1)$.

Similarly, $f_y(1, 1) = -4$ is the slope of the tangnet line to the parabola $C_2 : z = 3 - 2y^2$, $x = 1$ at $(1, 1, 1)$.

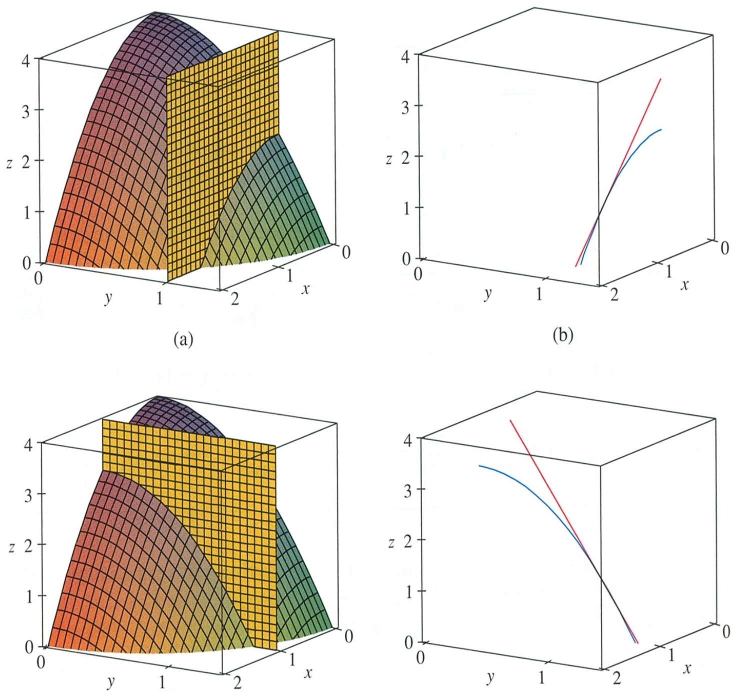


□

Note. We can express the curve C_1 as a vector equation $\mathbf{r}(t) = \langle \underset{x}{t}, \underset{y}{1}, \underset{z}{2 - t^2} \rangle$. Then the tangent vector is $\mathbf{r}'(t) = \langle 1, 0, -2t \rangle$.

At $(1,1,1)$, we have $t = 1$ and then $\mathbf{r}'(1) = \langle 1, 0, -2 \rangle$. The equation of the tangent line is

$$\mathbf{r}(1) + t\mathbf{r}'(1) = \langle 1 + t, 1, 1 - 2t \rangle.$$



Example 14.3.4. If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Proof. We can calculate the partial derivatives by the chain rule,

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{-x}{(1+y)^2}.$$

□

■ Implicit Differentiation

Recall that if the two variables x and y satisfy an equation $F(x, y) = 0$, then we can use the implicit differentiation to find the ralated rate of each other ($\frac{dy}{dx}$ or $\frac{dx}{dy}$).

By following the same idea, if three variables x, y and z satisfy an equation $F(x, y, z) = 0$, we want to find the related rates (partial derivatives) between any two variables.

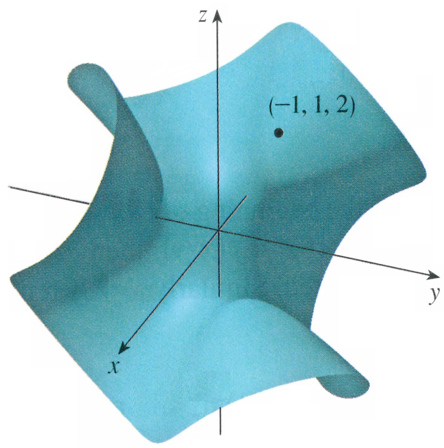
Example 14.3.5. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz + 4 = 0. \tag{14.1}$$

Proof.

Differentiating both sides of equation (14.1) with respect to x, we have

$$\frac{\partial}{\partial x}\left[x^3 + y^3 + z^3 + 6xyz + 4\right] = \frac{\partial}{\partial x}(0)$$



Then

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0 \quad \text{and hence}$$
$$\frac{\partial z}{\partial x}\left(3z^3 + 6xy\right) = -\left(3x^2 + 6yz\right).$$

We have

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

At the point $(-1, 1, 2)$, we have

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(-1,1,2)} = -\frac{5}{2} \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(-1,1,2)} = \frac{3}{2}.$$

□

■ Functions of Three or More Variables

- For a three variables function $f(x, y, z)$, fix y and z, the partial derivative of f with respect to x is defined by

$$f_x(x, y, z) = \lim_{h\to 0}\frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

(f_y and f_z have similar definition).

If $w = f(x, y, z)$, then $\frac{\partial w}{\partial x}$ can be interpreted as the rate of change of w with respect to x when y and z are fixed.

- for a n-variables function $f(x_1, x_2, \cdots, x_n)$,

$$f_{x_i}(x_1, x_2, \cdots, x_n) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots x_n)}{h}.$$

If $u = f(x_1, x_2, \cdots, x_n)$, then $\dfrac{\partial u}{\partial x_i} = \dfrac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$ is the partial deriveative of u with respect to x_i.

Note. Denote $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots 0)$. Then

$$f_{x_i}(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}.$$

Example 14.3.6. Let $f(x, y, z) = e^{xy} \ln z$, then

$$f_x(x, y, z) = e^{xy} \cdot y \ln z = ye^{xy} \ln z, \quad f_y(x, y, z) = xe^{xy} \ln z, \quad f_z(x, y, z) = e^{xy} \cdot \frac{1}{z}.$$

■ **Higher Derivatives**

When study a single variable function $f(x)$, we can regard its derivative $f'(x)$ as a new function and consider its second derivative $f''(x)$.

For a two variables function $f(x, y)$, we can also regard its partial derivatives $f_x(x, y)$ and $f_y(x, y)$ as new functions and consider the "*second partial derivatives*". Let $z = f(x, y)$. Then

$$\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = f_{11} \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = f_{12} \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} = f_{21} \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} = f_{22}
\end{aligned}$$

- **third partial derivatives**

$$\begin{aligned}
(f_{xy})_x &= f_{xyx} = \frac{\partial}{\partial x}\Big(\frac{\partial^2 f}{\partial y \partial x}\Big) = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial x \partial y \partial x} \\
(f_{xy})_y &= f_{xyy} = \frac{\partial}{\partial y}\Big(\frac{\partial^2 f}{\partial y \partial x}\Big) = \frac{\partial^3 f}{\partial^2 y \partial x} = \frac{\partial^3 z}{\partial^2 y \partial x}
\end{aligned}$$

Example 14.3.7. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Then the first partial derivatives of f are

$$f_x = 3x^2 + 2xy^3, \qquad f_y = 3x^2y^2 - 4y$$

and the second partial derivatives of f are

$$f_{xx} = 6x + 2y^3, \quad f_{xy} = 6xy^2, \quad f_{yx} = 6xy^2, \quad f_{yy} = 6x^2y - 4.$$

■ **Clairaut's Theorem**

Question: For a multi-variables function, does the second partial derivatives keep unchanged when the order of two partial differentiations exchange? For example, if $f(x,y)$ has all second partial derivatives, can we obtain

$$f_{xy} \stackrel{??}{=} f_{yx}.$$

In general, the answer is false.

Exercise. Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Check that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Question: What conditions of f can guarantee its second partial derivatives are equal when exchanging their order?

Theorem 14.3.8. *(Clairaut's Theorem) Suppose f is defined on a neighborhood D of (a,b). If the functions f_{xy} and f_{yx} are both continuous at (a,b), then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Proof. Consider

$$\begin{aligned} f_{xy}(a,b) &= \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k} \\ &= \lim_{k\to 0} \frac{\lim\limits_{h\to 0}\left[\frac{f(a+h,b+k)-f(a,b+k)}{h} - \frac{f(a+h,b)-f(a,b)}{h}\right]}{k} \\ &= \lim_{k\to 0}\lim_{h\to 0} \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{kh}. \end{aligned}$$

Define $g(y) = f(a+h,y) - f(a,y)$ Then $f_{xy}(a,b) = \lim\limits_{k\to 0}\lim\limits_{h\to 0} \dfrac{g(b+k)-g(b)}{kh}$.

Since f_y is defined on a neighborhood of (a,b), g is differentiable near b and, by the mean value theorem, $g(b+k) - g(b) = kg'(\xi)$ for some $\xi = \xi(k) \in (0,k)$. Then

$$f_{xy}(a,b) = \lim_{k\to 0}\lim_{h\to 0} \frac{g'\big(\xi(k)\big)}{h} = \lim_{k\to 0}\lim_{h\to 0} \frac{1}{h}\Big[f_y\big(a+h,b+\xi(k)\big) - f_y\big(a,b+\xi(k)\big)\Big].$$

Since f_y is differentiable with respect to x and by the mean value theorem again,

$$f_{xy}(a,b) = \lim_{k\to 0}\lim_{h\to 0} f_{yx}\big(a+\eta(h), b+\xi(k)\big)$$

where $\eta(h) \in (0,h)$ and $\xi(k) \in (0,k)$ and hence $\lim\limits_{h\to 0}\eta(h) = 0$ and $\lim\limits_{k\to 0}\xi(k) = 0$. Also, the continuity of f_{yx} at (a,b) implies that

$$f_{xy}(a,b) = \lim_{k\to 0}\lim_{h\to 0} f_{yx}\big(a+\eta(h), b+\xi(k)\big) = f_{yx}(a,b).$$

□

Remark. The Clairaut's Theorem still holds if the hypothesis is weaken that one of f_{xy} and f_{yx} is continuous at (a, b).

Example 14.3.9. Let $f(x, y) = \sin(3x + yz)$. Then

$$f_x = 3\cos(3x + yz), \quad f_{xx} = -9\sin(3x + yz), \quad f_{xy} = -3z\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz), \quad f_{xyx} = -9z\cos(3x + yz) = f_{xxy}.$$

❑ Partial Differential Equations

(Skip)

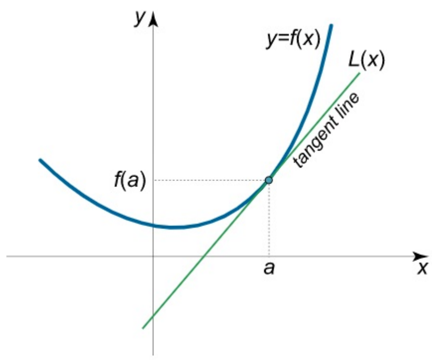
Homework 14.3. 5, 15, 19, 23, 25, 29, 33, 35, 39, 43, 47, 51, 59, 61, 85, 95, 100

14.4 Tangent Planes and Linear Approximations

❑ Tangent Planes

Recall that a single variable function $f(x)$ with derivative $f'(a)$ can be linearly approximated by its "tangent line" *

$$f(x) \approx L(x) = f(a) + f'(a)(x - a) \qquad \text{as } x \text{ is near } a$$



For a two variables function $f(x, y)$, we also expect that it can be linearly approximated by a certain "plane".

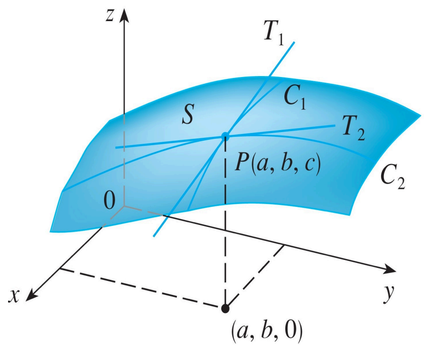
Suppose that

$f(x, y)$ is a two variables function which has continuous first partial derivatives;

S is the surface with equation $z = f(x, y)$ (the graph of f) and $P(a, b, c) \in S$;

C_1 and C_2 are the curves obtained by intersecting the vertical planes $y = b$ and $x = a$ with the sufrace S. Then $P \in C_1 \cap C_2$.

T_1 and T_2 are tangent lines to the curves C_1 and C_2 at the point P.

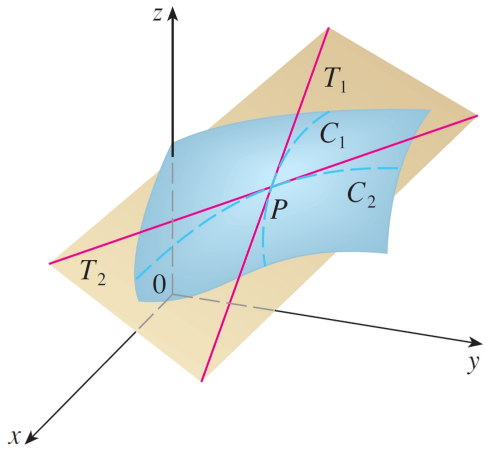


The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

*The figure is download from https://www.math24.net/linear-approximation/

Definition 14.4.1. The "*tangent plane*" to the surface S at P is defined to be the plane that contains both tangent lines T_1 and T_2.

Note. If C is any curve that lies on S and passes P, then the tangent line to C at P also lies on the tangent plane. Hence, we can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P.



The tangent plane contains the tangent line T_1 and T_2.

■ Equation of the tangent plane

Let the tangent plane to S passing throught $P(a, b, c)$ has equation

$$A(x - a) + B(y - b) + C(z - c) = 0 \tag{14.2}$$

We may assume that it is not a vertical tangent plane and hence $C \neq 0$. Dividing both sides of equation (14.3) by $-C$, the tangent plane has an equivalent equation

$$z - c = \alpha(x - a) + \beta(y - b) \qquad (\alpha = \frac{A}{-C} \text{ and } \beta = \frac{B}{-C}).$$

Since the intersection of the tangent plane and the vertical plane $y = b$ is the tangent line T_1, plugging $y = b$ into equation (14.3),

$$z - c = \alpha(x - a)$$

is the equation of the tangent line T_1. Then α is the slope of T_1 to the curve C_1 at (a, b, c) and hence $\alpha = f_x(a, b)$.

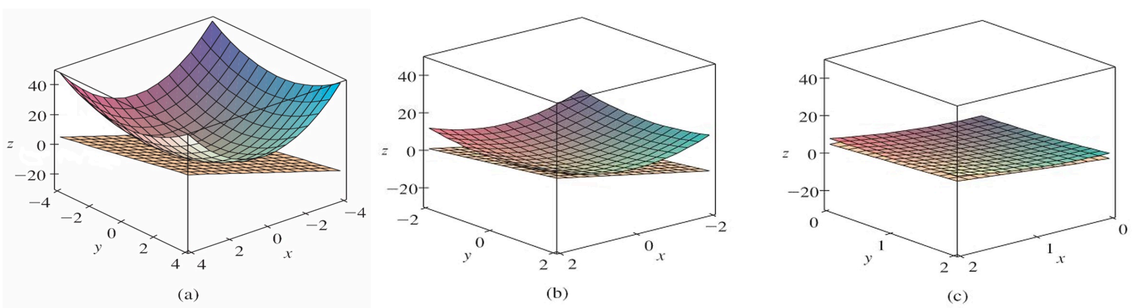
Similarly, $\beta = f_y(a, b)$. Therefore, the equation of the tangent plane to S at P is

$$z - c = f_x(a, b)\big(x - a\big) + f_y(a, b)\big(y - b\big).$$

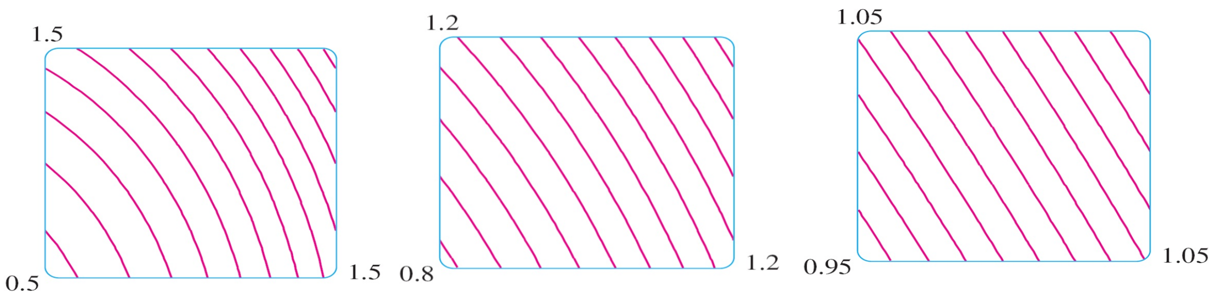
Example 14.4.2. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Proof. Let $f(x, y) = 2x^2 + y^2$. Then $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$. Hence, $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$. The equation of the tangent plane at $(1, 1, 3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1) \qquad \text{or} \qquad z = 4x + 2y - 3.$$



The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangnet plane as we zoom in toward (1,1,3).



Zooming in toward (1,1) on a contour map of $f(x,\ y) = 2x^2 + y^2$

□

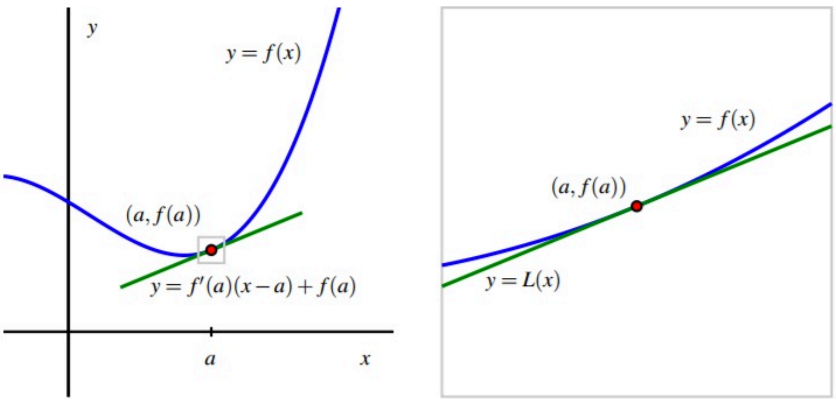
❑ Linear Approximations

We have studied the linear apporximation for a single variable function $f(x)$. We use the tangent line to the graph $y = f(x)$ at a to approxinate the value of f near a and the linearization for f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

and

$$f(x) \approx L(x) \qquad \text{as } x \text{ is close to } a.$$



For a two variable function $f(x, y)$, we expect to approximate its values, as (x, y) is near (a, b), by the tangnet plane at (a, b).

Suppose that $f(x, y)$ has continuous partial derivative. The tangnet plane to the surface $S : z = f(x, y)$ at $P\big(a, b, f(a, b)\big)$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Definition 14.4.3.

(a) We call the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

the "*linearization of f at* (a,b).

(b) The approximation $f(x,y) \approx L(x,y)$ is called the "*linear approximation*" or "tangent plane approximation" of f at (a,b).

Example 14.4.4. Find the linearization of $f(x,y) = 2x^2+y^2$ at $(1,1,3)$ and use it to approximate the value of $f(1.1, 0.95)$.

Proof. Compute $f_x(x,y) = 4x$ and $f_y(x,y) = 2y$ and hence $f_x(1,1) =$ and $f_y(1,1) = 2$. Then the linearization of f at $(1,1,3)$ is

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 3 + 4(x-1) + 2(y-1) = 4x + 2y - 3.$$

Also,

$$f(1.1, 0.95) \approx L(1.1, 0.95) = 3 + 4 \cdot 0.1 + 2 \cdot (-0.05) = 3.3.$$

□

We define tangent plane for surface $z = f(x,y)$, where f has continuous partial derivatives.

Question: What happens if f_x and f_y are not continuous? Consider the following example.

We define tangent plane for surface $z = f(x,y)$, where f has continuous partial derivatives.

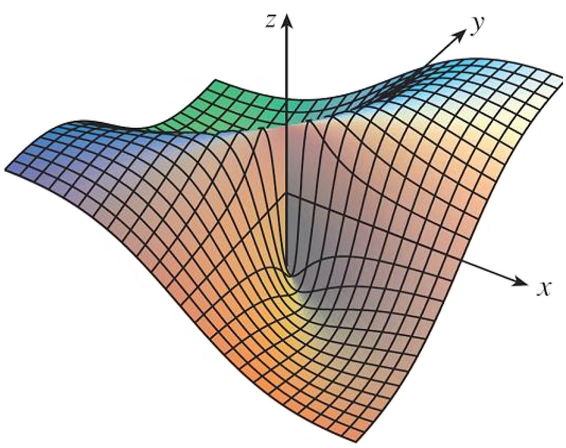
Example 14.4.5.

Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

Then $f_x(0,0) = 0 = f_y(0,0)$. For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$. Along $x = 0$,

$$\lim_{(x,y)\to(0,0),\ x=0} f_x(x,y) = \lim_{y\to 0} \frac{y^3}{y^4} = \infty.$$

Hence, f_x is continuous at $(0,0)$. Also, we can compute that f_y is not continuous at $(0,0)$. Observe that, for (x,y) on the line $x = y$, $f(x,y) = \dfrac{1}{2} \neq 0$. Therefore, f is not continuous at $(0,0)$. This implies that there is linear approximation of f at $(0,0)$.



$f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$,
$f(0,0) = 0$

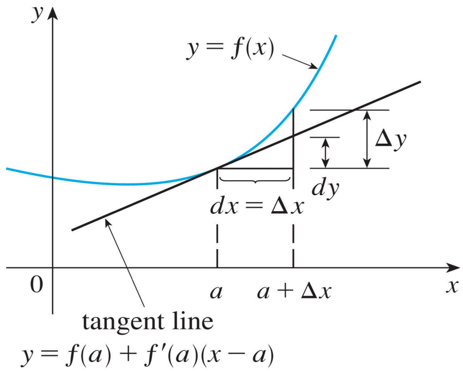
Note. This example says that for the linear approximation, the condition of the continuities of f_x and f_y are necessary.

❑ Differentiability

We recall the geometric meaning of linear approximation of $y = f(x)$. Let $\triangle y = f(a + \triangle x) - f(a)$. The rate of change of y with respect to x is

$$\frac{\triangle y}{\triangle x} = \frac{f(a + \triangle x) - f(a)}{\triangle x}.$$

If f is differentiable at a, then $\dfrac{\triangle y}{\triangle x} \to f'(a)$ as $\triangle x \to 0$.



Hence,

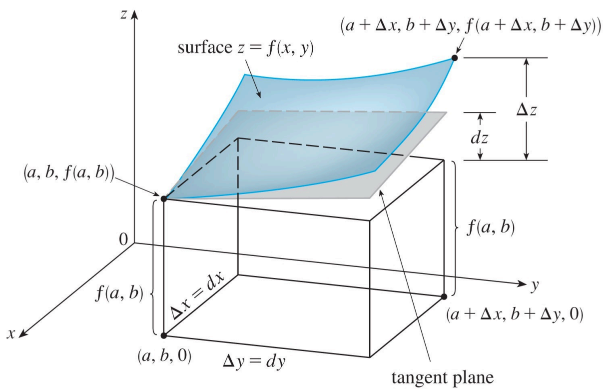
$$\underbrace{\triangle y}_{\text{increment in } y} = \underbrace{f'(a)\triangle x}_{\text{linear approximation}} + \underbrace{\varepsilon\triangle x}_{\text{error}} \quad \text{where } \varepsilon \to 0 \quad \text{as} \quad \triangle x \to 0.$$

(Note that $\varepsilon = \varepsilon(\triangle x)$ varies as $\triangle x$ varies.)

For a two variables function $z = f(x, y)$, as x changes from a to $a + \triangle x$ and y changes from b to $b + \triangle y$, the corresponding increment of z is

$$\begin{aligned} \triangle z &= f(a + \triangle x, b + \triangle y) - f(a, b) \\ &= \underbrace{f_x(a, b)\triangle x + f_y(a, b)\triangle y}_{\text{linear approximation}} + \underbrace{\varepsilon_1\triangle x + \varepsilon_2\triangle y}_{\text{error}} \end{aligned}$$

where $\varepsilon_1 = \varepsilon_1(\triangle x, \triangle y)$ and $\varepsilon_2 = \varepsilon_2(\triangle x, \triangle y)$. We expect that $\varepsilon_1, \varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.



$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Definition 14.4.6. Let $z = f(x, y)$. We call that f is "*differentiable*" at (a, b) if $\triangle z$ can be expressed in the form

$$\triangle z = f_x(a, b)\triangle x + f_y(a, b)\triangle y + \varepsilon_1\triangle x + \varepsilon_2\triangle y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.

Exercise. If $f(x, y)$ is differentiable at (a, b), then f is continuous at (a, b).

From Example 14.4.5, a two variables function $f(x, y)$ has all partial derivative at (a, b) cannot guarantee that it is differentiable there.

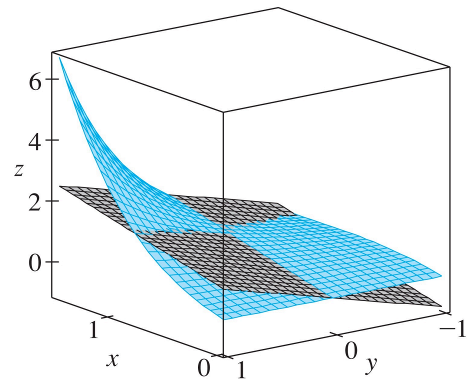
■ Sufficient condition for differentiability

Theorem 14.4.7. *If the partial derivative f_x and f_y exists near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 14.4.8. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1,-0.1)$.

Proof. Since $f_x(x,y) = e^{xy} + xye^{xy}$ and $f_y(x,y) = x^2e^{xy}$ are continuous functions, $f(x,y)$ is differentiable everywhere. Moreover, $f_x(1,0) = 1$ and $f_y(1,0) = 1$. The linearization of f at $(1,0)$ is

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + (x-1) + y \\ &= x + y. \end{aligned}$$

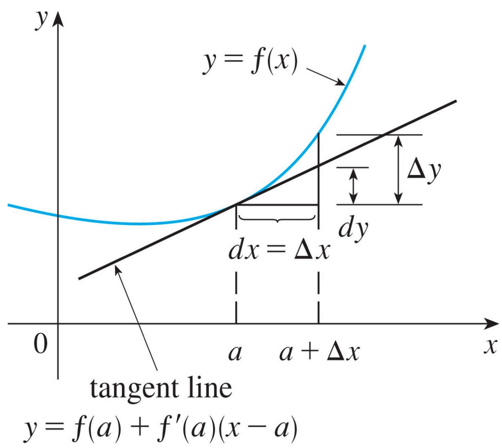


Then

$$f(1.1,-0.1) \approx L(1.1,-0.1) = 1.1 + (-0.1) = 1.$$

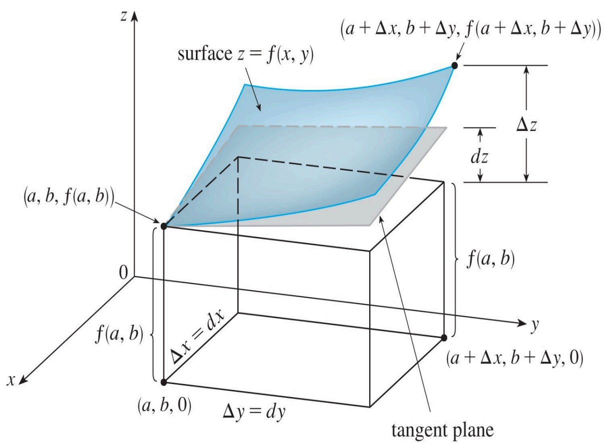
In fact, $f(1.1,-0.1) = 1.1e^{-0.1} \approx 0.98542$. □

❏ Differentials



Recall that for a differentiable single variable function $y = f(x)$, dx is the differantial of x and $dy = f'(x)\,dx$ is a differential of y.
The symbol $\triangle y$ denotes the change in height of y and dy represents the change in height of the tangent line when x changes $\triangle x = dx$. Hence, as (x,y) is near (a,b),

$$f(x,y) \approx f(a,b) + f'(a,b)\,dx = f(a,b) + dy.$$



For a differentiable fucntion of two variables $z = f(x,y)$, dx and dy are differentials of x and y respectively, and dz is the differenital of z which is called the "*total differential*". Then

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Taking $dx = \triangle x = x - a$ and $dy = \triangle y = y - b$, then

$$dz = f_x(x,y)(x-a) + f_y(x,y)(y-b).$$

As (x,y) is near (a,b),

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = f(a,b) + dz.$$

Example 14.4.9.

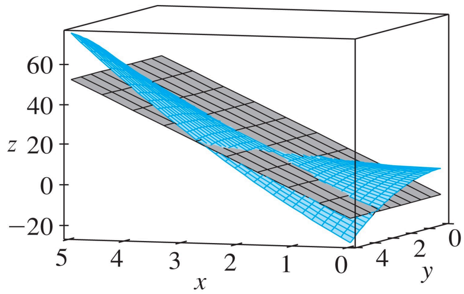
(a) If $z = f(x, y) = x^2 + 2xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.6, compare the values of $\triangle z$ and dz.

Proof.

(a) To find dz, $f_x(x, y) = 2x + 3y$ and $f_y(x, y) = 3x - 2y$. Then

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy.$$



(b) If x changes from x to 2.05 and y changes from 3 to 2.96, compare $\triangle z$ and dz.

$$\begin{aligned} \triangle z &= \ = f(2.05, 2.96) - f(2, 3) = 0.6449 \\ dz &= f_x(2, 3)(2.05 - 2) + f_y(2, 3)(2.96 - 3) = 0.65. \end{aligned}$$

□

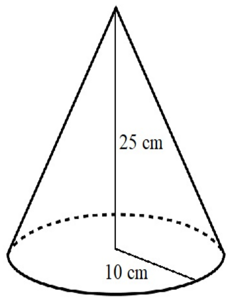
Example 14.4.10. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as ε cm in each.

(a) Use differentials to estimate the maximum error in the calculated volume of the cone.

Proof.

The volume of the cone is $V(r, h) = \frac{1}{3}\pi r^2 h$. Then

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \qquad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2.$$



The differential of V is

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = \frac{2\pi r h}{3}\, dr + \frac{\pi r^2}{3}\, dh.$$

When $|dr| \le \varepsilon$ and $|dh| \le \varepsilon$ and at $(r, h) = (10, 25)$, the differential of V is

$$\begin{aligned} \triangle V &\approx dV \le \frac{\partial V}{\partial r}(10, 25)\cdot\varepsilon + \frac{\partial V}{\partial h}(10, 25)\cdot\varepsilon \\ &= \frac{500\pi}{3}\cdot\varepsilon + \frac{100\pi}{3}\cdot\varepsilon = 200\pi\varepsilon \quad (\text{cm}^3) \end{aligned}$$

□

(b) What is the estimated maximum error in volume if the radius and height are measured with errors up to 0.1 cm?

Proof. Taking $\varepsilon = 0.1$ cm, then $dV = 200\pi(0.1) = 20\pi \approx 63 \ \ (\text{cm}^3)$. □

Note that the relative error is $\dfrac{dV}{V} \approx \dfrac{63}{2618} \approx 0.0214$ or 2.4%.

❑ Functions of Three or More Variables

■ Linear Approximation

The linearization of f at (a, b, c) is

$$f(x, y, z) \approx L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

■ Differentials

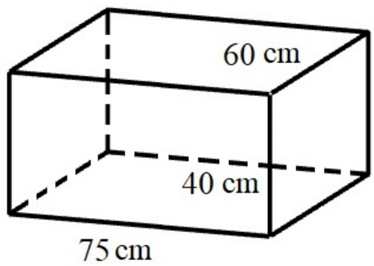
Let $w = f(x, y, z)$. Then

$$\begin{aligned} \triangle w &= f(x + \triangle x, y + \triangle y, z + \triangle z) - f(x, y, z) \\ dw &= f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz = \frac{\partial w}{dx}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz. \end{aligned}$$

Example 14.4.11. A rectangular box has length, width, and height 75cm, 60 cm and 40cm respectively. Use differentials to estimate the largest possible error when the volume of the box is calculatedas each measurement is correct to within ε cm.

(a) Use the differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Proof.



Let x, y and z denote the length, width and height of the box. The volume of the box is $V(x, y, z) = xyz$. Then

$$\frac{\partial V}{\partial x} = xy, \ \frac{\partial V}{\partial y} = xz, \ \frac{\partial V}{\partial z} = xy.$$

The differential of V is

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz.$$

When $|dx| \leq \varepsilon$, $|dy| \leq \varepsilon$ and $|dz| \leq \varepsilon$ and at $(x, y, z) = (75, 60, 40)$, the differential of V is

$$\triangle V \approx dV \leq (60)(40)\varepsilon + (75)(40)\varepsilon + (75)(60)\varepsilon = 9900\varepsilon \quad (\text{cm}^3).$$

□

(b) What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm?

Proof. Taking $\varepsilon = 0.2$, then $dV = 9900(0.2) = 1980$ cm^3 in the calculated volume. □

Note that this may seem like a large error, but the relative error is $\dfrac{dV}{V} = \dfrac{1980}{(75)(60)(40)} = 0.011$ or 1.1%.

Homework 14.4. 5, 9, 17, 21, 24, 27, 34, 36, 39, 43, 49, 54

14.5 The Chain Rule

Recall the for single variable functions $y = f(x)$, $x = g(t)$, $y = f\big(g(t)\big)$ is a composite function of variable t. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

■ **The Chain Rule: Case 1**

For a two variables function $z = f(x, y)$, if $x = g(t)$ and $y = h(t)$, then $z = f\big(g(t), h(t)\big)$ is indeirectly a function of t, say $z = z(t)$. Suppose that $z = f(x, y)$ is differentiable and, $x = g(t)$ and $y = h(t)$ are differentiable. Then

$$\begin{aligned}\triangle z &= f_x(x,y)\triangle x + f_y(x,y)\triangle y + \varepsilon_1\triangle x + \varepsilon_2\triangle y\\ &= f_x(x,y)\frac{\triangle x}{\triangle t}\triangle t + f_y(x,y)\frac{\triangle y}{\triangle t}\triangle t + \varepsilon_1\frac{\triangle x}{\triangle t}\triangle t + \varepsilon_2\frac{\triangle y}{\triangle t}\triangle t\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$. Since $x = g(t)$ and $y = h(t)$ are differentiable in t, we have $\dfrac{\triangle x}{\triangle t} \to \dfrac{dx}{dt}$ and $\dfrac{\triangle y}{\triangle t} \to \dfrac{dy}{dt}$ as $\triangle t \to 0$. Then, letting $\triangle t \to 0$,

$$\frac{\triangle z}{\triangle t} \to f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt} + \underbrace{\lim_{\triangle t\to 0}\varepsilon_1}_{=0}\cdot\frac{dx}{dt} + \underbrace{\lim_{\triangle t\to 0}\varepsilon_2}_{=0}\cdot\frac{dy}{dt}.$$

We obtain

$$\frac{dz}{dt} = \lim_{\triangle t\to 0}\frac{\triangle z}{\triangle t} = f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}.$$

Theorem 14.5.1. *(The Chain Rule: Case 1) (Two variables function) Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = x(t)$ and $y = y(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Remark. In Chapter 13, we studied the n vector-valued function $\mathbf{r}(t) = \langle x_1(t), \cdots, x_n(t)\rangle : I \to \mathbb{R}^n$. If $\mathbf{r}(t)$ is differentiable on I, then

$$\mathbf{r}'(t) = \langle x_1'(t), \cdots, x_n'(t)\rangle.$$

Hence, we have the chain rule for general multiple variables functions:

Suppose that $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. If $\mathbf{r} = \mathbf{r}(t)$ is a differentiable curve in D, then $f \circ \mathbf{r}$ is differentiable and

$$\frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big) = \nabla f\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t).$$

Proof. It suffices to prove the case $n = 2$ and the general cases are similar.

Since $x = x(t)$ and $y = y(t)$ are differentiable in t,

$$\triangle x = x(t + \triangle t) - x(t) = \frac{dx}{dt}\Delta t + \varepsilon_1 \Delta t \quad \text{and} \quad \triangle y = y(t + \triangle t) - y(t) = \frac{dy}{dt}\Delta t + \varepsilon_2 \Delta t$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\triangle t \to 0$ as well as

$$\lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t} = \frac{dx}{dt} \quad \text{and} \quad \lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t} = \frac{dy}{dt}.$$

Clearly, $\triangle x, \triangle y \to 0$ as $\triangle t \to 0$.

On the other hand, since f is differentiable,

$$\begin{aligned}\triangle z &= f(x + \triangle x, y + \triangle y) - f(x, y) \\ &= f_x(x, y)\triangle x + f_y(x, y)\triangle y + \varepsilon_3 \triangle x + \varepsilon_4 \triangle y\end{aligned}$$

where $\varepsilon_3, \varepsilon_4 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$. Then

$$\frac{\triangle z}{\triangle t} = f_x(x, y)\frac{\triangle x}{\triangle t} + f_y(x, y)\frac{\triangle y}{\triangle t} + \varepsilon_3 \frac{\triangle x}{\triangle t} + \varepsilon_4 \frac{\triangle y}{\triangle t}.$$

Taking limits as $\triangle t \to 0$, we have

$$\begin{aligned}\frac{dz}{dt} = \lim_{\triangle t \to 0} \frac{\triangle z}{\triangle t} &= f_x(x, y) \underbrace{\Big(\lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t}\Big)}_{=\frac{dx}{dt}} + f_y(x, y) \underbrace{\Big(\lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t}\Big)}_{=\frac{dy}{dt}} \\ &\quad + \underbrace{\Big(\lim_{\triangle t \to 0} \varepsilon_3\Big)}_{=0} \Big(\lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t}\Big) + \underbrace{\Big(\lim_{\triangle t \to 0} \varepsilon_4\Big)}_{=0} \Big(\lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t}\Big) \\ &= f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt} \\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.\end{aligned}$$

□

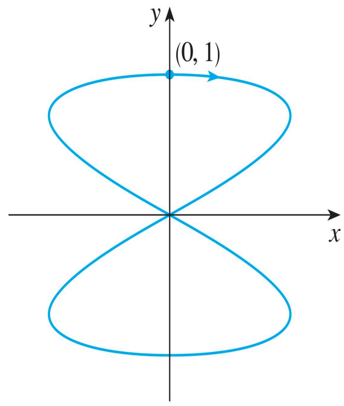
Example 14.5.2. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Proof.

Compute $\frac{\partial z}{\partial x} = 2xy + 3y^4$ and $\frac{\partial z}{\partial y} = x^2 + 12xy^3$. Then

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dz}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy + xy^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t) \\
&= (2\sin 2t\cos t + 3\cos^4 t)(2\cos 2t) \\
&\qquad +(\sin^2 2t + 12\sin 2t\cos^3 t)(-\sin t).
\end{aligned}$$

At $t = 0$, $\left.\frac{dz}{dt}\right|_{t=0} = 6.$



The curve $x = \sin 2t$, $y = \cos t$

Note that $\frac{dz}{dt}$ represents the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equation $r(t) = \langle \sin 2t, \cos t\rangle$. □

Example 14.5.3. The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the reate at whcih the ressure is changing when the temperature is $300K$ and increasing at a reate of 0.1 K/s and the volume is $100L$ and increasing at a rate of 0.2 L/s.

Proof. From the equation $PV = 8.31T$, we can express P as a function of variables V and T. That is, $P = 8.31\frac{T}{V}$. By the Chain Rule,

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = 8.31 \cdot \frac{1}{V} \cdot \frac{dT}{dt} + 8.31\left(-\frac{T}{V^2}\right) \cdot \frac{dV}{dt}$$

The hypothesis indicates that $\frac{dT}{dt} = 0.1$ and $\frac{dV}{dt} = 0.2$. We want to find $\left.\frac{dP}{dt}\right|_{(T,V)=(300,100)}$. Then

$$\left.\frac{dP}{dt}\right|_{(T,V)=(300,100)} = 8.31\left[\frac{1}{100} \cdot 0.1 + \left(-\frac{300}{100^2}\right) \cdot 0.2\right] = -0.04155 \quad (KPa/s)$$

□

Example 14.5.4. Compute the rate of change of $f(x, y, z) = x^2y + z\cos z$ along the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.

Proof. Compute

$$\nabla f(x, y, z) = \langle 2xy, x^2, \cos z - z\sin z\rangle \quad \text{and} \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle.$$

Then

$$\begin{aligned}\frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big) &= \nabla f\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\\ &= \langle 2t^3, t^2, \cos t^3 - t^3\sin t^3\rangle\cdot\langle 1, 2t, 3t^2\rangle\\ &= 4t^3 + 3t^2\cos t^3 - 3t^5\sin t^3.\end{aligned}$$

□

Remark. (1) Suppose that $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$ and $\mathbf{r}(t) = \langle x_1(t), \cdots, x_n(t)\rangle$. Then

$$\nabla f(\mathbf{x}) = \langle\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x})\rangle \quad \text{and} \quad \mathbf{r}'(t) = \langle x_1'(t), \cdots, x_n'(t)\rangle$$

Hence,

$$\begin{aligned}\frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big) &= \nabla f\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\\ &= \langle\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x})\rangle\cdot\langle x_1'(t), \cdots, x_n'(t)\rangle\\ &= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\big(\mathbf{r}(t)\big)x_i'(t)\\ &= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\big(\mathbf{r}(t)\big)\frac{dx_i}{dt}(t)\end{aligned}$$

(2) Recall that the directional derivative of f at (a, b) in the direction $\mathbf{u}$ (unit vector) is

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b)\cdot\mathbf{u}.$$

Let the plane curve $\mathbf{r}(t)$ pass $\langle a, b\rangle$ when $t = t_0$ (that is, $\mathbf{r}(t_0) = \langle a, b\rangle$). Then

$$\frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big)\Big|_{t=t_0} = \nabla f\big(\mathbf{r}(t_0)\big)\cdot\mathbf{r}'(t_0) = \|\mathbf{r}'(t_0)\|D_{\mathbf{u}}f(a, b)$$

where $\mathbf{u} = \dfrac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|}$. This means that the rate of change of the composite function $f\big(\mathbf{r}(t)\big)$ at $t = t_0$ is equal to $\|\mathbf{r}'(t_0)\|$ multiple of the directional derivative of f at $\mathbf{r}(t_0)$ in the direction $\mathbf{r}'(t_0)$.

Corollary 14.5.5. *If $x = x(t)$ and $y = y(t)$ are twice differentiable at t and if $z = f(x, y)$ is* *twice differentiable at* $\big(x(t), y(t)\big)$, *then* $z = f\big(x(t), y(t)\big)$ *is twice differentiable at* t *and*

$$\frac{d^2z}{dt^2} = \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \left(\frac{dy}{dt}\right)^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}.$$

Proof. (Exercise) □

■ **The Chain Rule: Case 2**

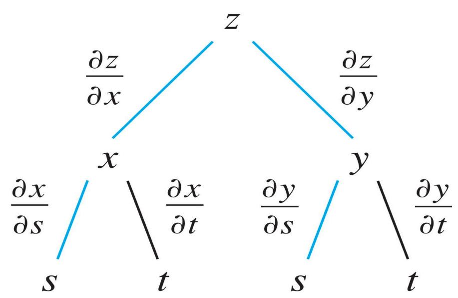
Let $z = f(x, y)$, $x = x(s, t)$ and $y = y(s, t)$ be differentiable functions. Then $z = z(s, t) = f\big(x(s, t), y(s, t)\big)$ is indirectly a function of s and t. Consider the partial derivative of z with respect to t. From the discuss in Section 14.3, fixing s (as a constant w.r.t t) and regarding z as a function of t. We can use the idea of Case1 to find the partial derivative of z with respect to t.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Theorem 14.5.6. *(The Chain Rule: Case 2) Suppose that* $z = f(x, y)$ *is a differentiable function of* x *and* y, *where* $x = x(s, t)$ *and* $y = y(s, t)$ *are differentiable functions of* s *and* t. *Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

The tree diagram is



If $x_i = x_i(s, t)$ *are differentiable at* (s, t) *for* $i = 1, \cdots n$ *and* $z = f\big(x_1, \cdots, x_n\big)$ *is differentiable at* $\big(x_1(t), x_n(t)\big)$ *then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial s} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial s} = \sum_{i=1}^{n}\frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t} = \sum_{i=1}^{n}\frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t}$$

Example 14.5.7. If $z = e^x \sin y$, where $x = st$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Proof. Compute that

$$\frac{\partial z}{\partial x} = e^x \sin y, \ \frac{\partial z}{\partial y} = e^x \cos y$$

and

$$\frac{\partial x}{\partial s} = t^2, \ \frac{\partial x}{\partial t} = 2st, \ \frac{\partial y}{\partial s} = 2st, \ \frac{\partial y}{\partial t} = s^2.$$

Then

$$\begin{aligned}\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} &= e^2 \sin y \cdot t^2 + e^2 \cos y \cdot 2st \\ &= t^2 e^{st} \sin(s^2 t) + 2ste^{st} \cos(s^2 t).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} &= e^2 \sin y \cdot 2st + e^2 \cos y \cdot s^2 \\ &= 2ste^{st} \sin(s^2 t) + s^2 e^{st} \cos(s^2 t).\end{aligned}$$

□

Corollary 14.5.8. *Suppose that $z = f(x, y)$ is a twice differentiable function of x and y, where $x = x(s, t)$ and $y = y(s, t)$ are twice differentiable functions of s and t. Then*

$$\begin{aligned}\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}\Big(\frac{\partial z}{\partial s}\Big) &= \frac{\partial}{\partial s}\Big[\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\Big] \\ &= \Big(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial s}\Big)\frac{\partial x}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} \\ &+ \Big(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\Big)\frac{\partial y}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2}\end{aligned}$$

Example 14.5.9. Let $u = f(s^2 + t^2, st)$ Find $\frac{\partial^2 u}{\partial s \partial t}$.

Proof.

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}(s^2 + t^2, st) \cdot 2t + \frac{\partial f}{\partial y}(s^2 + t^2, st) \cdot s.$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial s \partial t} = \frac{\partial}{\partial s}\Big(\frac{\partial u}{\partial t}\Big) &= \frac{\partial^2 f}{\partial x^2}(s^2 + t^2, st)(2s)(2t) + \frac{\partial^2 f}{\partial y \partial x}(s^2 + t^2, st)(2t^2) \\ &+ \frac{\partial^2 f}{\partial x \partial y}(s^2 + t^2, st)2s \cdot s + \frac{\partial^2 f}{\partial y^2}(s^2 + t^2, st)t \cdot s + \frac{\partial f}{\partial y}(s^2 + t^2, st) \cdot 1.\end{aligned}$$

□

■ **Chain Rule: General Version**

Suppose that u is a differentiable function of n variables $x_1, \cdots, x_n$ and each x_i is a differenbitable function of m variables $t_1, \cdots, t_m$. Then u is a differentiable function of $t_1, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

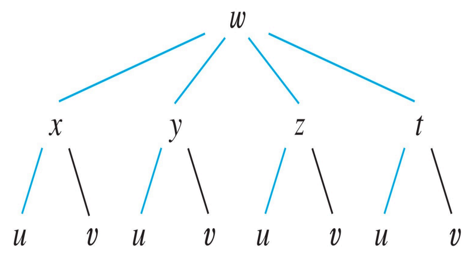
for each $i = 1, 2, \cdots, m$.

Example 14.5.10. Let $w = f(x, y, z, t)$, $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$. Then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.$$



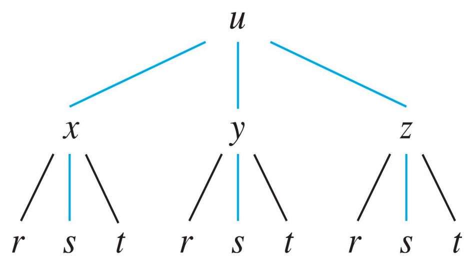
Example 14.5.11. If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$ and $t = 0$.

Proof.

$$\frac{\partial u}{\partial x} = 4x^3y, \quad \frac{\partial u}{\partial y} = x^4 + 2yz^3, \quad \frac{\partial u}{\partial z} = 3y^2z^2$$

and

$$\frac{\partial x}{\partial s} = re^t, \quad \frac{\partial y}{\partial s} = 2rse^{-t}, \quad \frac{\partial z}{\partial s} = r^2 \sin t.$$



Then

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= 4x^3y \cdot re^t + (x^4 + 2yz^3) \cdot 2rse^{-t} + 3y^2z^2 \cdot r^2 \sin t.\end{aligned}$$

When $(r, s, t) = (2, 1, 0)$, $x = 2$, $y = 2$ and $z = 0$. Hence,

$$\left.\frac{\partial u}{\partial s}\right|_{(r,s,t)=(2,1,0)} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = 192.$$

□

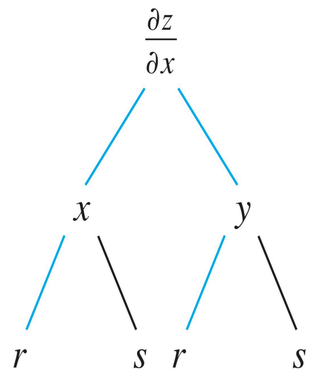
Example 14.5.12. If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$.

Proof.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s).$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) &= 2r\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\right] + 2\frac{\partial z}{\partial x} \\ &\quad +2s\left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right] \\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 4s^2\frac{\partial^2 z}{\partial y^2} + 8sr\frac{\partial^2 z}{\partial x \partial y}.\end{aligned}$$



Note that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ since f has continuous second partial derivatives. □

❏ Implicit Differentiation

Recall that if the two variable x and y have a relation, for example $xy^2 + x\sin y = 1$, we can find $\dfrac{dy}{dx}$. By differentiating of both sides,

$$\frac{d}{dx}\left(xy^2 + x\sin y\right) = \frac{d}{dx}(1)$$

we have

$$\frac{dy}{dx} = -\frac{y^2 + \sin y}{2xy + x\cos y}.$$

In general, for the equation $F(x, y) = 0$ where F is differentiable, we can regard y as a function of x. That is, $y = f(x)$ and then $F\big(x, f(x)\big) = 0$. To find $\dfrac{dy}{dx}$,

$$\frac{\partial}{\partial x}\Big(F(x, y)\Big) = \frac{\partial}{\partial x}(0).$$

We have

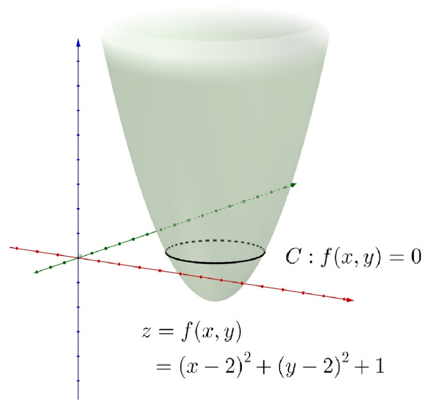
$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

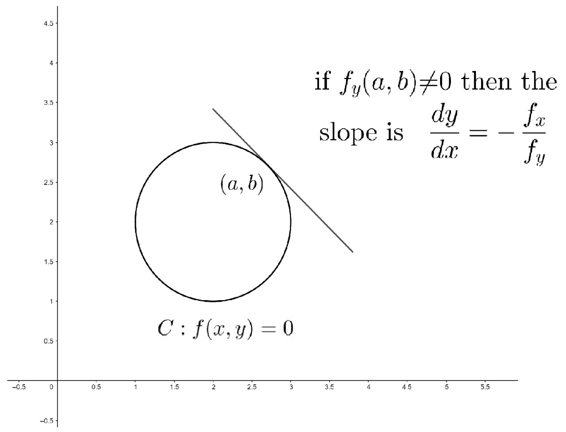
and then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Note. The "*Implicit Function Theorem*" give conditions under which this assumption is valid: if F is defined on a dist containing (a, b) where $F(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivtive of y with respect to x is

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$





Example 14.5.13. Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Proof. Let $F(x, y) = x^3 + y^3 - 6xy$. Then $F_x = 3x^2 - 6y$ and $F_y = 3y^2 - 6x$. We have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}.$$

□

Question: If $z = f(x, y)$ or $F(x, y, z) = 0$, how to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$?

For $F(x, y, z) = 0$, we can regard z as a function of x and y, say $z = f(x, y)$. Then $F\big(x, y, f(x, y)\big)$ for all $x, y \in Dom(f)$. Find $\dfrac{\partial z}{\partial x}$. Consider

$$\frac{\partial}{\partial x}\Big(F(x, y, z)\Big) = \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(0) = 0.$$

Therefore,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{provided } F_z \neq 0.$$

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$ provided $F_z \neq 0$.

Example 14.5.14. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Proof. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then

$$F_x = 3x^2 + 6yz,\ F_y = 3y^2 + 6xz,\ F_z = 3z^2 + 6xy.$$

We have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x^2 + 2yz}{z^2 + 2xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

□

We give the Implicit Function Theorem here. It will be discussed in the course of Advanced Calculus.

Theorem 14.5.15. *(Implicit Function Theorem) If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y and F_z aer continuous inside the sphere, then the equation $F(x, y, z) = 0$ define z as a function of x and y near the point (a, b, c) and this function is differentiable and*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Homework 14.5. 4, 7, 12, 15, 18, 21, 28, 29, 34, 38, 39, 43, 52, 60

14.6 Directional Derivatives and the Gradient Vector

❑ Directional Derivatives

In Section 14.3, we studied the partial derivatives for a two variables function $z = f(x, y)$. The partial derivative

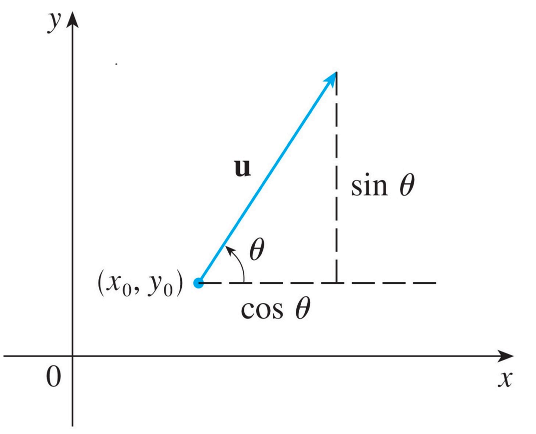
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

represents the rate of change of z in the x-direction (in the direction of the unit vector **i**). Similarly,

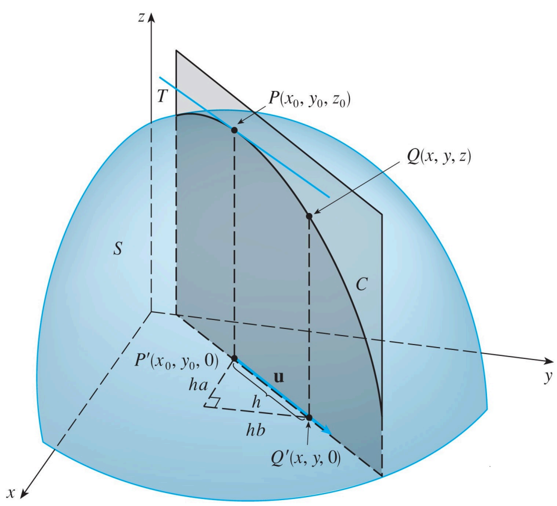
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

represents the rate of change of z in the y-direction (in the direction of the unit vector **j**).

Question: How about the rate of change of z at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$.



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$



Let $P(x_0, y_0, z_0)$ lie on a surface S. The vertical plane that passes through P in the direction of **u** intersects S in a curve C. The slope of the tangent line T to C at the point P is the rate of change of z in the direction $\mathbf{u}$.

Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector and $z = f(x, y)$. Consider the quotient difference of z in the direction $\mathbf{u}$

$$\frac{\triangle z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

Taking $h \to 0$, we obtain the rate of change of z in the direction $\mathbf{u}$.

Definition 14.6.1.

(a) Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function and $(x_0, y_0) \in D$. The "*directional derivatives*" of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

(b) In general, let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function, $\mathbf{a} \in D$ and $\mathbf{u}$ be a unit vector. The directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{u}$ is the limit

$$\lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

if it exists and is denoted by $D_{\mathbf{u}} f(\mathbf{a})$.

Remark. (1) In the above definition, the direction $\mathbf{u}$ is a "unit" vector. Hence, if we want to compute the directional derivative of f in the direction $\mathbf{v}$, which is not a unit vector, we should normalize $\mathbf{v}$ by $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$.

(2) If $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, then $D_{\mathbf{i}} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{\mathbf{j}} f(x_0, y_0) = f_y(x_0, y_0)$. The partial derivative of f with respect to x_i is a special directional derivative in the direction x_i.

(3) If $\mathbf{u} = \langle 0, \cdots, 0, 1, 0, \cdots, 0\rangle$, then $D_{\mathbf{u}} f(\mathbf{a}) = f_{x_i}(\mathbf{a})$. The partial derivative of f with respect to x_i is a special directional derivative in the direction x_i.

To compute the directional derivative $D_{\mathbf{u}} f(x_0, y_0)$, there are two common methods:

(i) By the definition

(ii) Under certain assumptions, we can use the following theorem.

Theorem 14.6.2. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$D_{\mathbf{u}} f(x, , y) = f_x(x, y)a + f_y(x, y)b.$$

Proof. Let $g(h) = f(x_0 + ha, y_0 + hb)$ where $x = x_0 + ha$ and $y = y_0 + hb$. Then

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \qquad \text{(exists since } f \text{ is differentiable)} \\ &= \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0) \end{aligned}$$

Also, by the Chain Rule,

$$\begin{aligned} g'(h) &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} \\ &= f_x(x_0 + ha, y_0 + hb)a + f_y(x_0 + ha, y_0 + hb)b \end{aligned}$$

Therefore, putting $h = 0$,

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

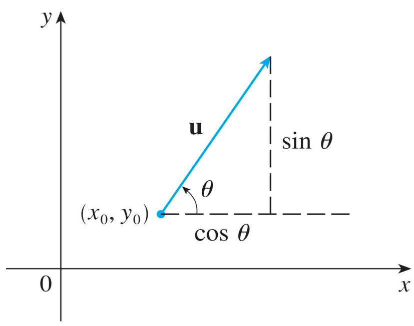
□

Note. In particular, if f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Moreover, if $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$, then

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Theorem 14.6.3. *If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$, then f has a directional derivative at $\mathbf{a}$ in every direction $\mathbf{u}$ where $\mathbf{u}$ is a unit vector and*

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Proof.

Recall that f is differentiable at $\mathbf{a}$. Then

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}|}{|\mathbf{h}|} = 0.$$

Let $\mathbf{h} = t\mathbf{u}$ and then $|\mathbf{h}| = |t||\mathbf{u}| = |t|$. We have

$$\frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} = \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}}{t} + \frac{\nabla f(\mathbf{a}) \cdot \mathbf{h}}{t}.$$

Hence,

$$\begin{aligned}\lim_{t\to 0}\left|\frac{f(\mathbf{a}+t\mathbf{u})-f(\mathbf{a})}{t}-\nabla f(\mathbf{a})\cdot\mathbf{u}\right| &= \lim_{t\to 0}\left|\frac{f(\mathbf{a}+t\mathbf{u})-f(\mathbf{a})}{t}-\frac{\nabla f(\mathbf{a})\cdot(t\mathbf{u})}{t}\right| \\ &= \lim_{\mathbf{h}\to\mathbf{0}}\frac{|f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-\nabla f(\mathbf{a})\cdot\mathbf{h}|}{|\mathbf{h}|} \\ &= 0. \quad (\text{since } f \text{ is differentiable at } \mathbf{a})\end{aligned}$$

Therefore,

$$D_{\mathbf{u}}f(\mathbf{a}) = \lim_{t\to 0}\frac{f(\mathbf{a}+t\mathbf{u})-f(\mathbf{a})}{t} = \nabla f(\mathbf{a})\cdot\mathbf{u}.$$

□

Remark. If f is differentiable and $\mathbf{u}$ is a unit vector, then

$$D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a})\cdot\mathbf{u}.$$

This means that the directional derivative (the rate of change of f) in the direction of a unit vector $\mathbf{u}$ is the scalar projection of the gradient vector $\nabla f(\mathbf{a})$ onto $\mathbf{u}$.

Example 14.6.4. Find the directional derivative $D_{\mathbf{u}}f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_{\mathbf{u}}f(1,2)$?

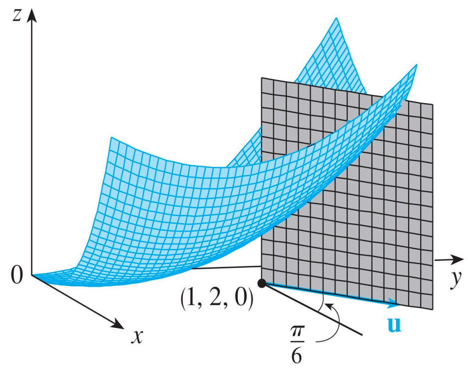
Proof. The gradient of f is

$$\nabla f = \langle f_x, f_y\rangle = \langle 3x^2-3y, -3x+8y\rangle.$$

Hence, the directional derivative is

$$\begin{aligned}D_{\mathbf{u}}f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (3x^2-3y)\cos\frac{\pi}{6} + (-3x+8y)\sin\frac{\pi}{6} \\ &= \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8-3\sqrt{3})y]\end{aligned}$$

and $D_{\mathbf{u}}f(1,2) = \dfrac{13-3\sqrt{3}}{2}$.



□

❑ The Gradient Vector

Note. If $f(x,y)$ is a differentiable function of x and y, then the directional derivative of f at (x_0,y_0) in the unit vector $\mathbf{u} = \langle a,b\rangle$ is

$$D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b = \big\langle f_x(x_0,y_0), f_y(x_0,y_0)\big\rangle\cdot\langle a,b\rangle.$$

Definition 14.6.5. If f is a function of two variables x and y, then the "*gradient*" of f is the vector function, "∇f", defined by

$$\nabla f(x,y) = \left\langle f_x(x,y), f_y(x,y)\right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notation: Denote "*grad* f" or "∇f" and read "del f".

Remark. (1) The gradient of f, $\nabla f(x,y)$ is a vector.

(2) If f is differentiable and $\mathbf{u}$ is a unit vector, then

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot \mathbf{u}.$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar prejection of the gradient vecctor onto $\mathbf{u}$.

Example 14.6.6. If $f(x,y) = \sin x + e^{xy}$, then

$$\begin{aligned}\nabla f(x,y) &= \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle\\ \nabla f(0,1) &= \langle 2, 0\rangle.\end{aligned}$$

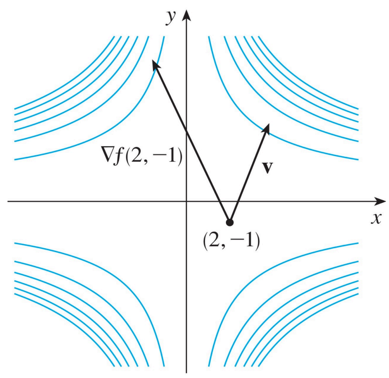
Example 14.6.7. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Proof. The gradient of f is

$$\nabla f = \langle f_x, f_y\rangle = \langle 2xy^3, 3x^2y^2 - 4\rangle.$$

Let $\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$. The directional derivative is

$$D_{\mathbf{u}} f(2,-1) = f_x(2,-1)\cdot\frac{2}{\sqrt{29}} + f_y(2,-1)\cdot\frac{5}{\sqrt{29}} = \frac{32}{\sqrt{29}}.$$



□

■ Function of Three Variables

Let $f(x,y,z)$ be a three variables function and $\mathbf{u}$ be a unit vector. The vector function $D_{\mathbf{u}} f(x,y,z)$ can be interpreted as the rate of change of the function in the direction of $\mathbf{u}$.

Definition 14.6.8. The "*directional derivative*" of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0,y_0,z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

if this limit exists.

Note. If we use vector notatin, then the directional derivative can be written as

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0\rangle$ (or $\langle x_0, y_0, z_0\rangle$)

Remark. If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c\rangle$, then

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

The "*gradient*" of f is

$$\nabla f(x, y, z) = \big\langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\big\langle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Example 14.6.9. Let $f(x, y, z) = x \sin yz$.

(a)

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \langle \sin yz, xz \cos yz, xy \cos yz\rangle.$$

(b) At $(1, 3, 0)$, for the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The unit vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$. Then the directional derivative at $(1, 3, 0)$ in the direction $\mathbf{v}$ is

$$\begin{aligned} D_{\mathbf{u}}f(1, 3, 0) &= \Big\langle \sin 0, 0 \cos 0, 3 \cos 0\Big\rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle \\ &= \langle 0, 0, 3\rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle = -\frac{3}{\sqrt{6}}. \end{aligned}$$

❏ Differentiability and Partial Derivatives

From Definition 14.4.6, we can prove that a differentiable function f havs (all) partial derivatives. In fact, it has directional derivatives in every direction. But the converse is false. There indeed exists a function which has all directional derivatives but it is not differentiable.

On the other hand, Theorem **??** says that continuity of all partial derivatives implies differentiability of f. We hope to understand the connection between the partial derivatives and differentiability.

Theorem 14.6.10. *If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$, then all partial derivatives of f exist at $\mathbf{a}$ and*

$$\nabla f(\mathbf{a}) = \Big\langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \Big\rangle.$$

Proof. Since f is differentiable at $\mathbf{a}$, the gradient vector $\nabla f(\mathbf{a})$ exists and denote

$$\nabla f(\mathbf{a}) =< \alpha_1, \alpha_2, \cdots, \alpha_n > .$$

The partial derivative of f with respect to x_i is

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot < 0, \cdots, 0, 1, 0, \cdots, 0 >= \alpha_i$$

for $i = 1, 2 \cdots, n$. Hence $\nabla f(\mathbf{a}) = \left\langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right\rangle$. □

Note. If f is differentiable at $\mathbf{a}$, then we can explicitly write the form of $\nabla f(\mathbf{a})$.

Conclusion: Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Then

All partial derivatives of f exist and are continuous at $\mathbf{a}$

$$\Downarrow$$

f is differentiable at $\mathbf{a}$ and $\nabla f(\mathbf{a})$ exists and $\nabla f(\mathbf{a}) = \left\langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right\rangle$.

$$\Downarrow$$

All partial derivatives of f exist and the directional derivative $D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$

Note. All the converse of the above arrows are false.

❑ Maximizing the Directional Derivatives

Suppose that $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$. Then all directional derivatives of f at $\mathbf{a}$ exist and

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$$

for any unit vector $\mathbf{u}$.

Question: In which direction does f change fastest and what is the maximum rate of change?

Observe that the rate of change of f in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u} = |\nabla f(\mathbf{a})| \underbrace{|\mathbf{u}|}_{=1} \cos\theta = |\nabla f(\mathbf{a})| \cos\theta$$

where θ is the angle between the two vectors $\nabla f(\mathbf{a})$ and $\mathbf{u}$. Hence, the maximum value of $D_{\mathbf{u}} f(\mathbf{a})$ occurs when $\theta = 0$.

Theorem 14.6.11. *Suppose that f is differentiable at $\mathbf{a}$. Then*

(a) The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ is $|\nabla f(\mathbf{a})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{a})$. That is, the function f at $\mathbf{a}$ increases fastest in the same direction of $\nabla f(\mathbf{a})$.

(b) Similarly, the minimum value of the direction derivative $D_{\mathbf{u}} f(\mathbf{a})$ is $-|\nabla f(\mathbf{a})|$ and it occurs when $\mathbf{u}$ has the opposite direction to the gradient vector $\nabla f(\mathbf{a})$. That is, the function f at $\mathbf{a}$ decreases fastest in the opposite direction to $\nabla f(\mathbf{a})$.

(c) The function does not change in the direction of $\mathbf{u}$ which is perpendicular to $\nabla f(\mathbf{a})$.

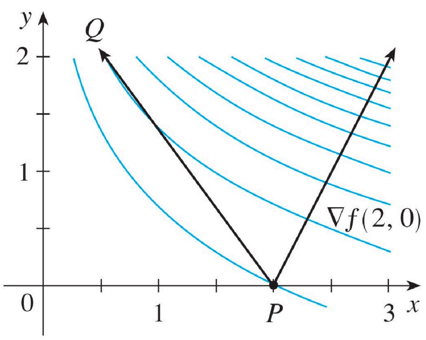
Example 14.6.12. Let $f(x, y) = xe^y$.

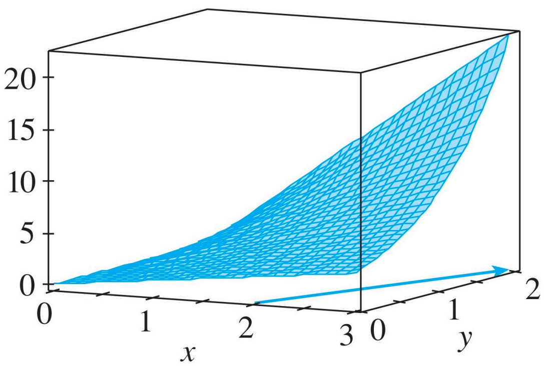
(a) Find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

Proof. The vector $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2\rangle$ and $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$. The gradient of f is $\nabla f(x, y) = \langle e^y, xe^y\rangle$ and $\nabla f(2, 0) = \langle 1, 2\rangle$. Hence, the rate of change of f in the direction $\overrightarrow{PQ}$ is $D_{\mathbf{u}} f(1, 2) = \langle 1, 2\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle = 1$. □

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Proof. f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2\rangle$ and the maximum rate of change is $|\nabla f(2, 0)| = |\langle 1, 2\rangle| = \sqrt{5}$. □





Example 14.6.13. Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2},$$

where T is measured in degree Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Proof. The gradient of T is $\nabla T(x,y,z) = \dfrac{160}{(1+x^2+2y^2+3z^2)^2}(-x\mathbf{i}-2y\mathbf{j}-3z\mathbf{k})$ and then $\nabla T(1,1,-2) = \frac{5}{8}(-\mathbf{i}-2\mathbf{j}+6\mathbf{k})$.

The temperature increases fastest in the direction of the gradient vector $\nabla T(1,1,-2) = \frac{5}{8}(-\mathbf{i}-2\mathbf{j}+6\mathbf{k})$ or $-\mathbf{i}-2\mathbf{j}+6\mathbf{k}$. The maximum rate of increase is

$$|\nabla T(1,1,-2)| = \frac{5}{8}|-\mathbf{i}-2\mathbf{j}+6\mathbf{k}| = \frac{5\sqrt{41}}{8} \approx 4 \quad (^oC/m).$$

□

❑ Tangent Plane to Level Surfaces

Recall: In Section14.4, we have learned that the equation of the tangent plane to the surface $S : z = f(x,y)$ at $P(x_0,y_0,z_0)$ is

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0). \tag{14.3}$$

Define $F(x,y,z) = z - f(x,y)$. Then

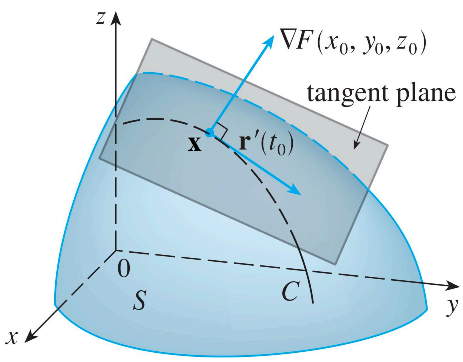
$$S = \{(x,y,z) \mid z = f(x,y)\} = \{(x,y,z) \mid z - f(x,y) = 0\} = \{(x,y,z) \mid F(x,y,z) = 0\}$$

is a level surface of F when the value is equal to 0. Hence, (14.3) also interprets the equation of the tangnet plane to the level surface of F at P.

From the same spirit as above, we consider a differentiable function $F(x,y,z)$ of three variables x, y and z. Let S be a level surface with equation $F(x,y,z) = k$ and $\mathbf{x} = \langle x_0,y_0,z_0\rangle \in S$. To find the tangent plane to S at $\mathbf{x}$, it suffices to find the normal vector of S at $\mathbf{x}$.

Theorem 14.6.14. *Let $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable and $S \subset D$ be a level surface of F. If $\mathbf{x} = \langle x_0,y_0,z_0\rangle \in S$ and $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then $\nabla f(\mathbf{x})$ is perpendicular to S at $\mathbf{x}$.*

Proof. In order to prove $\nabla f(\mathbf{x})$ is perpendicular to S at $\mathbf{x}$, it suffices to show that the vector $\nabla f(\mathbf{x})$ is perpendicular to any curve on S passing $\mathbf{x}$ (the tangent vector to the curve at $\mathbf{x}$).



Let $C : \mathbf{r}(t) = \langle x(t),y(t),z(t)\rangle$ be a differentiable curve that lies on S and passes through $\mathbf{x} = \langle x_0,y_0,z_0\rangle$ when $t = t_0$. Let S be the level surface with equation $F(x,y,z) = k$. Then

$$F\big(\mathbf{r}(t)\big) = F\big(x(t),y(t),z(t)\big) = k.$$

Hence,

$$\begin{aligned} 0 = \frac{d}{dt}\Big[F\big(\mathbf{r}(t)\big)\Big] &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle \\ &= \nabla F\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t) \end{aligned}$$

Taking $t = t_0$, $\nabla F(\mathbf{x}) \perp \mathbf{r}'(t_0)$.

Note that $\mathbf{r}'(t_0)$ is a tangent vector lying on the tangent plane. Since C is an arbitrary curve on S, any vector on the tangent plane (to S at $\mathbf{x}$) is perpendicular to $\nabla F(\mathbf{x})$. Therefore, $\nabla F(\mathbf{x})$ is the normal vector of the tangent plane to S at $\mathbf{x}$. □

Note. (1) Let S be the level surface with equation $F(x, y, z) = k$ and $\mathbf{x} = \langle x_0, y_0, z_0\rangle \in S$. If $\nabla F(\mathbf{x}) \neq \mathbf{0}$, it is natural to define the tangent plane to the level surface S at $\mathbf{x}$ as the plane that passes through $\mathbf{x}$ and has normal vector $\nabla F(\mathbf{x})$. The equation of the tangent plane is

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

That is,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

(2) Consider the special case that the surface S with equation $z = f(x, y)$ which is the graph of a function f of two variables. Let $F(x, y, z) = f(x, y) - z$. Then S is with the equation $F(x, y, z) = 0$. Also,

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = f_y(x_0, y_0), \quad \text{and} \quad F_z(x_0, y_0, z_0) = -1.$$

The equation of the tangent plane to S at (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1)(z - z_0) = 0.$$

Example 14.6.15. Find the equation of the tangne tplane at the point $(-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Proof.

Let $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Then the ellipsoid is the level surface (with $k = 3$) of $F(x, y, , z)$. Then

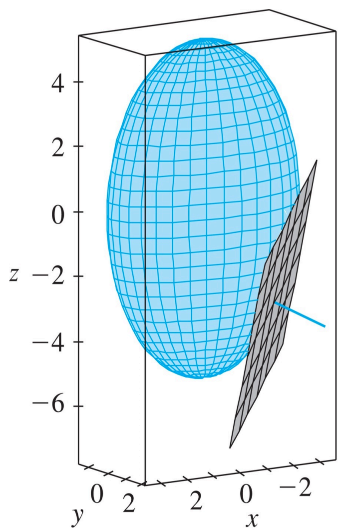
$$F_x = \frac{x}{2}, \quad F_y = 2y \quad \text{and} \quad F_z = \frac{2z}{9}.$$

Hence, $F_x(-2, 1, 3) = -1$, $F_y(-2, 1, 3) = 2$ and $F_z(-2, 1, -3) = -\frac{2}{3}$. The equation of the tangnet plane is

$$-(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

or

$$3x - 6y + 2z + 18 = 0.$$



□

❏ Normal Line

The normal line to S at $\mathbf{x}$ is the line passing through $\mathbf{x} = \langle x_0, y_0, z_0\rangle$ and perpendicular to the tangent plane. The direction of the normal line is the gradient vector $\nabla F(\mathbf{x})$. The symmetric equation are

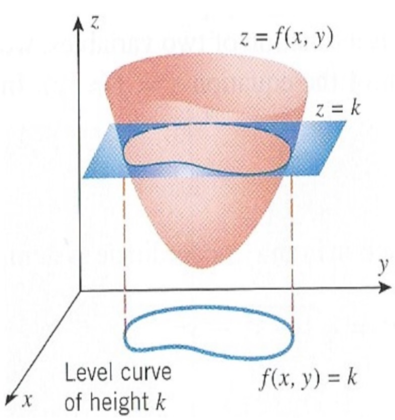
$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

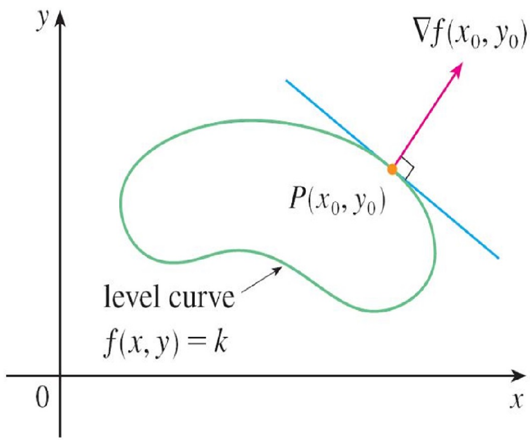
Example 14.6.16. As the above example, the equation of the normal line is

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}.$$

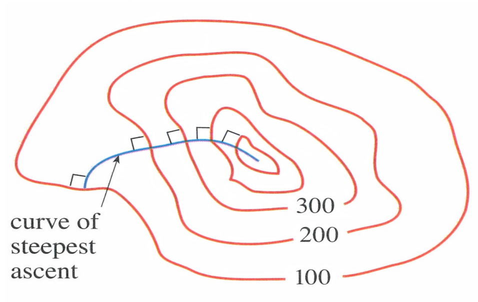
❑ Significance of the Gradient Vector

Consider the function $f(x, y)$ of two variables.

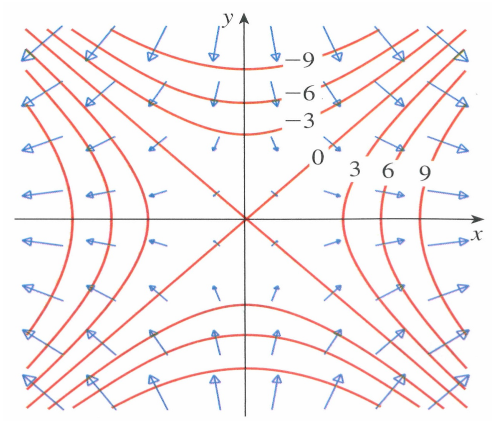




- The gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Intuitively, it is because the values of f remain constant as we move along the level curve.



A curve of steepest ascent is with direction $\nabla f(x, y)$. It is perpendicular to all of the contour lines.



a gradient vector field for the function $f(x, y) = x^2 - y^2$

- $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through (x_0, y_0).

- For a plane curve $C : y = f(x)$, define $F(x, y) = y - f(x)$. Then C is a level curve of F. If $(x_0, y_0) \in C$, then $\nabla F(x_0, y_0)$ is the normal vector of C at (x_0, y_0).

Example 14.6.17. Let C be the curve defined by $C = \{(x, y) \mid x^2 + y^3 = 9\}$. Find the tangent line of C at $(1, 2)$.

Proof. Let $f(x, y) = x^2 + y^3$. Then C is a level curve of f (with $k = 9$). The gradient vector $\nabla f(1, 2) = \langle \frac{\partial f}{\partial x}(1, 2), \frac{\partial f}{\partial y}(1, 2)\rangle = \langle 2, 12\rangle$ is the normal vector of C at $(1, 2)$. Hence, the tangent vector of C at $(1, 2)$ is $\langle 12, -2\rangle$ (perpendicular to $\langle 2, 12\rangle$). The equation of the tangent line to C at $(1, 2)$ is

$$\langle x - 1, y - 2\rangle \cdot \langle 2, 12\rangle = 0 \quad \text{or} \quad 2(x - 1) + 12(y - 2) = 0.$$

□

Homework 14.6. 6, 9, 12, 13, 16, 19, 21, 24, 35, 39, 45, 47, 51, 57, 60, 64, 67

14.7 Maximum and Minimum Values

In the present section, we will study the extreme values of two variables function $f(x, y)$. Recall that, of a single variable funciton $f(x)$, we find the critical points as candinates and determine the extreme values by first derivative test or second derivative test. For a muti-variables functions, we also want to find the critical points by considering the directional derivatives.

Definition 14.7.1. Let f be a two variables function on D. We say that

(1) f has a local maximum (minimum) at (a, b) if

$$f(x, y) \leq f(a, b) \qquad \big(f(x, y) \geq f(a, b)\big)$$

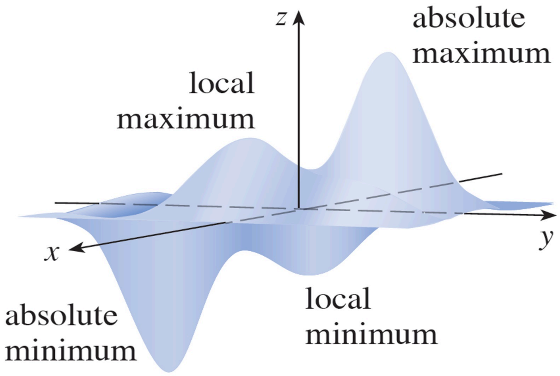
when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all point (x, y) in some dist center (a, b)]. The number $f(a, b)$ is called a "*local maximum (minimum) value*".

(2) f has an absolute maximum (minimum) at (a, b)if

$$f(x, y) \leq f(a, b) \qquad \big(f(x, y) \geq f(a, b)\big)$$

for all $(x, y) \in D$. The number $f(a, b)$ is called an "*absolute maximum (minimum) values*".

(3) The maximum and minimum values of f are called the "*extreme values of f*".



Question: How to find the extreme values of f?

Theorem 14.7.2. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exists there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.* $(\nabla f(a, b) = \mathbf{0})$

Proof. Let $g(x) = f(x, b)$. If f has a local maximum or minimum at (a, b), g has a local maximum or minimum at a. Thus, $0 = g'(a) = f_x(a, b)$. Similarly, $f_y(a, b) = 0$.

□

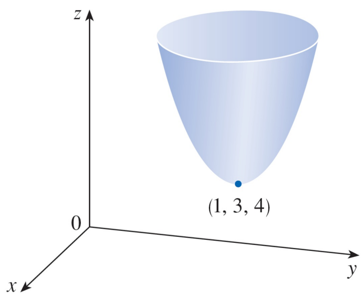
Note. The geometric interpretation is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

Definition 14.7.3. We call that point (a, b) a "*critical point*" of f if either (1) $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or (2) one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Example 14.7.4. Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the critical point of f.

Proof.

The partial derivatives $f_x(x, y) = 2x - 2$ and $f_y(x, y) = 2y - 6$. Therefore, $f_x(x, y) = 0$ when $x = 1$ and $f_y(x, y) = 0$ when $y = 3$. The point $(1, 3)$ is a critical point of f. In fact, $f(x, y) = 4 + (x - 1)^2 + (y - 3)$ ahs a local and an absolute maximum at $(1, 3)$.



$z = x^2 + y^2 - 2x - 6y + 14$

□

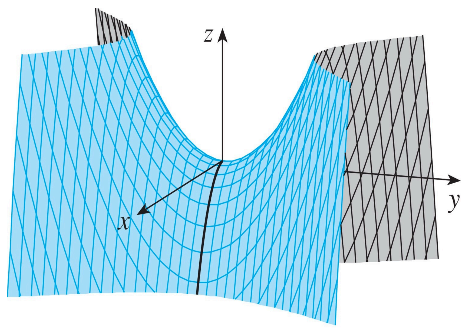
Remark. The above theorem says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, not all critical points give rise to maximum or minima.

Example 14.7.5. Find the extreme values of $f(x, y) = y^2 - x^2$.

Proof.

The partial derivatives $f_x = -2x$ and $f_y = 2y$. Then $f_x = 0$ when $x = 0$ and $f_y = 0$ when $y = 0$. The point $(0, 0)$ is a critical point of f. But $f(0, 0)$ is neither a local maximum nor a local minimum.

Indeed, on the x-axis, $f(x, y) = -x^2 < 0$ if $x \neq 0$ and on the y-axis, $f(x, y) = y^2$ if $y \neq 0$.



$z = y^2 - x^2$

Note. Near the origin the graph has the shape of a saddle and so $(0, 0)$ is called a "*saddle point*" of f.

□

❑ Second Derivative Test

Theorem 14.7.6. *Suppose that f_{xx}, f_{xy}, f_{yx} and f_{yy} are continuous near (a, b) and $f_x(a, b) = f_y(a, b) = 0$ (that is, (a, b) is a critical point of f). Let*

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$ and $f(a, b)$ is not a local maximum or minimum.

Note. (1) In case(c), (a, b) is called a "*saddle point*" of f.

(2) If $D = 0$, the test is inconclusive, f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

(3)

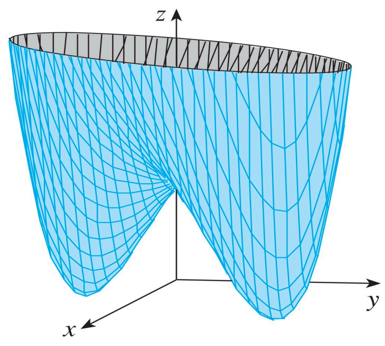
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 14.7.7. Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

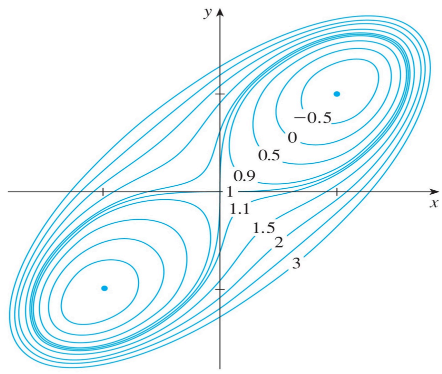
Proof. The first and second partial derivatives of f are $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4 = f_{yx}$ and $f_{yy} = 12y^2$. Then $f_x = 0$ when $x^3 = y$ and $f_y = 0$ when $y^3 = x$. We can solve the critical points of f are $(0, 0)$, $(1, 1)$ and $(-1, -1)$, and

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16.$$

- At $(0, 0)$, $D(0.0) = -16 < 0$. Then f has neither a local maximum nor a local minimum at $(0, 0)$.
- At $(1, 1)$, $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$. Then $f(1, 1) = -1$ is a local minimum of f.
- At $(-1, -1)$, $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$. Then $f(-1, -1) = -1$ is a local minimum of f.



$z = x^4 + y^4 - 4xy + 1$



□

Example 14.7.8. Find and classify the critical points of the function $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$. Also find the highest points on the graph of f.

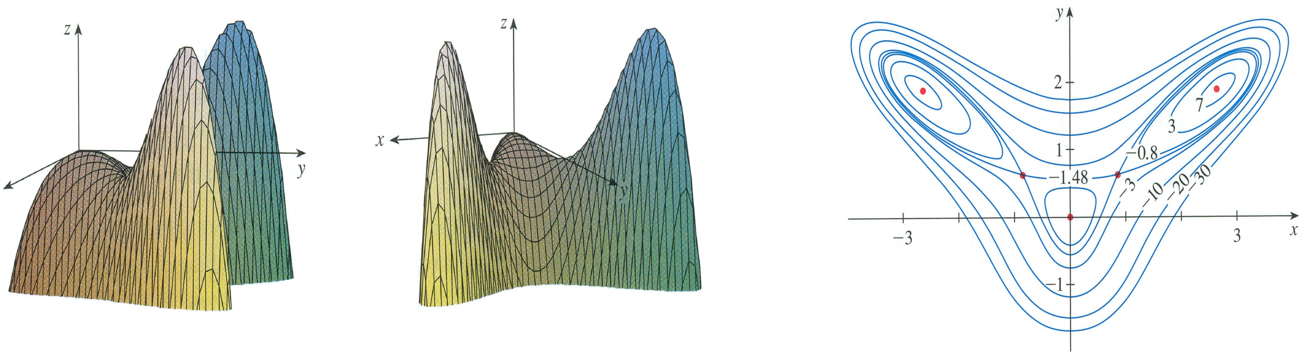
Proof. The first and second partial derivatives of f are

$$f_x = 20xy - 10x - 4x^3,\ f_y = 10x^2 - 8y - 8y^3,\ f_{xx} = 20y - 10 - 12x^2,\ f_{xy} = f_{yx} = 20x,\ f_{yy} = -8 - 24y^2.$$

To find the critical points of f by solving $f_x = 0$ and $f_y = 0$, we have $(x,y) = (0,0),\ (\pm 2.64, 1.90),\ (\pm 0.86, 0.65)$.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0,0)$	0	-10	80	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

The highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.



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Example 14.7.9. Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Proof. Let (x,y,z) be a point on the plane $x + 2y + z = 4$. The distance from (x,y,z) to $(1,0,-2)$ is

$$d(x,y,z) = \sqrt{(x-1)^2 + y^2 + (z+2)^2}.$$

Taking $z = 4 - x - 2y$, then $d = \sqrt{(x-1)^2 + y^2 + (-x-2y+6)^2}$. Consider $f(x,y) = d^2(x,y) = (x-1)^2 + y^2 + (-x-2y+6)^2$. The first and second partial derivatives of f are

$$f_x = 4x + 4y - 14,\ f_y = 4x + 10y - 24,\ f_{xx} = 4,\ f_{xy} = f_{yx} = 4,\ f_{yy} = 10.$$

To find the critical point of f by solving $f_x = 0$ and $f_y = 0$, the point $(x,y) = (\frac{11}{6}, \frac{5}{3})$ is the only critical point of f. Also, $D = 4 \cdot 10 - 4^2 = 24 > 0$ and $f_{xx} = 4 > 0$. By the second derivatives test, $f(x,y)$ has a local minimium at $(\frac{11}{6}, \frac{5}{3})$. Then $d(\frac{11}{6}, \frac{5}{3}) = \frac{5}{\sqrt{6}}$. In fact, it is the absolute minimum.

□

Example 14.7.10. A rectangle box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

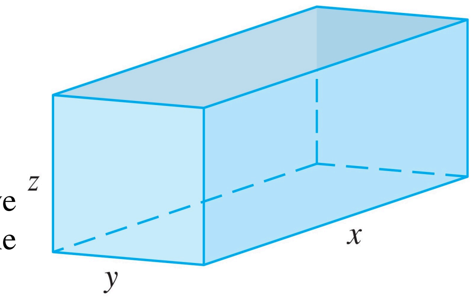
Proof. Let x, y and z be the length, width and height of the box. Then the volume of the box is $V(x,y,z) = xyz$ and the area of the four sides and the bottom is $2xz + 2yz + xy = 12$. Hence $z = \dfrac{12 - xy}{2(x+y)}$ and we can rewrite the volume function

$$V(x,y) = \frac{12xy - x^2y^2}{2(x+y)}.$$

Consider

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \quad \text{and} \quad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}.$$

The critical point of V is $(2,2)$. We can use the second derivative test to check that V has a local maximum at $(2,2,1)$. Then the maximum volume of the box is $4m^3$.



□

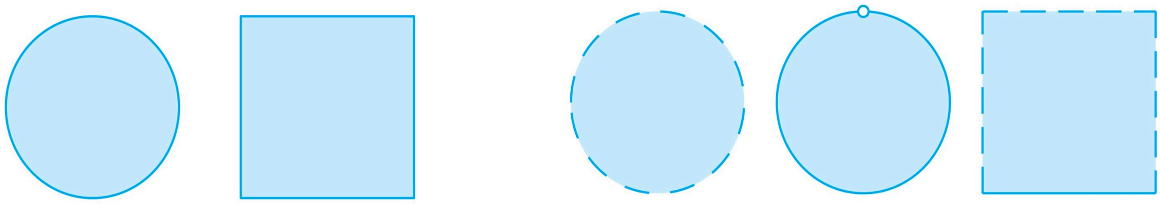
❑ Absolute Maximum and Minimum Values

Question: Under what conditions does a function $f(x,y)$ have (absolute) extreme values?

Recall that, for a single variable function $f(x)$, we have the "*Extreme Value Theorem*" that if f is continuous on a closed interval $[a,b]$, then f has an absolute maximum value and an absolute minimum value.

Question: How about two variables function $f(x,y)$?

Heuristically, corresponding to the "closed interval" in $\mathbb{R}$, a "close set" in $\mathbb{R}^2$ is a set contains all its boundary points. Also, a bounded set in $\mathbb{R}^2$ is a set that is contained within some disk.



(a) Closed sets (b) Sets that are not closed

■ Extreme Value Theorem for Functions of Two Variables

Theorem 14.7.11. *If f is continuous on a closed and bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some point (x_1,y_1) and (x_2,y_2) in D.*

Note. If $f(x,y)$ has an extreme value at (x_1,y_1), then (x_1,y_1) is either a critical point of f or a boundary point of D.

Question: How to find the absolute maximum value or minimum value of a continuous function $f(x, y)$ on a closed and bounded set D?

■ **Strategy:**

(1) Find the values of f at the critical point of f in D.

(2) Find the extreme value of f on the boundary of D.

(3) Check the values in (1) and (2). The largest value is the absolute maximum value and the smallest value is the absolute minimum.

Example 14.7.12. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

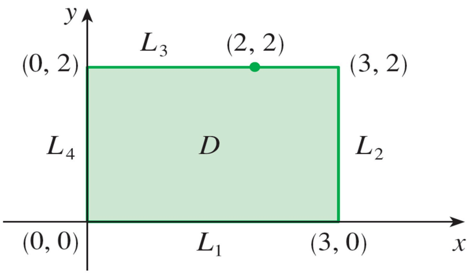
Proof. Since f is a polynomial on the closed and bounded set D, there exists absolute maximum and minimum values in D.

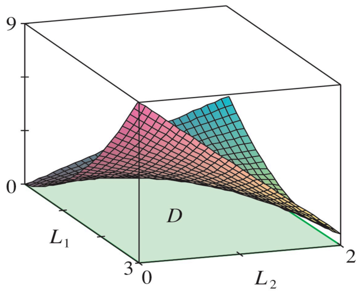
First of all, we find the critical points of f in the interior of D. The partial derivatives of f are $f_x = 2x - 2y$ and $f_y = -2x + 2$. Hence, $(1, 1)$ is a critical point of f in D and $\boxed{f(1,1)=1}$.

Next, we consider the candinates of extreme point on the boundary D. The boundary of D consists of four lines L_1, L_2, L_3 and L_4.

- For $(x, y) \in L_1$, $0 \le x \le 3$ and $y = 0$, $f(x, 0) = x^2$ is increasing. On L_1, f has a local maximum $\boxed{f(3,0)=9}$ and a local minimum $\boxed{f(0,0)=0}$.
- For $(x, y) \in L_2$, $x = 3$ and $0 \le y \le 2$, $f(3, y) = -4y + 9$ is decreasing. On L_2, f has a local maximum $\boxed{f(3,0)=9}$ and a local minimum $\boxed{f(3,2)=1}$.
- For $(x, y) \in L_3$, $0 \le x \le 3$ and $y = 2$, $f(x, 2) = x^2 - 4x + 4 = (x - 2)^2$. On L_3, f has a loca maximum $\boxed{f(0,2)=4}$ and a local minimum $\boxed{f(2,2)=0}$.
- For $(x, y) \in L_4$, $x = 0$ and $0 \le y \le 2$, $f(0, y) = 2y$ is increasing. On L_4, f has a local maximum $\boxed{f(0,2)=4}$ and a local minimum $\boxed{f(0,0)=0}$.

Hence, f has an absolute maximum value $f(3, 0) = 9$ and an absolute minimum value $f(0, 0) = f(2, 2) = 0$. □





$f(x, y) = x^2 - 2xy + 2y$

Homework 14.7. 3, 5, 9, 13, 17, 21, 35, 38, 40, 45, 51, 55, 59

14.8 Lagrange Multipliers

In the present section, we will study the Lagrange's method to maximize or minimize a general function $f(\mathbf{x})$ subject to a constraint (or side condition) of the form $g(\mathbf{x}) = k$. The method works for n variables functions but we will only consider 2 or 3 variables functions in this section.

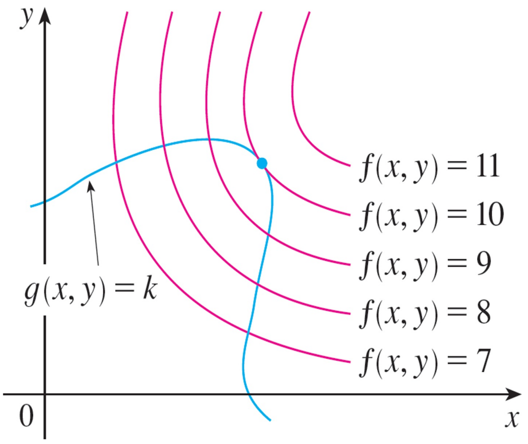
■ Geometric basis of Lagrange's method (for two variables functions)

Let $f(x, y)$ and $g(x, y)$ be two differentiable functions. The goal is to find the maximum (or minimum) of $f(x, y)$ subject to the constraint $g(x, y) = k$. For (x, y) satisfies $g(x, y) = k$, the point (x, y) lies on the level curve of $g(x, y)$ with the value k.

We want to find a point(s) (x_0, y_0) on the level curve $C = \{(x, y) \mid g(x, y) = k\}$ such that

$$f(x_0, y_0) \geq f(x, y) \quad \text{for all } (x, y) \in C. \tag{14.4}$$

Suppose that $(x_0, y_0) \in C$ satisfying (14.4) and $f(x_0, y_0) = M$. Then (x_0, y_0) is also on the level curve $C_1 = \{(x, y) \mid f(x, y) = M\}$. Moreover, since (x_0, y_0) is the maximum point, *the two level curve C and C_1 must be tangent each other at (x_0, y_0)*.



Since C and C_1 are level curves of g and f respectively, the gradient vectors $\nabla g \perp C$ and $\nabla f \perp C_1$. Then $\nabla g(x_0, y_0)$ is parallel to $\nabla f(x_0, y_0)$. Therefore, there exists a number λ ("*Lagrange multiplier*") such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Conclusion: The candidnate point(s) where the extreme values occur must satisfy

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) & \text{for some number } \lambda \\ g(x, y) = k \end{cases}$$

■ Lagrange methods for three variables functions

For finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, by the same argument as above, if the maximum value of f is $f(x_0, y_0, z_0) = M$ where (x_0, y_0, z_0) lies on the level surface $S = \{(x, y, z) \mid g(x, y, z) = k\}$. Then the level surface $\{(x, y, z) \mid f(x, y, z) = M\}$ is tangent to S at (x_0, y_0, z_0). We have

$$\nabla f(x_0, y_0, z_0) \,/\!/\, \nabla g(x_0, y_0, z_0).$$

(Intuitive veiwpoint) Let S be the level surface with equation $g(x, y, z) = k$. For every curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ lie on S, the tangent vector $\mathbf{r}'(t) \perp \nabla g\big(\mathbf{r}(t)\big)$ for every t.

Suppose that f has an extreme value at $P(x_0, y_0, z_0) \in S$ and $\mathbf{r}(t)$ is a curve on S passing P, say $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Consider the function $h(t) = f\big(\mathbf{r}(t)\big)$ which has maximum value at t_0.

Then $0 = h'(t_0) = \nabla f\big(\mathbf{r}(t_0)\big) \cdot \mathbf{r}'(t_0)$. We have $\nabla f\big(\mathbf{r}(t_0)\big) \perp \mathbf{r}'(t_0)$. Also, $\mathbf{r}'(t_0) \perp \nabla g\big(\mathbf{r}(t_0)\big)$. Then $\nabla f(x_0, y_0, z_0) \mathbin{/\!/} \nabla g(x_0, y_0, z_0)$. This implies that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \qquad \text{for some number } \lambda.$$

This number λ is called a "*Lagrange multiplier*".

❑ Method of Lagrange Multiplier

To find the maximu and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assume that these extreme value exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$). We solve this problem by following the below steps.

(a) Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad \text{and} \qquad g(x, y, z) = k.$$

(b) Evaluate f at all the points (x, y, z) that result from Step(a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

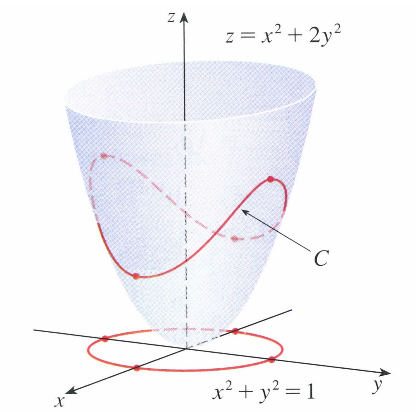
Example 14.8.1. Find the extreme values of the function $f(x, y) = x^2+2y^2$ on the circle $x^2+y^2 = 1$.

Proof. Let $g(x, y) = x^2 + y^2$. Then

$$\nabla f(x, y) = \langle 2x, 4y \rangle \quad \text{and} \quad \nabla g(x, y) = \langle 2x, 2y \rangle.$$

Consider

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y) = 1 \end{cases} \Rightarrow \begin{cases} 2x = 2\lambda x & (1) \\ 4y = 2\lambda y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$



By Equation(1), either $\lambda = 1$ or $x = 0$.

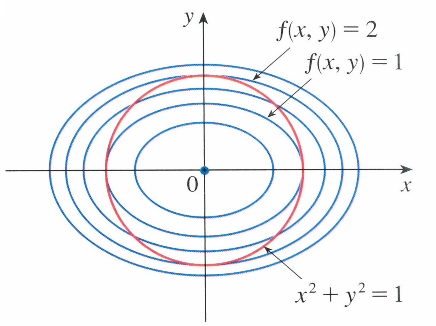
(i) If $\lambda = 1$, by Equation(2), $y = 0$. Then $x = \pm 1$ by Equation(3).

(ii) If $x = 0$, then $y = \pm 1$ by Equation(3) and $\lambda = 2$ by Equation(2).

Consider

$$\underbrace{f(1, 0) = 1,\ f(-1, 0) = 1}_{minimum} \text{ and } \underbrace{f(0, 1) = 2,\ f(0, -1) = 2}_{maximum}.$$

The maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$.



□

Example 14.8.2. A rectangle box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box.

Proof. Let the length, width and height of the box be x, y and z. Then the volume of the box is

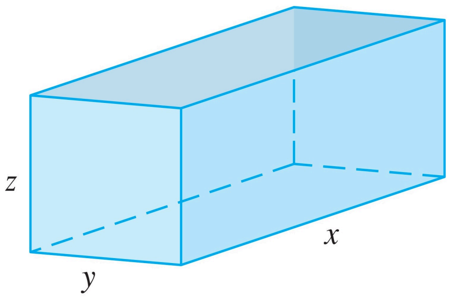
$$V(x, y, z) = xyz.$$

The area of the four sides and the bottom is

$$g(x, y, z) = 2xz + 2yz + xy = 12.$$

To find the maximum of V subject to the constraint $g(x, y, z) = 12$. The gradient vector of V and g are

$$\nabla V = \langle yz, xz, xy\rangle \quad \text{and} \quad \nabla g = \langle y + 2z, x + 2z, 2x + 2y\rangle.$$



Consider

$$\begin{cases} \nabla V = \lambda \nabla g \\ g(x, y, z) = 12 \end{cases} \Rightarrow \begin{cases} yz = \lambda(y + 2z) \\ xz = \lambda(x + 2z) \\ xy = \lambda(2x + 2y) \\ 2xz + 2yz + xy = 12 \end{cases} \Rightarrow \begin{cases} xyz = \lambda(xy + 2xz) & (1) \\ xyz = \lambda(xy + 2yz) & (2) \\ xyz = \lambda(2xz + 2yz) & (3) \\ 2xz + 2yz + xy = 12 & (4) \end{cases}$$

The number $\lambda \neq 0$; otherwise, we obtain $xy = xz = yz = 0$ and hence $g(x, y, z) = 0$ which contradicts the constraint. Also, Equations(1),(2) and (3) imply that

$$2xz + xy = 2yz + xy = 2xz + 2yz \quad \Rightarrow \quad xz = yz.$$

This says that either $x = y$ or $z = 0$.

(i) If $z = 0$, then $xy = 0$ and hence $x = y = 0$ which contradicts $g(x, y, z) = 12$.

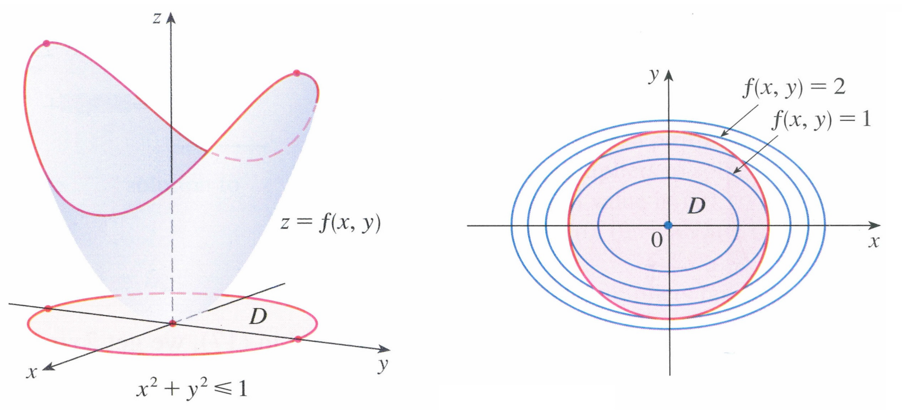
(ii) If $x = y$ and $z \neq 0$, then $2xz + x^2 = 4xz$ and then $x = 2z = y$. Also, from Equation(4), we obtain $x = y = 2$ and $z = 1$.

The maximum volume of the box is 4m^3. □

Example 14.8.3. Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Proof. (1) Find the extreme values of f inside the disk $x^2 + y^2 \leq 1$.
Consider $f_x = 2x = 0$ and $f_y = 4y = 0$. Then the critical point of f is $(0, 0)$. Moreover, $f_{xx} = 2$, $f_{xy} = f_{yx} = 0$ and $f_{yy} = 4$ and hence $D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0$. Also, $f_{xx} > 0$. By the second derivative test, $f(0, 0)$ is a local minimum.

(2) Combining with the previous example, $f(0, 0) = 0$, $f(\pm 1, 0) = 1$ and $f(0, \pm 1) = 2$. Hence, the maximum value of f on the disk $x^2 + y^2 \leq 1$is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$.



□

Example 14.8.4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$

Proof. Let $f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)$ and $g(x, y, z) = x^2 + y^2 + z^2$. Then

$$\nabla f = \langle 2(x-3), 2(y-1), 2(z+1)\rangle \quad \text{and} \quad \nabla g = \langle 2x, 2y, 2z\rangle.$$

Consider

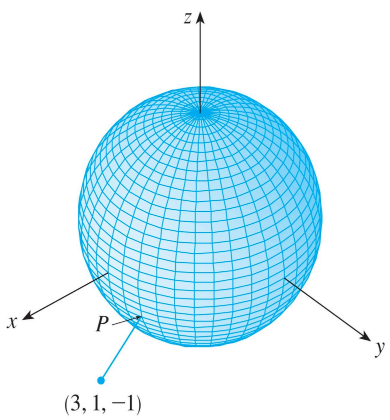
$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 4 \end{cases} \Rightarrow \begin{cases} 2x - 6 = 2\lambda 2x \\ 2y - 2 = 2\lambda y \\ 2z + 1 = 2\lambda z \\ x^2 + y^2 + z^2 = 4 \end{cases} \Rightarrow \begin{cases} (1-\lambda)x = 3 & (1) \\ (1-\lambda)y = 1 & (2) \\ (1-\lambda)z = -1 & (3) \\ 2xz + 2yz + xy = 12 & (4) \end{cases}$$

Clearly, $\lambda \neq 1$, $x \neq 0$, $y \neq 0$ and $z \neq 0$. Consider

$$\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = 3 \Rightarrow x = 3y \quad \text{and} \quad \frac{(2)}{(3)} \Rightarrow \frac{y}{z} = -1 \Rightarrow z = -y.$$

By (4), we have

$$(3y)^2 + y^2 + (-y)^2 = 4 \Rightarrow y = \pm\frac{2}{\sqrt{11}}.$$



Then

$$(x, y, z) = \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{or} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right).$$

Taking these two poinits into $f(x, y, z)$ the closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and the farthest point is $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

□

Remark. In the example, the line passes through the origin and the point $(3, 1, -1)$ has parametric equation $x = 3t$, $y = t$ and $z = -t$. The line intersection the sphere $x^2 + y^2 + z^2 = 4$ when $t = \pm\dfrac{2}{\sqrt{11}}$. Then we can also solve the closest and the farthest points.

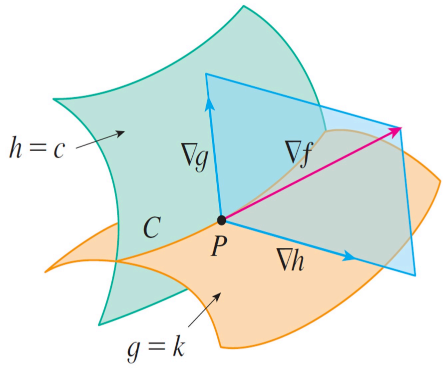
❏ Two Constraints

Find the maximum and minimum values of $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$.

Let C be the intersection of the two level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. Find $P(x_0, y_0, z_0) \in C$ such that $f(x_0, y_0, z_0)$ ahs extreme value along C.

To find the level surface $S = \{(x, y, z) \mid f(x, y, z) = M\}$ which tangnet to C. Then , at the intersection of C and S, $\nabla f \perp C$. We have

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$



Example 14.8.5. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Proof. Let $g(x, y, z) = x - y + z$ and $h(x, y, z) = x^2 + y^2$ Then

$$\nabla f = \langle 1, 2, 3\rangle, \quad \nabla g = \langle 1, -1, 1\rangle \quad \text{and} \quad \nabla h = \langle 2x, 2y, 0\rangle.$$

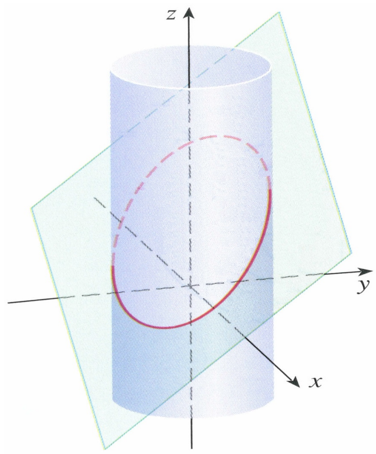
Consider

$$\begin{cases} \langle 1, 2, 3\rangle = \lambda\langle 1, -1, 1\rangle + \mu\langle 2x, 2y, 0\rangle \\ x - y + z = 1 \\ x^2 + y^2 = 1 \qquad (*) \end{cases} \Rightarrow \begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \end{cases} \Rightarrow \begin{cases} \lambda = 3 \\ x = -\dfrac{1}{\mu} \\ y = \dfrac{5}{2\mu} \end{cases}$$

Taking into $(*)$, we have $\mu = \pm\dfrac{\sqrt{29}}{2}$. Hence,

$$(x, y, z) = \left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}}\right) \quad \text{or} \quad \left(\frac{2}{-\sqrt{29}}, \frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}}\right).$$

Therefore, the maximum value of f is $3 + \sqrt{29}$.



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Homework 14.8. 5, 10, 14, 20, 21, 23, 25, 29, 33, 39, 47, 56

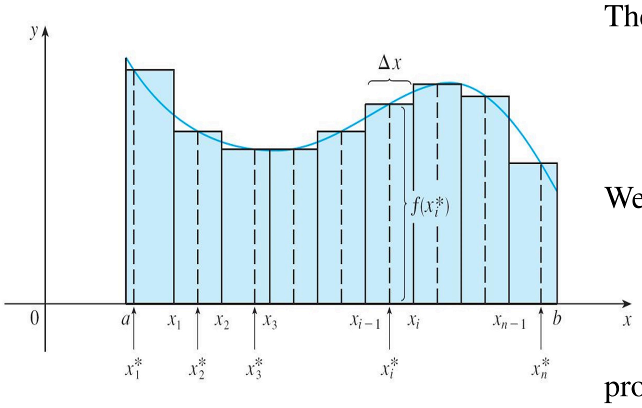
Chapter 15

Multiple Integrals

In the present chapter, we will extend the idea of a definite integral to double or triple integrals of functions of two or three variables.

15.1 Double Integrals over Rectangles

Recall: Compute the area under the graph of a single variable function $y = f(x)$ over $[a, b]$ where $f(x) \geq 0$. Dividing $[a, b]$ into n subintervals $[x_{i-1}, i]$ of equal width $\triangle x = \dfrac{b-a}{n}$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.



The Riemann sum is

$$\sum_{i=1}^{n} f(x_i^*)\triangle x \quad \text{where} \quad \triangle x = \frac{b-a}{n}.$$

We define the definite integral

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x$$

provided the limit exists.

❑ Volumes and Double Integrals

Consider a function $f(x, y)$ of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; a \le x \le b,\ c \le y \le d \right\}.$$

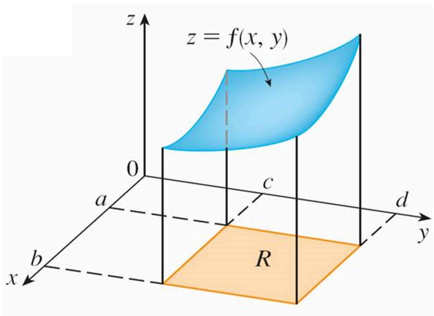
Suppose that $f(x, y) \ge 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

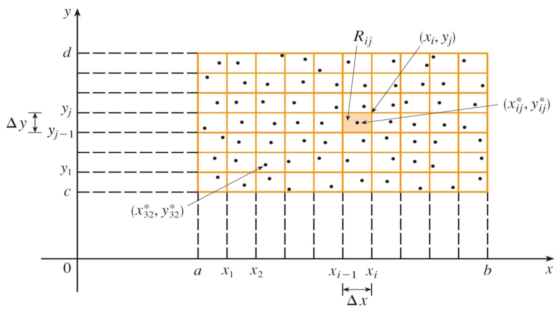
$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; 0 \le z \le f(x, y),\ (x, y) \in R \right\}.$$

To find the volume of S. Taking a partition P of R

$$\begin{aligned} a = x_0 < x_1 < x_2 < \cdots < x_m = b, \quad & \triangle x_i = x_i - x_{i-1} = \frac{b-a}{m} \\ c = y_0 < y_1 < y_2 < \cdots < y_n = d, \quad & \triangle y_j = y_j - y_{j-1} = \frac{d-c}{n} \end{aligned}$$

Let $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and $\triangle A = \triangle A_{ij} = \triangle x_i \triangle y_j =$ the area of R_{ij}. Let (x^*_{ij}, y^*_{ij}) be a *sample point* in R_{ij}.

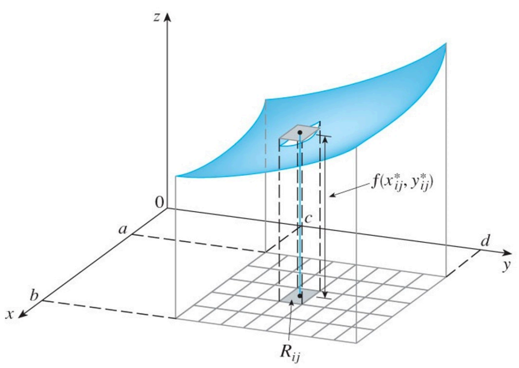


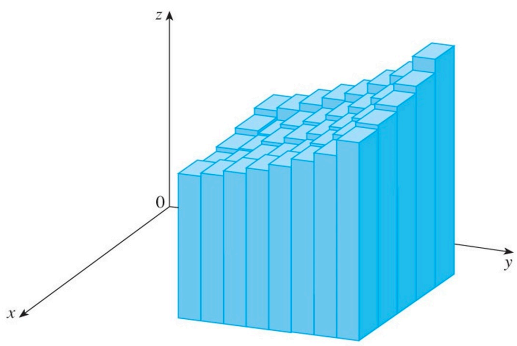


Dividing R into subrectangles

The volume of the solid under the graph of f over R_{ij} is approximated by volume of the rectangular box with base $\triangle A_{ij}$ and height $f(x^*_{ij}, y^*_{ij})$ whose volume is $f(x^*_{ij}, y^*_{ij})\triangle A_{ij}$. Then the approximation to the total volume of S is

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \triangle A_{ij} \quad \text{(double Riemann Sum)}$$





Note. (1) The approximation becomes better as m and n become larger. We expect that

$$V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle A.$$

(2) The double limit is that we can make the double sum as close as we like to the volume V [for any choice of (x_{ij}^*, y_{ij}^*) in R_{ij}] by taking m and n sufficiently large.

Definition 15.1.1. Let f be a function defined on a rectangle $R = [a, b] \times [c, d]$.

(1) The "*double integral*" of f over R is

$$\iint_R f(x, y)\, dA = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \triangle A$$

if the limit exists.

(2) A function f is called "*integrable*" over R if the above limit exists.

Definition 15.1.2. (Precise Definition) The limit L in the equation means that for every $\varepsilon > 0$ there exists an integer N such that

$$\left| L - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle A \right| < \varepsilon$$

for all $m, n > N$ and for any choice of sample points $(x_{ij}^*, y_{ij}^*) \in R_{ij}$. Denote the number L by

$$\iint_R f(x, y)\, dA.$$

Remark. (1) If f is continuous on R, then f is integrable over R.

(2) If f is integrable over R, then f is "almost" continuous on R (not too discontinuous).

(3) If f is bounded on R and continuous there except on a finite number of smooth curves, then f is integrable over R.

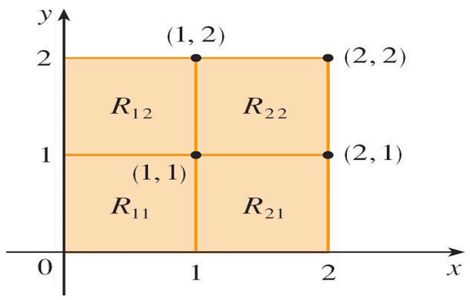
■ Volume

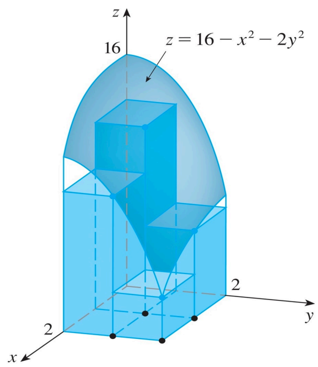
Definition 15.1.3. If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA.$$

Example 15.1.4. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Devide R into four equal squares and choose the sample point to be the upper right corner of eahc R_{ij}. Sketch the solid and the approximating rectangular boxes.

Proof. Set $m = n = 2$ and $\triangle x = \frac{2-0}{2} = 1$ and $\triangle y = \frac{2-0}{2} = 1$. We have $\triangle A = \triangle x \triangle y = 1$.



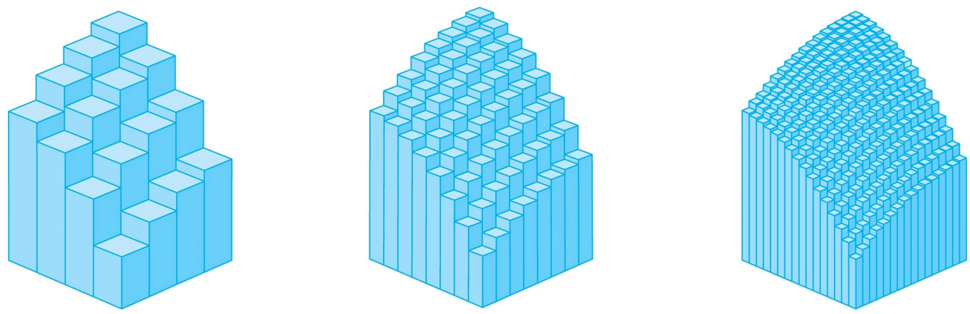


The volume is approximated by

$$
\begin{aligned}
V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \triangle A \\
&= f(1, 1) \triangle A + f(1, 2) \triangle A + f(2, 1) \triangle A + f(2, 2) \triangle A \\
&= (13 + 7 + 10 + 4) \times 1 = 34.
\end{aligned}
$$

□

Note. As m and n becomes larger, the approximation becomes better. The exact volume of the solid is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

The Riemann sum approximation volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

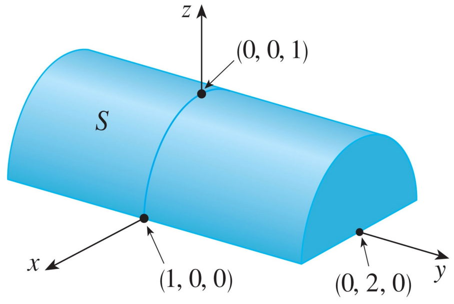
Example 15.1.5. If $R = [-1, 1] \times [-2, 2]$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

Proof. Compute the integral by interpreting it as a volume.

Let $z = \sqrt{1-x^2}$. Tehn $x^2 + z^2 = 1$ and $z \geq 0$. The solid S lies below the circular cylinder $x^2 + z^2 = 1$ and the double integral is equal to the volume of S. That is,

$$\iint_R \sqrt{1-x^2}\, dA = \frac{1}{2}\pi(1)^2 \times r = 2\pi.$$



□

❑ The Midpoint Rule (for Double Integrals)

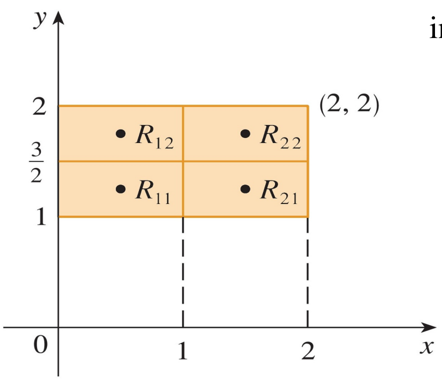
Choose the sample points (x^*_{ij}, y^*_{ij}) as the midpoints in R_{ij}. That is, $x^*_{ij} = \dfrac{x_{i-1}+x_i}{2} = \bar{x}_i$ and $y^*_{ij} = \dfrac{y_{j-1}+y_j}{2} = \bar{y}_j$ and

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\triangle A$$

Example 15.1.6. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral

$$\iint_R (x - 3y^2)\, dA \qquad \text{where} \quad R = [0,2]\times[1,2].$$

Proof.



The midpoints are $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$ and $\bar{y}_2 = \frac{7}{4}$, and $\triangle A = \triangle x \triangle y \times \frac{1}{2} = \frac{1}{2}$. The approximation of the double integral is

$$\iint_R (x-3y^2)\, dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\,\triangle A \quad \text{where } f(x,y) = x - 3y^2$$

$$= f\left(\frac{1}{2},\frac{5}{4}\right)\triangle A + f\left(\frac{1}{2},\frac{7}{4}\right)\triangle A + f\left(\frac{3}{2},\frac{5}{4}\right)\triangle A + f\left(\frac{3}{2},\frac{7}{4}\right)\triangle A$$

$$= \left[\left(-\frac{67}{16}\right) + \left(-\frac{139}{16}\right) + \left(-\frac{51}{16}\right) + \left(-\frac{123}{16}\right)\right] \times \frac{1}{2}$$

$$= -\frac{95}{8} = -11.875.$$

□

Number of subrectangles	Midpoint Rule approximation
1	-11.5000
4	-11.8750
16	-11.9687
64	-11.9922
256	-11.9980
1024	-11.9995

Note. (1) The exact value of the double integral is -12.

(2) $f(x,y)$ is not always positive. The double integral is not the volume.

(3) As m and n become larger, the approximation becomes better.

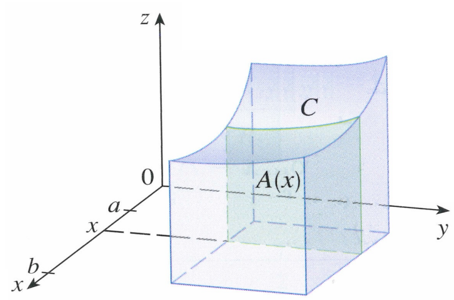
❑ Iterated Integrals

It is usually difficult to evaluate single integrals directly from the definition. Recall that for a single variable function $f(x)$, we use the Fundamental Theorem of Calculus to evaluate the integral $\int_a^b f(x)\,dx$.

Question: How to evaluate a double integral?

Suppose that f is a function of two variables that is integrable on $R = [a,b] \times [c,d]$.

- The integral $\int_c^d f(x,y)\,dy$ means that x is held fixed and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$.
- The procedure is called "*partial integration with respect to y*".



Note. The integral $\int_c^d f(x,y)\,dy$ is a number that depends on the value of x. Define

$$A(x) = \int_c^d f(x,y)\,dy.$$

If We integrate the function $A(x)$ with repsect to x from $x = a$ to $x = b$.

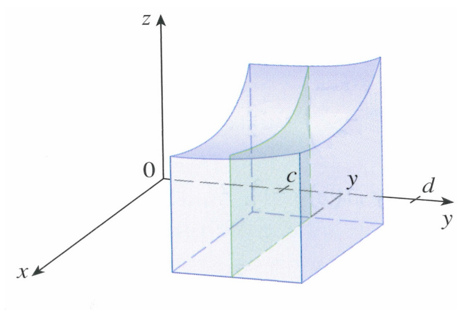
$$\begin{aligned}\int_a^b A(x)\,dx &= \int_a^b \Big[\int_c^d f(x,y)\,dy\Big]dx \\ &= \int_a^b \int_c^d f(x,y)\,dydx.\end{aligned}$$

The last integral is called the "*iterated integral*".

Simlarly we can also consider the iterated integral

$$\int_c^d \int_a^b f(x,y)\, dxdy = \int_c^d \Big[\int_a^b f(x,y)\, dx\Big] dy.$$

means that fixing each $y \in [c,d]$ and integrating the function f with respect to x from $x = a$ to $x = b$. After that, integrating the resulting function of y with respect to y from $y = c$ to $y = d$.



Example 15.1.7. (a)

$$\begin{aligned}\int_0^3 \int_1^2 x^2 y\, dydx &= \int_0^3 \Big[\frac{1}{2}x^2y^2\Big|_{y=1}^{y=2}\Big]\, dx = \frac{3}{2}\int_0^3 x^2\, dx \\ &= \frac{3}{2}\cdot\frac{1}{3}x^3\Big|_0^3 = \frac{27}{2}.\end{aligned}$$

(b)

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y\, dxdy &= \int_1^2 \Big[\frac{1}{3}x^3y\Big|_{x=0}^{x=3}\Big]\, dy = 9\int_1^2 y\, dy \\ &= \frac{9}{2}y^2\Big|_1^2 = \frac{27}{2}.\end{aligned}$$

Remark. (1) In this example, the two iterated integrals are equal under the exchange of the order of integrations.

(2) In general cases, the two iterated integrals

$$\int_a^b \int_c^d f(x,y)\, dydx \quad \text{and} \quad \int_c^d \int_a^b f(x,y)\, dxdy$$

may not be equal.

Question: When are the two iterated integrals equal to each other? How to evaluate the double integral $\iint_R f(x,y)\, dA$?

❑ The Fubini's Theorem

Theorem 15.1.8. *If f is continuous on $R = [a,b]\times[c,d]$, then*

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy.$$

More generally, the equalities are still ture if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exists.

Intuition: If $f(x, y) \geq 0$, the double integral $\iint_R f(x, y)\, dA$ is the volume V of the solid S the lies above R and under the graph of f with equation $z = f(x, y)$. On the other hand, the function $A(x) = \int_c^d f(x, y)\, dy$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leq y \leq d$. Also, $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis.

In Section 5.2,

$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x, y)\, dydx.$$

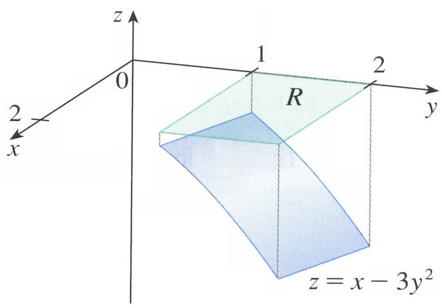
A similar argument, using cross-sections perpendicular to the y-axis, we have

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dxdy.$$

Example 15.1.9. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$ where $R = [0, 2] \times [1, 2]$.

Proof. Since $f(x, y) = x - 3y^2$ is continuous on R, by the Fubini's Theorem,

$$\begin{aligned} \iint_R x - 3y^2\, dA &= \int_0^2 \int_1^2 x - 3y^2\, dydx \\ &= \int_0^2 \left[xy - y^3 \Big|_{y=1}^{y=2} \right] dx \\ &= \int_0^2 [x - y\, dx = \frac{1}{2}x^2 - 7x]_0^2 \\ &= -12. \end{aligned}$$



Also,

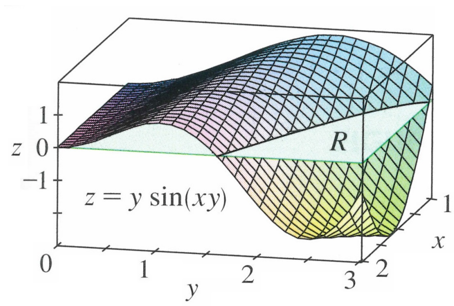
$$\begin{aligned} \iint_R x - 3y^2\, dA &= \int_1^2 \int_0^2 x - 3y^2\, dxdy = \int_1^2 \left[\frac{1}{2}x^2 - 3xy^2 \Big|_{x=0}^{x=2} \right] dy \\ &= \int_1^2 [\, - 6y^2\, dy = 2y - 2y^3]_1^2 = -12. \end{aligned}$$

□

Example 15.1.10. Evaluate $\iint_R y \sin(xy)\, dA$ where $= [1, 2] \times [0, \pi]$.

Proof. Since $f(x, y) = y \sin(x, y)$ is continuous on R, by the Fubini's Theorem,

$$\begin{aligned}\iint_R y\sin(xy)\,dA &= \int_0^{\pi}\int_1^2 y\sin(xy)\,dxdy\\ &= \int_0^{\pi}\Big[y\big(-\cos(xy)\big)\cdot\frac{1}{y}\Big|_{x=1}^{x=2}\Big]\,dy\\ &= -\int_0^{\pi}\cos(2y)-\cos y\,dy\\ &= \Big[-\frac{1}{2}\sin(2y)-\sin y\Big]_0^{\pi}=0\end{aligned}$$



□

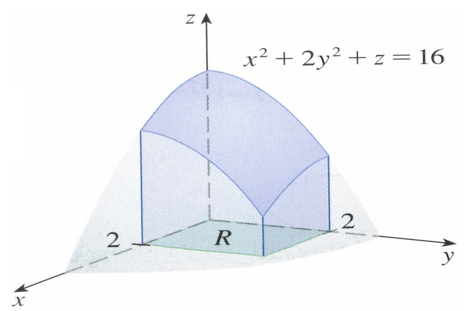
Note. If exchanging the order of the iterated integral, it is difficult to compute $\int_1^2\int_0^{\pi} y\sin(xy)\,dydx$.

Example 15.1.11. Find the volume of the solid that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$ the plane $x = 2$ and $y = 2$, and the three coordinate planes.

Proof. The domain of the integarl is $R = [0, 2] \times [0, 2]$.

Observe the graph of the paraboloid $z = 16-x^2-2y^2$ and the volume of the solid is

$$\begin{aligned}V &= \iint_R 16-x^2-2y^2\,dA\\ &= \int_0^2\int_0^2 16-x^2-2y^2\,dxdy\\ &= \int_0^2 \frac{88}{3}-4y^2\,dy\\ &= 48.\end{aligned}$$



□

■ **Special Case**

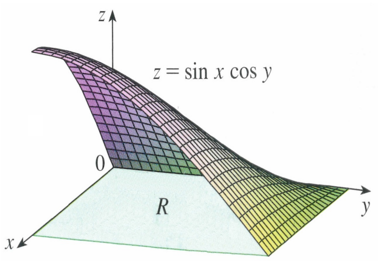
Suppose $f(x,y) = g(x)h(x)$ on $R = [a, b] \times [c, d]$. By the Fubini's Theorem,

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d\int_a^b g(x)h(y)\,dxdy = \int_c^d\Big[\int_a^b g(x)\underbrace{h(y)}_{\text{as a constant when integratingw.r.t } x}\,dx\Big]dy\\ &= \int_c^d h(y)\underbrace{\Big[\int_a^b g(x)\,dx\Big]}_{\text{a constant number}}\,dy\\ &= \int_a^b g(x)\,dx\int_c^d h(y)\,dy.\end{aligned}$$

Example 15.1.12. Evaluate $\iint_R \sin x\cos y\,dA$ where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Proof.

$$\iint_R \sin x \cos y \, dA = \int_0^{\frac{\pi}{2}} \sin x \, dx \int_0^{\frac{\pi}{2}} \cos y \, dy = 1 \times 1 = 1.$$



□

❑ Properties of Double Integrals

Theorem 15.1.13. *Suppose that $f(x, y)$ and $g(x, y)$ are integrable over R and c is a constant.*

(1) $\iint_R f(x, y) \pm g(x, y) \, dA = \iint_R f(x, y) \, dA \pm \iint_R g(x, y) \, dA$

(2) $\iint_R cf(x, y) \, dA = c \iint_R f(x, y) \, dA.$

(3) If $f(x, y) \geq g(x, y)$ for every $(x, y) \in R$, then

$$\iint_R f(x, y) \, dA \geq \iint_R g(x, y) \, dA.$$

❑ Average Value

Recall: Suppose that $f(x)$ is a singe variable function on $[a, b]$. The average of f on $[a, b]$ is $f_{avg} = \frac{1}{b-a} \int_a^b f(x) \, dx.$

Similarly, for a two variable function $f(x, y)$ defined on R, we define the "*average value*" of f on R by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) \, dA$$

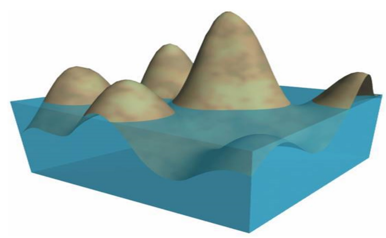
where $A(R)$ is the area of R.

Note. (1) If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{avg} = \iint_R f(x, y) \, dA$$

says that the box with base R and height f_{avg} has the same volume as the solid that lies under the graph of f.

(2) If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{avg}, then your can use them to fill in the valleys so that the region becomes complete flat.



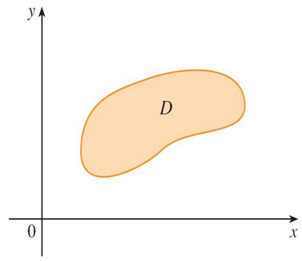
Homework 15.1. 11, 14, 15, 18, 21, 26, 29, 31, 34, 45, 47, 53, 55

15.2 Double Integrals over General Regions

In Section 15.1, we have learned the double integrals over a rectanglur region $R = [a, b] \times [c, d]$. In the present section, we consider the double integrals over general regions.

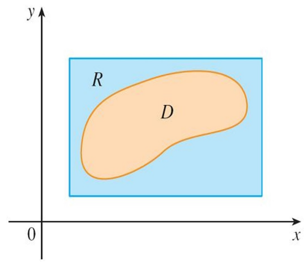
Let $f(x, y)$ be defined on a general region D.

Question: How to use the techniques for double integrals in Section 15.1 to find the double integrals $\iint_D f(x, y)\, dA$?



Choose a rectangle R which contains D. Define

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in R \backslash D. \end{cases}$$



Then F is a two variables function defined on R.

Definition 15.2.1. If F is integrable over R, then we define the "*double integral of f over D*" by

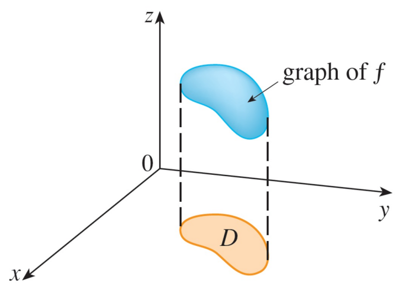
$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA.$$

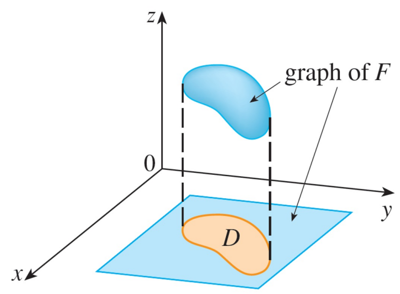
Remark. (1) This definition is reasonable since $F(x, y)$ contributes nothing to the integral when $F(x, y) = 0$ outside D.

(2) It doesn't matter what rectangle R we use as long as it contains D.

(3) If $f(x, y) \geq 0$ on D, $\iint_D f(x, y)\, dA$ is interpreted as the volume of the solid that lies above D and under the surface $z = f(x, y)$.

(4) F is likely to have discontinuities at the boundary points of D. If f is continuous on D and boundary curve of D is "well-behaved", then $\iint_R F(x, y)\, dA$ exists and hence $\iint_D f(x, y)\, dA$ exists.

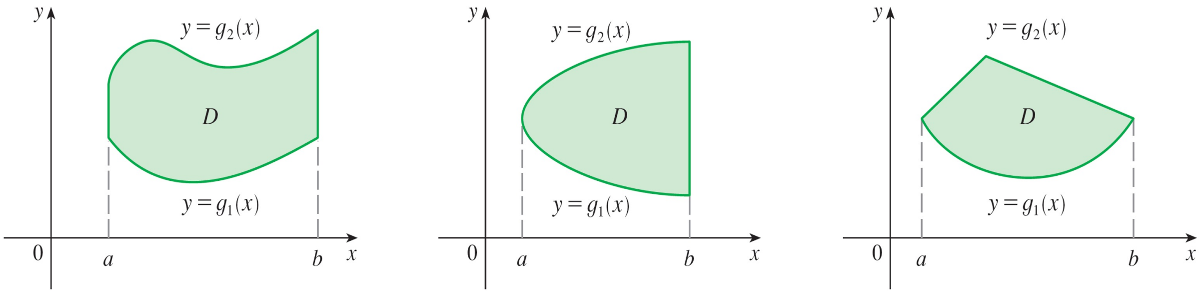




■ **Double Integrals over general regions**

We will consider the double integrals over some nice regions.

Type I: Let $D = \left\{(x,y) \;\middle|\; a \le x \le b,\ g_1(x) \le y \le g_2(x)\right\}$ where g_1 and g_2 are continuous on $[a,b]$.



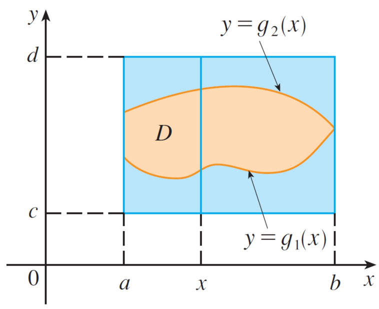
Some type I regions

Choose a rectangle $R = [a,b] \times [c,d]$ containing D and let $F(x,y)$ be the function defined as above. Then

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA \overset{(Fubini)}{=} \int_a^b \int_c^d F(x,y)\,dydx.$$

Since $F(x,y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$,

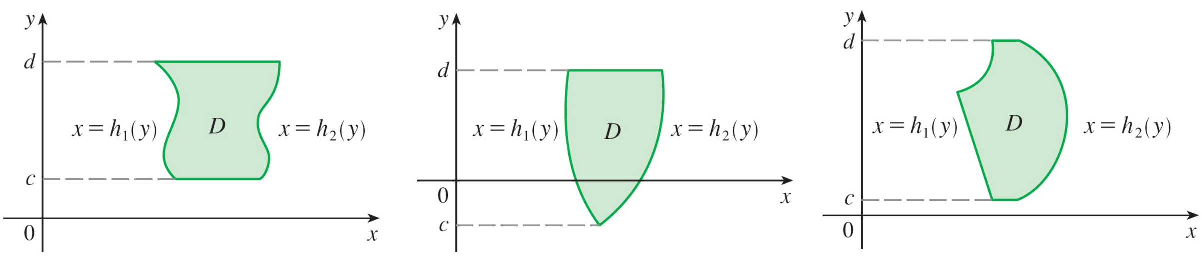
$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$



Conclusion: If f is continuous on a Type I region D such that $D = \left\{(x,y) \;\middle|\; a \le x \le b,\ g_1(x) \le y \le g_2(x)\right\}$ then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx.$$

Type II: Let $D = \left\{(x,y) \;\middle|\; c \le y \le d,\ h_1(y) \le x \le h_2(y)\right\}$ where h_1 and h_2 are continuous on $[c,d]$.



Some type II regions

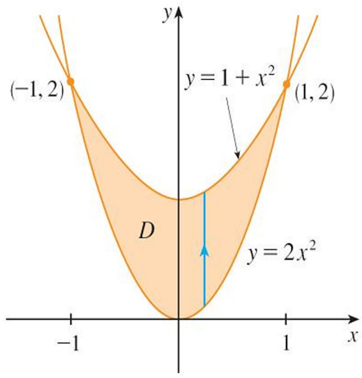
Conclusion: If f is continuous on a Type II region D, then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy.$$

Example 15.2.2. Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Proof.

Find the intersection of $y = 2x^2$ and $y = 1 + x^2$. We have $2x^2 = 1 + x^2$ and hence $x = \pm 1$. The region $D = \left\{(x, y) \,\middle|\, -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\right\}$. The double integral

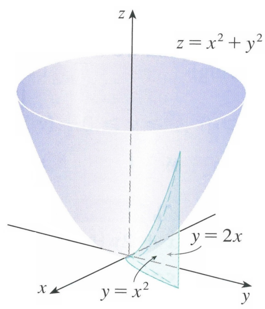


$$\begin{aligned}
\iint_D x + 2y\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y\, dydx \\
&= \int_{-1}^{1} \left[xy + y^2 \Big|_{y=2x^2}^{y=1+x^2} \right] dx \\
&= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1\, dx \\
&= \frac{32}{15}.
\end{aligned}$$

□

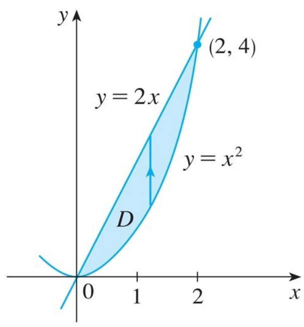
Example 15.2.3.

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



Proof. Find the intersections of the line $y = 2x$ and the parabola $y = x^2$. We have $2x = x^2$ and then $x = 0$ and 2.
(**Solution 1**) The Type I region $D = \left\{(x, y) \,\middle|\, 0 \le x \le 2,\ x^2 \le y \le 2x\right\}$ and the volume of the region is

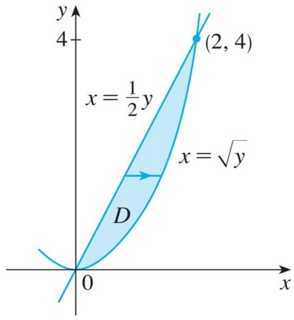
$$\begin{aligned}
V &= \iint_D (x^2 + y^2)\, dA \\
&= \int_0^2 \int_{x^2}^{2x} x^2 + y^2\, dydx \\
&= \int_0^2 -\frac{x^6}{3} - x^4 + \frac{14}{3}x^3\, dx = \frac{216}{35}.
\end{aligned}$$



D as a type I region

(**Solution 2**) The Type II region $D = \left\{(x, y) \mid 0 \le y \le 4,\ \frac{1}{2}y \le x \le \sqrt{x}\right\}$ and the volume of the region is

$$\begin{aligned} V &= \iint_D (x^2 + y^2)\, dA \\ &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} x^2 + y^2\, dxdy \\ &= \int_0^4 \frac{1}{3}y^{\frac{3}{2}} + y^{\frac{5}{2}} - \frac{13}{24}y^3\, dy = \frac{216}{35}. \end{aligned}$$



D as a type II region

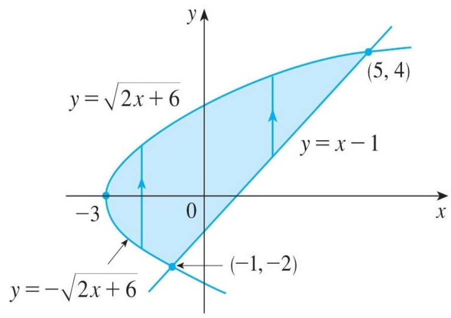
□

Example 15.2.4. Evaluate $\iint_D xy\, dA$, where D is teh region bounded by teh line $y = x - 1$ and teh parabola $y^2 = 2x + 6$.

Proof. If D is expressed as Type I region, we should divide D into two subregions and the double integral is

$$\begin{aligned} \iint_D xy\, dA &= \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\, dydx \\ &\quad + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\, dydx \end{aligned}$$

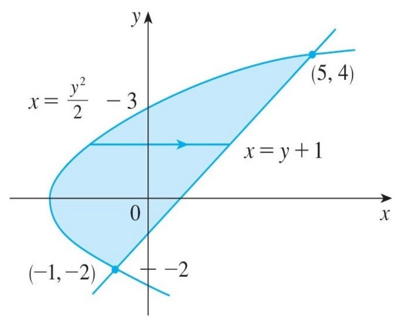
The iterated integrals are complicated.



(a) *D* as a type I region

We express D as Type II region and the double integral is

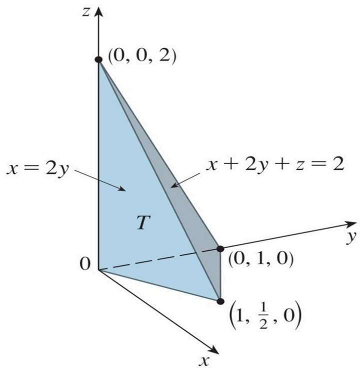
$$\begin{aligned} \iint_D xy\, dA &= \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dxdy \\ &= \frac{1}{2}\int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy = 36 \end{aligned}$$

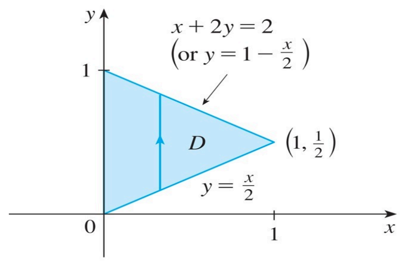


(b) *D* as a type II region

□

Example 15.2.5. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.





Proof. The plane $x + 2y + z = 2$ intersects xy-plane in the line $x + 2y = 2$. Then the region $D = \left\{(x, y) \,\middle|\, 0 \le x \le 1,\ \frac{1}{2}x \le y \le 1 - \frac{1}{2}x\right\}$ and the volume of the tetrahedron is

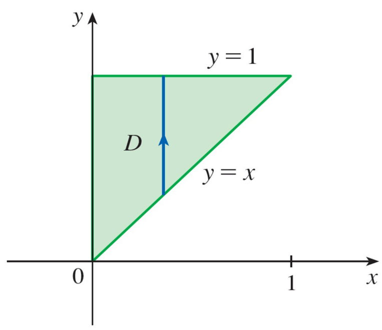
$$\begin{aligned}\iint_D 2 - x - 2y \, dA &= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} 2 - x - 2y \, dydx \\ &= \frac{1}{2}\int_0^1 x^2 - 2x + 1 \, dy = \frac{1}{3}.\end{aligned}$$

□

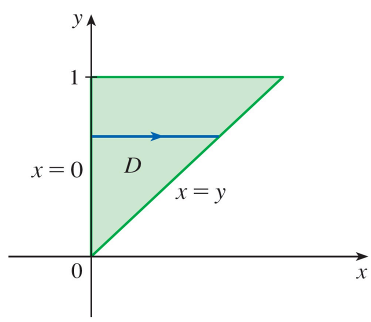
Example 15.2.6. Evaluate the iterated integral $\displaystyle\int_0^1 \int_x^1 \sin(y^2) \, dydx$.

Proof. The iterated integral is expressed as Type I case. We can check that the direct computation is difficult. Hence, we try to rewrite it as Type II case

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2) \, dydx &= \iint_D \sin(y^2) \, dA = \int_0^1 \int_0^y \sin(y^2) \, dxdy \\ &= \int_0^1 y \sin(y^2) \, dy = \frac{1}{2}(1 - \cos 1).\end{aligned}$$



D as a type I region



D as a type II region

□

❏ Properties of Double Integrals

Theorem 15.2.7. *Let $f(x,y)$ and $g(x,y)$ be integrable over D and c be a constant.*

(1) $\displaystyle\iint_D \left[f(x,y) \pm g(x,y)\right] dA = \iint_D f(x,y)\, dA \pm \iint_D g(x,y)\, dA.$

(2) $\displaystyle\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA.$

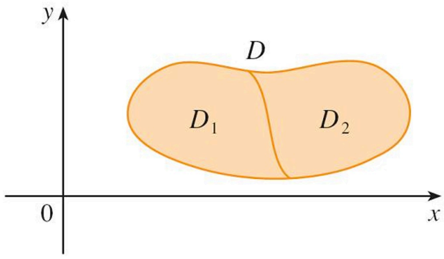
(3) If $f(x,y) \geq g(x,y)$ on D, then

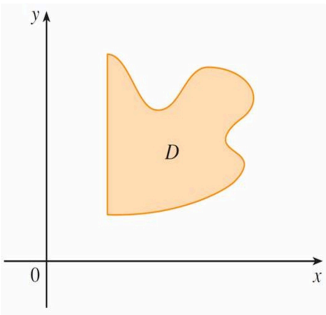
$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

(4) If $D = D_1 \cup D_2$ where D_1 and D_2 don't overlap except on their boundaries, then

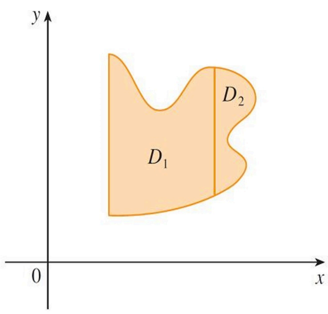
$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

Note. The above equality is also true even if D_1 and D_2 are not Type I or Type II.





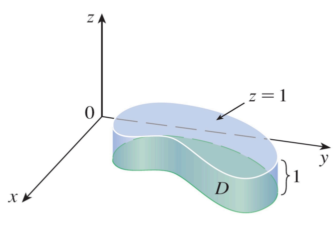
(a) D is neither type I nor type II.



(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

(5) The area of the region D is equal to

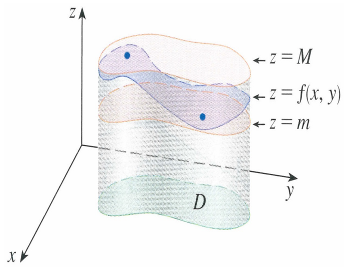
$$\iint_D 1\, dA = A(D).$$



Cylinder with base D and height 1

(6) If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$mA(D) \le \iint_D f(x,y)\,dA \le MA(D).$$



Example 15.2.8. Estimate $\iint_D e^{\sin x \cos y}\,dA$ where $D = \{(x,y) \mid 0 \le x^2 + y^2 \le 4\}$.

Proof. Since $-1 \le \sin x \cos y \le 1$, we have $e^{-1} \le e^{\sin x \cos y} \le e^1$ for all $(x,y) \in D$. Then

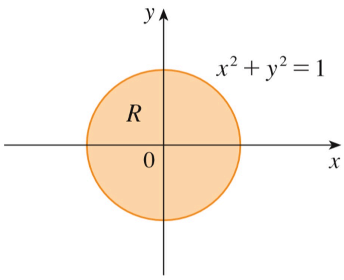
$$\frac{4\pi}{e} = e^{-1}A(D) \le \iint_D e^{\sin x \cos y}\,dA \le eA(D) = 4\pi e.$$

□

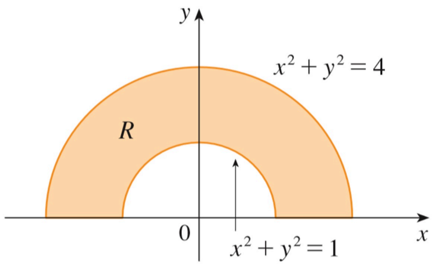
Homework 15.2. 5, 10, 11, 14, 19, 23, 27, 32, 35, 38, 46, 56, 59, 61, 64, 67, 71, 74

15.3 Double Integrals in Polar Coordinates

So far, we can only evaluate the double integrals over rectangles, Type I or Type II regions. Now, we want to evaluate the double integrals over the region R which is described using polar coordinates



(a) $R = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

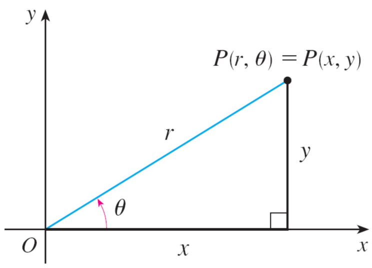


(b) $R = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

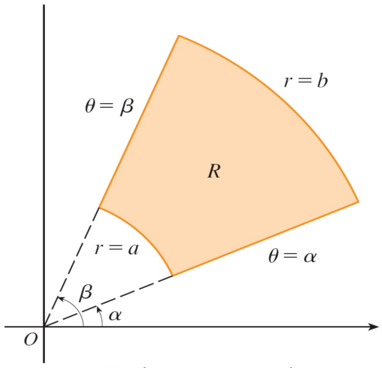
Recall: The Change of the variables between Cartesian coordinates (x,y) and polar coordinates (r,θ).

polar coordinates (r,θ) $\longleftrightarrow$ rectangular coordinates $(x,y$

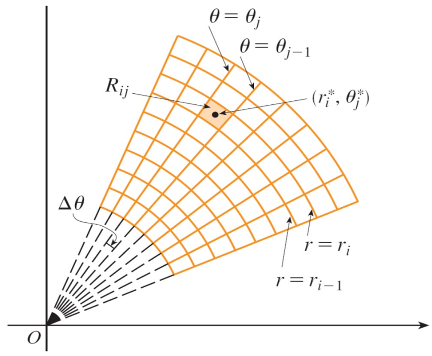
$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$



Compute the double integral $\iint_R f(x,y)\,dA$ where $R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$ is a polar rectangle.



Polar rectangle



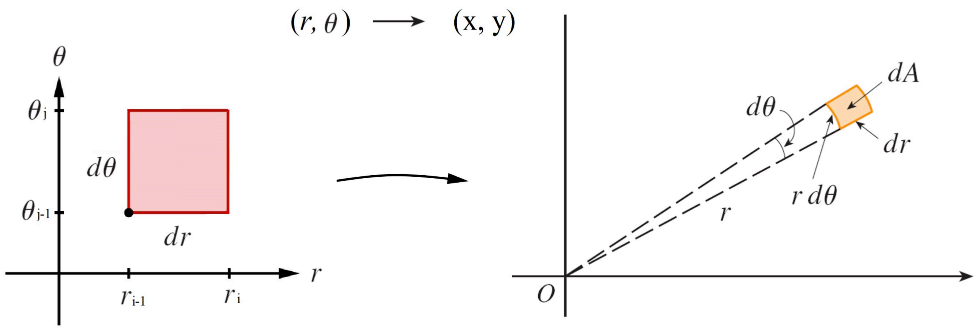
Dividing R into polar subrectangles

- Divide $[a,b]$ into m subintervals and divide $[\alpha,\beta]$ into n subintervals $a = r_0 < r_1 < \cdots < r_m = b$ and $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$ where $\triangle r = \dfrac{b-a}{m}$ and $\triangle\theta = \dfrac{\beta-\alpha}{n}$.

- Let $R_{ij} = \left\{(r,\theta) \;\middle|\; r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\right\}$. Choose $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ and $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$. The area of R_{ij} is

$$\triangle A_{ij} = \frac{1}{2} r_i^2 \triangle\theta - \frac{1}{2} r_{i-1}^2 \triangle\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\triangle\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\triangle\theta = r_i^* \triangle r \triangle\theta.$$

■ Change of Areas between Polar Transformation



For a continuous function $f(x,y)$ defined on R, the Riemann Sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*\right)\triangle A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \triangle r \triangle\theta.$$

Define $g(r,\theta) = r f(\cos\theta, r\sin\theta)$. Then the above Riemann Sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\triangle r \triangle\theta.$$

Then

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)r_i^*\triangle r\triangle\theta \\
&= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\theta_j^*)\triangle r\triangle\theta \\
&= \int_\alpha^\beta\int_a^b g(r,\theta)\,drd\theta \\
&= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta.
\end{aligned}$$

❑ Change to Polar Coordinates in a Double Integral

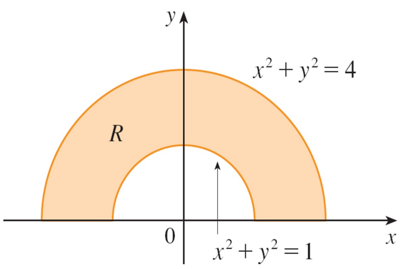
If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta.$$

Example 15.3.1. Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Proof.

$$\begin{aligned}
\iint_R (3x+4y^2)\,dA &= \int_0^\pi\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)r \\
&= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta \\
&= \int_0^\pi 7\cos\theta + 15\sin^2\theta\,d\theta \\
&= \frac{15}{2}\pi.
\end{aligned}$$



$R = \{(r,\theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

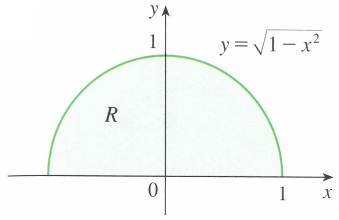
□

Example 15.3.2. Evaluate the double integral

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dydx$$

Proof. The iterated integral is a double integral over the region

$$R = \left\{(x,y) \,\middle|\, -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\right\}.$$



To express R in polar coordinates,

$$R = \left\{(r,\theta) \,\middle|\, 0 \le \theta \le \pi,\ 0 \le r \le 1\right\}.$$

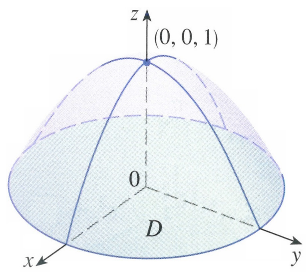
The double integral is

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dydx = \int_0^{\pi}\int_0^1 r^2 \cdot r\,drd\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \frac{1}{4}\int_0^{\pi} d\theta = \frac{\pi}{4}.$$

□

Example 15.3.3. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Proof.



The xy-plane intersects the paraboloid in the circle $x^2+y^2 = 1$. Let $R = \left\{(x,y) \,\middle|\, x^2+y^2 \le 1\right\}$. In polar coordinates, D is given by

$$D = \left\{(r,\theta) \,\middle|\, 0 \le r \le 1,\ 0 \le \theta \le 2\pi\right\}.$$

The volume of the solid is

$$V = \iint_D (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)r\,drd\theta = \frac{\pi}{2}.$$

□

Note. If using rectangular coordinates,

$$V = \iint_D (1-x^2-y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dydx.$$

It is difficult to find $\int (1-x^2)^{\frac{3}{2}}\,dx$.

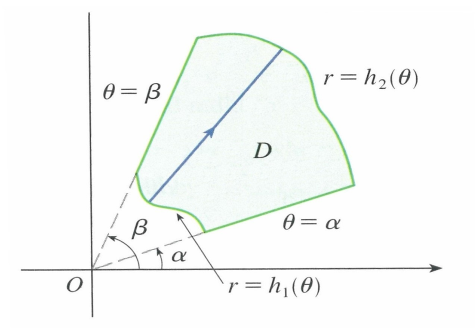
❑ Polar Region

If f is continuous on a polar region of the form

$$D = \left\{(r,\theta) \,\middle|\, \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\right\},$$

then

$$\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(f\cos\theta, r\sin\theta)r\,drd\theta.$$



$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$

In particular, the area of the region D bounde by $\theta = \alpha$, $\theta = \beta$ and $r = h(\theta)$ is

$$\begin{aligned} A(D) &= \iint_D 1\, dA = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr d\theta \\ &= \int_\alpha^\beta \frac{1}{2}\big(h(\theta)\big)^2\, d\theta. \end{aligned}$$

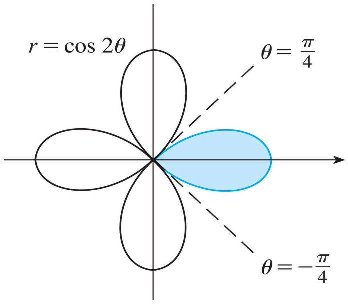
Example 15.3.4. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Proof.
The region enclosed by one loop of the four-leaved rose is

$$D = \left\{(r,\theta) \;\middle|\; -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le r \le \cos 2\theta\right\}.$$

The area of the region is

$$\begin{aligned} A(D) &= \iint_D 1\, dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\, dr d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{\pi}{8}. \end{aligned}$$



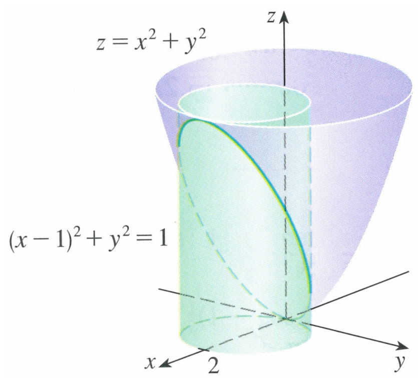
□

Example 15.3.5. Find teh volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Proof.

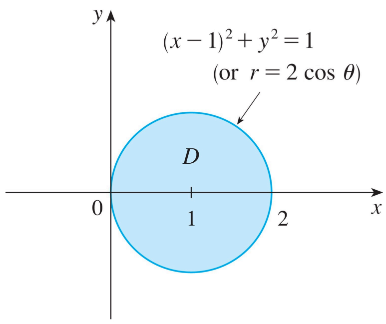
The solid lies above the disk $D = \left\{(x,y) \mid x^2 + y^2 - 2x \le 0\right\}$ with boundary $x^2+y^2 = 2x$. In polar coordinate, the circle becomes $r^2 = 2r\cos\theta$ and this implies $r = 2\cos\theta$. We have

$$D = \left\{(r,\theta) \;\middle|\; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\right\}.$$



The volume of the solid is

$$\begin{aligned} V &= \iint_D x^2 + y^2\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\, dr d\theta \\ &= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 8\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{3}{2}\pi. \end{aligned}$$



□

Example 15.3.6. Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\, dx.$

Proof. Consider

$$\begin{aligned}
\Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Big)^2 &= \Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}\, dy\Big) = \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy \\
&= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} \qquad \text{where } D_a = \{(x,y) \mid x^2+y^2 \le a^2\} \\
&= \lim_{a\to\infty} \int_0^{2\pi}\int_0^a e^{-r^2}\cdot r\, drd\theta \\
(u = r^2) \qquad &= \lim_{a\to\infty} \int_0^{2\pi} \frac{1}{2}\int_0^{\sqrt{a}} e^{-u}\, dud\theta = \lim_{a\to\infty} \pi(1-e^{-\sqrt{a}}) \\
&= \pi.
\end{aligned}$$

Hence, $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$ □

Homework 15.3. 10, 13, 15, 18, 22, 26, 33, 36, 41, 49

15.4 Applications of Double Integrals

(Skip)

Homework 15.4.

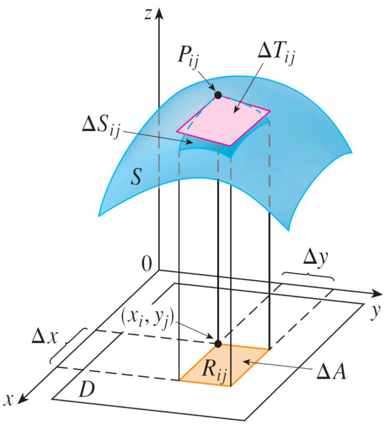
15.5 Surface Area

Recall: In Sec8.2, we study to find the surface area of a special type of surface - a surface of revolution. In the present section, we compute the area of a surace with equation $z = f(x,y)$, the graph of a function of two variables.

Let $f(x,y)$ be a function with continuous partial derivatives. Assume $f(x,y) \ge 0$ and $D = Dom(f)$ is a rectangle Let S be the graph of f which is a surface with equation $z = f(x,y)$. To find the area of S above D by following steps:

(1) We divide D into small rectangles R_{ij} with area $\triangle A = \triangle x \triangle y$. Let (x_i, y_j) be the corner of R_{ij} closest to the origin. Then the point $P_{ij} = P_{ij}\big((x_i, y_j, f(x_i, y_j)\big)$ lies on S.

(2) Let S_{ij} be the part of S that lies above R_{ij} with area $\triangle S_{ij}$ and T_{ij} be the tangent plane to S at P_{ij}. Hence, it is an approximation of S near P_{ij}. The area $\triangle T_{ij}$ of the part of this tangent plane that lies directly above R_{ij} satisfies



$$\triangle S_{ij} \approx \triangle T_{ij}$$

(3) The approximation to the total area of S is

$$A(S) = \sum_{i=1}^{m} \sum_{j=1}^{n} \triangle S_{ij} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \triangle T_{ij}.$$

Definition 15.5.1. The surface area of S is defined by

$$A(S) = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \triangle T_{ij}.$$

■ **To find $\triangle T_{ij}$**

Let $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ be the vectors that start at P_{ij} and lie along the side of the parallelogram with area $\triangle T_{ij}$. Then

$$\triangle T_{ij} = \left|\mathbf{a}_{ij} \times \mathbf{b}_{ij}\right|.$$

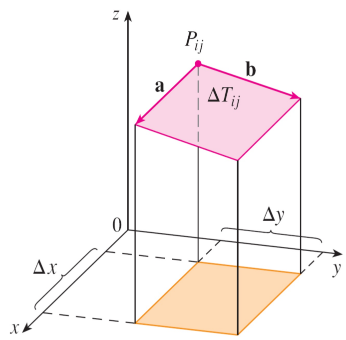
Note. The partial derivatives $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\mathbf{a}$ and $\mathbf{b}$.

Hence,

$$\begin{aligned} \mathbf{a}_{ij} &= \triangle x\, \mathbf{i} + f_x(x_i, y_j) \triangle x\, \mathbf{k} \\ \mathbf{b}_{ij} &= \triangle y\, \mathbf{j} + f_y(x_i, y_j) \triangle y\, \mathbf{k} \end{aligned}$$

We have

$$\begin{aligned} \mathbf{a}_{ij} \times \mathbf{b}_{ij} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \triangle x & 0 & f_x(x_i, y_j) \triangle x \\ 0 & \triangle y & f_y(x_i, y_j) \triangle y \end{vmatrix} \\ &= -f_x(x_i, y_j) \triangle x \triangle y\, \mathbf{i} - f_y(x_i, y_j) \triangle x \triangle y\, \mathbf{j} + \triangle x \triangle y\, \mathbf{k} \\ &= \triangle A \left[- f_x(x_i, y_j)\, \mathbf{i} - f_y(x_i, y_j)\, \mathbf{j} + \mathbf{k} \right] \end{aligned}$$



Then

$$\triangle T_{ij} = \left|\mathbf{a}_{ij} \times \mathbf{b}_{ij}\right| = \sqrt{\left[f_x(x_i, y_j)\right]^2 + \left[f_y(x_i, y_j)\right]^2 + 1}\ \triangle A.$$

Therefore,

$$\begin{aligned} A(S) &= \lim_{m,\, n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \triangle T_{ij} = \lim_{m,\, n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left|\mathbf{a}_{ij} \times \mathbf{b}_{ij}\right| \\ &= \lim_{m,\, n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{\left[f_x(x_i, y_j)\right]^2 + \left[f_y(x_i, y_j)\right]^2 + 1^2}\ \triangle A \\ &= \iint_D \sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\, dA. \end{aligned}$$

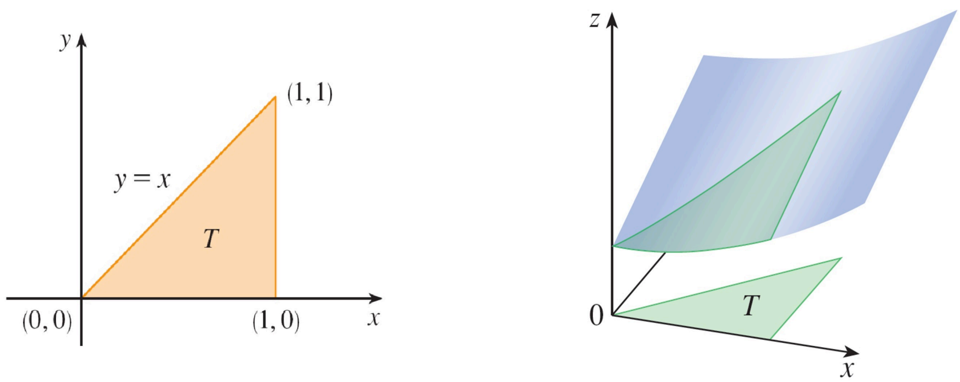
Theorem 15.5.2. *The area of the surface with equation $z = f(x, y)$ for $(x, y) \in D$, where f_x and f_y are continuous, is*

$$A(S) = \iint_D \sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\, dA.$$

Note. In Section 8.1, the arc length formula is

$$L = \int_a^g \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example 15.5.3. Find teh surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vetices $(0, 0)$, $(1, 0)$ and $(1, 1)$.



Proof. The triangular region is $T = \left\{(x, y)\ \middle|\ 0 \le x \le 1,\ 0 \le y \le x\right\}$. Let $f(x, y) = x^2 + 2y$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 2$. The surface area is

$$\begin{aligned} A(S) &= \iint_T \sqrt{(2x)^2 + 2^2 + 1}\, dA = \iint_T \sqrt{4x^2 + 5}\, dA \\ &= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dydx \\ &= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \frac{1}{12}(27 - 5\sqrt{5}). \end{aligned}$$

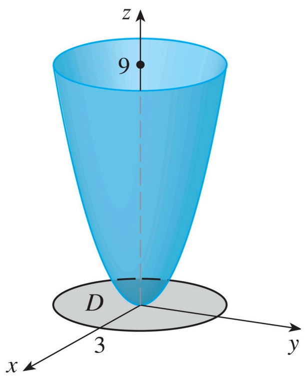
□

Example 15.5.4. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Proof. The plane $z = 9$ intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Let $D = \{(x,y) \mid x^2 + y^2 \le 9\}$ and let $f(x,y) = x^2 + y^2$. Then $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$.

The surface area is

$$\begin{aligned} A(S) &= \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA \\ &= \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1} \cdot r\, dr d\theta \\ &= \frac{\pi}{6}(37\sqrt{37} - 1). \end{aligned}$$



□

Homework 15.5. 5, 8, 11, 14, 17, 25

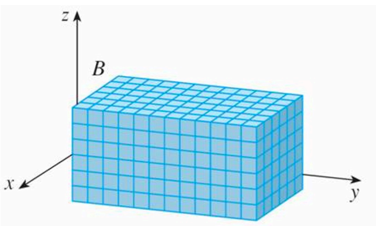
15.6 Triple Integrals

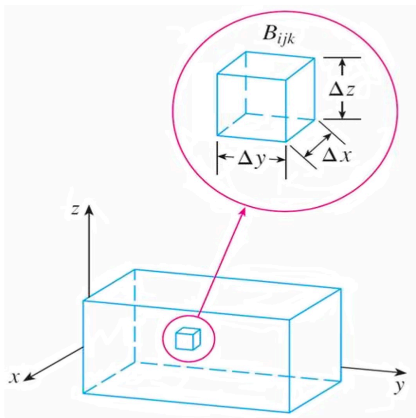
Let $f(x,y,z)$ be defined on a rectangular box $B = [a,b] \times [c,d] \times [r,s]$.

■ The Triple integral of f over B

Divide B into sub-boxes by

$$\begin{array}{ll} a = x_0 < x_1 < x_2 < \cdots < x_\ell = b & \text{with equal width } \triangle x = \frac{b-a}{\ell} \\ c = y_0 < y_1 < y_2 < \cdots < y_m = b & \text{with equal width } \triangle y = \frac{d-c}{m} \\ r = z_0 < z_1 < z_2 < \cdots < z_n = s & \text{with equal width } \triangle z = \frac{s-r}{n} \end{array}$$





$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$

Each sub-box has volume $\triangle V = \triangle x \triangle y \triangle z$. The "*triple Riemann sum*" is

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \triangle V$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \in B_{ijk}$ is a sample point.

Definition 15.6.1. The "*triple integral of f over B*" is

$$\iiint_B f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right)\triangle V$$

if this limit exists.

Remark. (1) If f is continuous on B, then the triple integral exists.

(2) The limit exists for arbitrary choice of the sample points. For the convenience, we can choose $x_{jik} = x_i$, $y_{ijk} = y_j$ and $z_{ijk} = z_k$. Then

$$\iiint_B f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_i, y_j, z_k\right)\triangle V.$$

❑ Fubini's Theorem for Triple Integrals

Theorem 15.6.2. *(Fubini's Theorem) If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then*

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dxdydz.$$

Note. The *iterated integral* "$\int_r^s \int_c^d \int_a^b f(x,y,z)\,dxdydz$" means that when taking the integrations from inner to outer. Firstly, fix y and z and integrating f with respect to x. After taking the values of lower and upper limit for x, fixing z and integrating with respect to y. Then, after taking the values of lower and upper limits for y, integrating with respect to z.

Remark. If f is continuous on B, we can exchange the order of integration. For example,

$$\iiint_B f(x,y,z)\,dv = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dydxdz.$$

and other 5 cases are equal.

Example 15.6.3. Evaluate the triple integral $\iiint_B xyz^2\,dV$ where B is the rectangular box $B = [0,1]\times[-1,2]\times[0,3]$.

Proof. Since $f(x,y,z) = xyz^2$ is continuous on B, by the Fubini's Theorem,

$$\begin{aligned}\iiint_B xyz^2\,dV &= \int_0^3\int_{-1}^2\int_0^1 xyz^2\,dxdydz\\ &= \int_0^3\int_{-1}^2 \frac{yz^2}{2}\,dydz\\ &= \int_0^3 \frac{3z^2}{4}\,dz = \frac{27}{4}.\end{aligned}$$

□

❑ Triple Integral over a General Bounded Region

Idea: Suppose that $f(x, y, z)$ is defined on a bounded region E. Choose a rectangular box $B = [a, b] \times [c, d] \times [r, s]$ such that $E \subseteq B$. Define $F(x, y, z)$ on B by

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \backslash E. \end{cases}$$

Define

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

Remark. The integral exists if f is continuous on E and the boundary of E is "*reasonably smooth*".

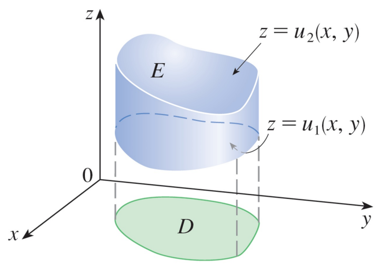
■ Different Types of Regions

From now on, we only consider those functions which are continuous on certain simple types of regions.

Type I: $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ where D is the projection of E onto the xy-plane.

Fix $(x, y) \in D$. Let $k(x,y) = \displaystyle\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$. Then k is continuous on D.

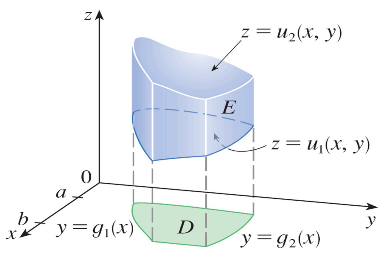
$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \iint_D k(x,y)\,dA \\ &= \iint_D \Big[\int_{u_{(x,y)}}^{u_2(x,y)} f(x,y,z)\,dz \Big] dA. \end{aligned}$$



A type I solid region

In particular, if $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

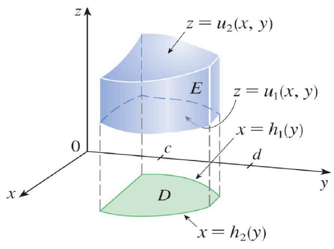
$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \iint_D k(x,y)\,dA \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} k(x,y)\,dydx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dzdydx. \end{aligned}$$



A type I solid region where the repjection D is a type I plane region

Similarly, if $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dzdxdy.$$



A type I solid region with a type II projection

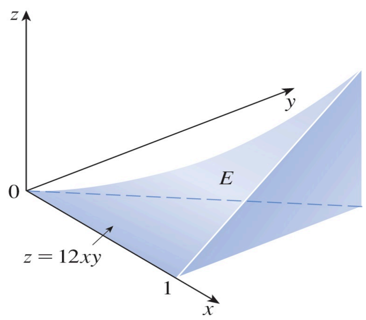
Example 15.6.4. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

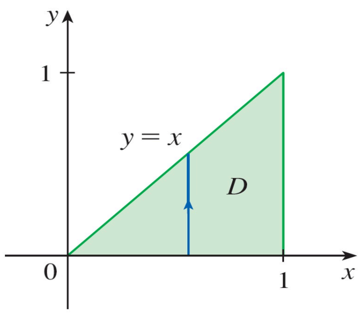
Proof. The region D is the projection of the solid E onto xy-plane. Then

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x\}.$$

The lower boundary of the tetrahedron is $z = 0$ and the upper boundary is the plane $x+y+z = 1$. Then

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le z \le 1 - x - y,\ (x,y) \in D\} \\ &= \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}. \end{aligned}$$

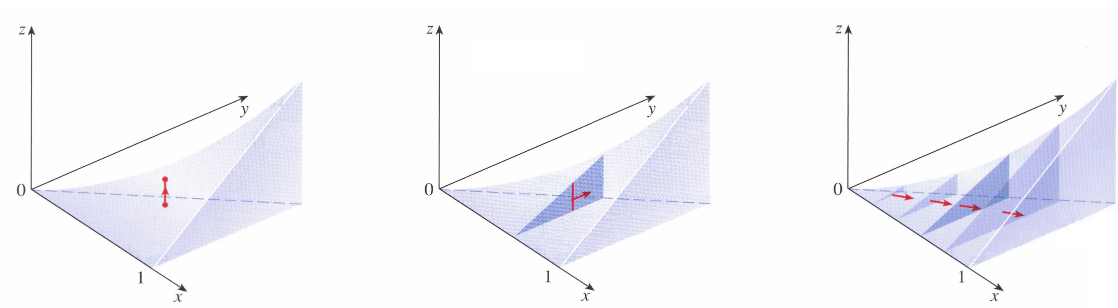




The triple integral over E is

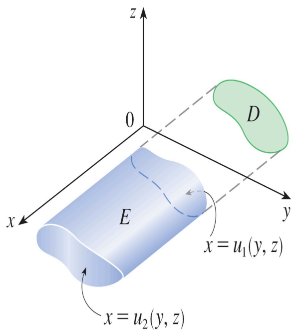
$$\begin{aligned} \iiint_E z\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dzdydx \\ &= \frac{1}{2}\int_0^1 \int_0^{1-x} (1 - x - y)^2\,dydx \\ &= \frac{1}{6}\int_0^1 (1 - x)^3\,dx = \frac{1}{24}. \end{aligned}$$

□



z varies from 0 to xy while x and y are constant. y varies from 0 to x while x is constant. x varies from 0 to 1.

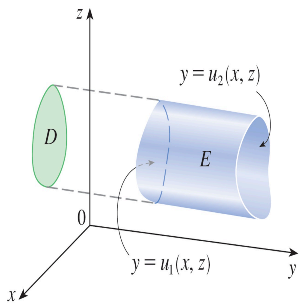
Type II:



$E = \left\{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\right\}$ where D is the projection of E onto the yz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA.$$

Type III:

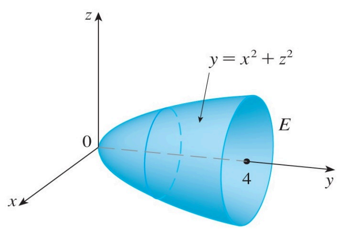


A type III region

$E = \left\{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\right\}$ where D is the projection of E onto the xz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA.$$

Example 15.6.5.

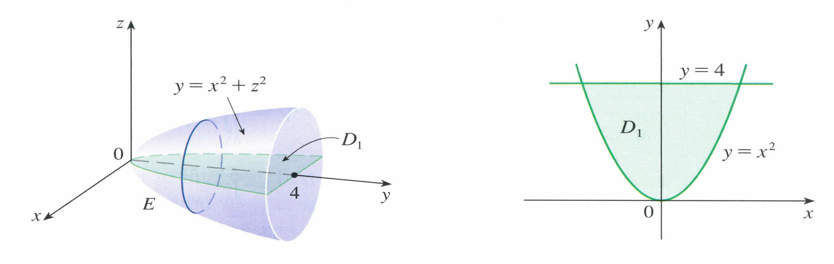


Region of integration

Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Proof. **Solution 1:**
D_1 is the projection of E onto xy-plane. For $(x, y, z) \in E$, $(x, y) \in D$ and $-\sqrt{y - x^2} \le z \le \sqrt{y - x^2}$. Then

$$E = \left\{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\right\}.$$



Region of integration   Projection onto the xy-plane

The integral is

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dzdydx.$$

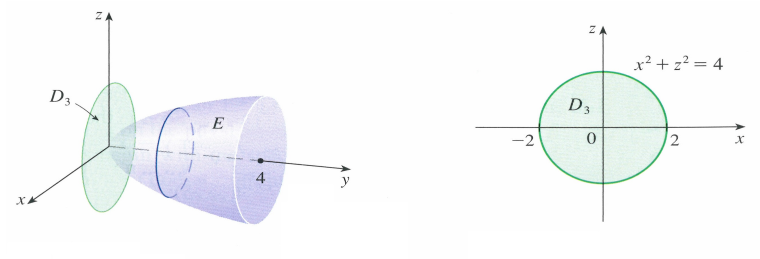
The above integral is difficult to evaluate.

Solution 2:
D_3 is the projection of E onto xz-plane.

$$D_3 = \left\{(x,z) \;\middle|\; -2 \le x \le 2,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}\right\}.$$

For $(x,y,z) \in E$, $(x,z) \in D_3$ and $x^2+z^2 \le y \le 4$. Then

$$E = \left\{(x,y,z) \;\middle|\; -2 \le x \le 2,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2},\ x^2+z^2 \le y \le 4\right\}.$$



Region of integration   Projection onto the xz-plane

The integral is

$$\begin{aligned}
\iiint_E \sqrt{x^2+z^2}\,dV &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dydzdx \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\,dzdx \\
\begin{pmatrix} x = r\cos\theta \\ z = r\sin\theta \end{pmatrix} \quad &= \int_{0}^{2\pi}\int_{0}^{2} (4-r^2)\sqrt{r^2}\cdot r\,drd\theta \\
&= \frac{128}{15}\pi.
\end{aligned}$$

□

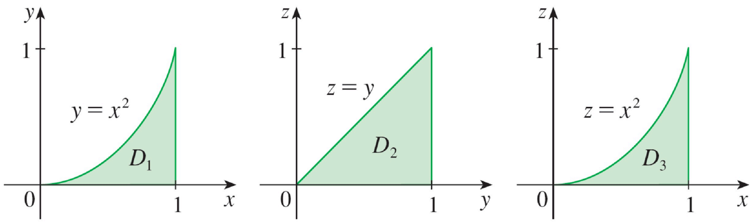
Remark. From the above example, formally, an triple integral may have several expressions. Some are easy to compute but some are difficult.

Example 15.6.6. Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dzdydx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x then z, and then y.

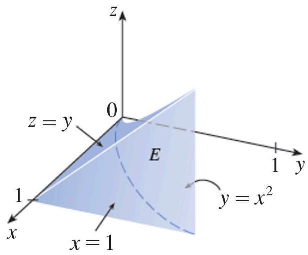
Proof.

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dzdydx = \iiint_E f(x,y,z)\, dV$$

where $E = \left\{(x,y,z) \mid 0 \le x \le,\ 0 \le y \le x^2,\ 0 \le z \le y\right\}$.



Projection of E



The solid E

$$\begin{aligned}\iiint_E f(x,y,z)\, dV &= \iint_{D_2} \int_{\sqrt{y}}^1 f(x,y,z)\, dxdA = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dxdzdy \\ &= \iint_{D_3} f(x,y,z)\, dydA = \int_0^1 \int_0^{x^2} \int_z^1 f(x,y,z)\, dydzdx.\end{aligned}$$

□

❑ Applications of Triple Integrals

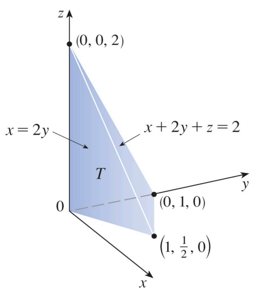
If $f(x,y,z) \ge 0$, it is difficult to visualize the triple integral $\iiint_E f(x,y,z)\, dV$.

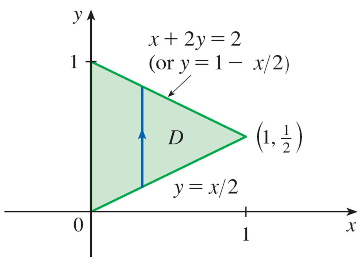
■ Volume

Let $f(x,y,z) = 1$ for all points in E. Then the"*volume*" of E is

$$V(E) = \iiint_E 1\, dV.$$

Example 15.6.7. Find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.





Proof. Let D be the projection of T onto the xy-plane. Then

$$D = \left\{(x, y) \;\middle|\; 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2}\right\}.$$

For $(x, y, z) \in T$, $(x, y) \in D$ and $0 \le z \le 2 - x - 2y$. Then

$$T = \left\{(x, y, z) \;\middle|\; 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2},\ 0 \le z \le 2 - x - 2y\right\}.$$

The volume of T is

$$V(T) = \iiint_T 1\, dV = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} 1\, dzdydx = \frac{1}{3}.$$

□

❏ Application (the center of mass)

(Skip)

Homework 15.6. 6, 8, 11, 14, 17, 20, 23, 27(a), 3136, 37, 41, 57

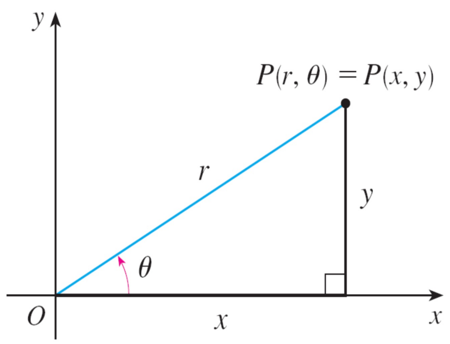
15.7 Triple Integrals in Cylindrical Coordinates

Recall : In plane geometry,

Cartesian Coordinate $\longleftrightarrow$ Polar Coordinate

$$(x, y) \longleftrightarrow (r, \theta)$$

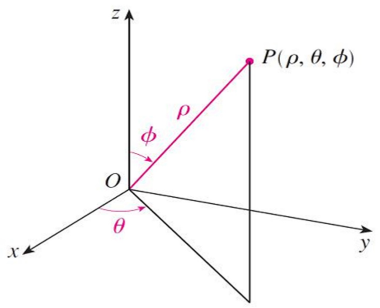
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$



❏ Cylindrical Coordinates

Let P be a point in three dimensional space. Regard the Cartesian coordinate (x, y, z) as $\Big(\underbrace{(x, y)}_{\substack{\textit{convert}\\ \textit{to polar}\\ \textit{coordinate}}}, z\Big)$

Write $P(r, \theta, z)$ where r ane θ are polar coordinates of the projection of P onto xy-plane and z is the directed distance from the xy-plane to P.



The spherical coordinates of a point

rectangular coordinates $\longleftrightarrow$ cylindrical coordintes

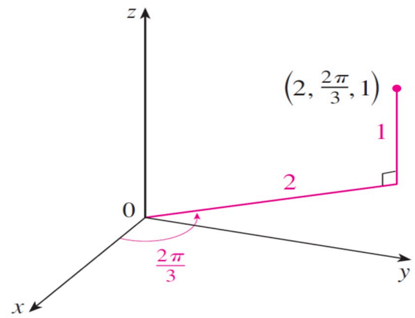
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \iff \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

Example 15.7.1.

(a) Plot the point with cylindrical coordinates $(\underbrace{2}_{r}, \underbrace{\frac{2\pi}{3}}_{\theta}, 1)$ and fidn its rectangular coordinates

Proof. Consider $x = 2\cos\frac{2\pi}{3} = -1$ and $y = 2\sin\frac{2\pi}{3} = \sqrt{3}$ and $z = 1$. □



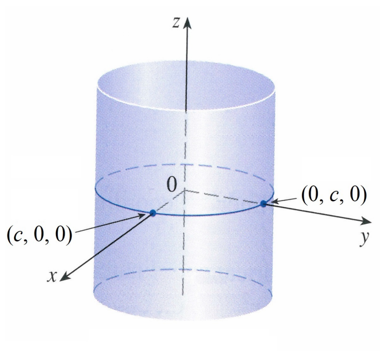
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Proof. In the cylindrical coordinates, $r = \sqrt{(3)^2 + (-3)^2} = 3\sqrt{2}$, $\tan\theta = \frac{-3}{3} = -1$ and $z = -7$. Then $\theta = \frac{7\pi}{4} + 2n\pi$ and $(r, \theta, z) = (3\sqrt{2}, \frac{7\pi}{4}, -7)$. □

Note.

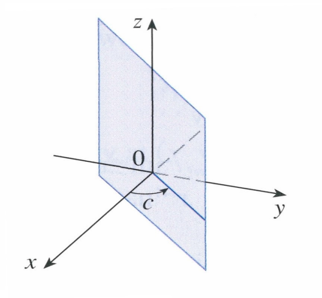
Cylindrical coordinates are useful in problems that involves symmetry about an axis, andthe z-axis is chosen to coincide with this axis of symmetry. For example, consider the circular cylinder

Cartesian equation: $x^2 + y^2 = c$
Cylindrical equation: $r = c$.

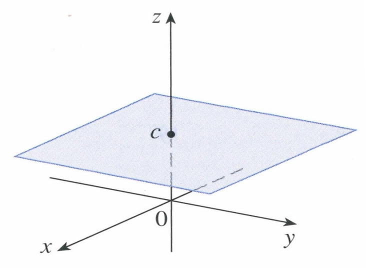


$r = c$, a cylinder

The graph of the equation $\theta = c$ is a vertical plane through the origin and the graph of the equation $z = c$ is a horizontal plane.



$\theta = c$, a vertical plane

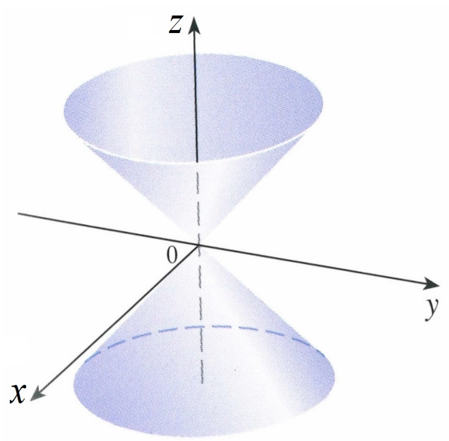


$z = c$, a horizontal plane

Example 15.7.2. Describe the surface whose equation in cylindrical coordinates is $z = r$.

Proof.

- The coordinate z is the height of the point on the surface. Hence, from the equation $z = r$, each point on the surface has height r which is the distance from the point to the z-axis.
- The coordinate θ does not appear (since it can vary from 0 to 2π).
- The horizontal trace in the plane $z = k$ ($k > 0$) is a circle of radius k.
- The rectangular coordinates $z^2 = x^2 = x^2+y^2$.



$z = r$, a cone

□

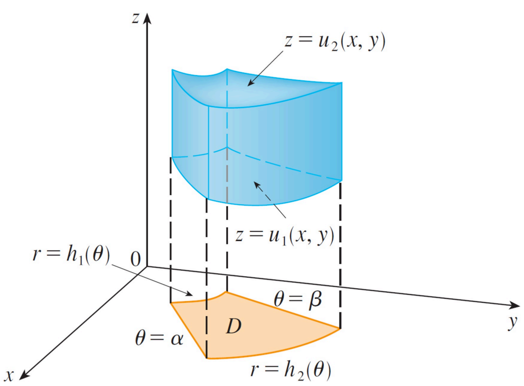
❑ Triple Integrals with Cylindrical Coordinates

Let E be a Type I region

$$E = \left\{(x,y,z) \,\middle|\, (x,y) \in D,\ u_1(x,u) \le z \le u_2(x,y)\right\}$$

where $D = \left\{(r,\theta) \,\middle|\, \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\right\}$. The triple integral over E is

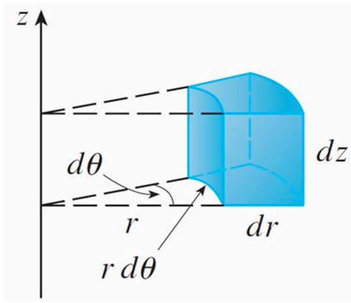
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\right] dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta, z) r\,dz dr d\theta.$$



Remark.

Convert a triple integral from rectangular to cylindrical coordinates. The appropriate limits of integration for x, y and z are replaced by z, r and θ. The infinitesimal volume dV is converted from

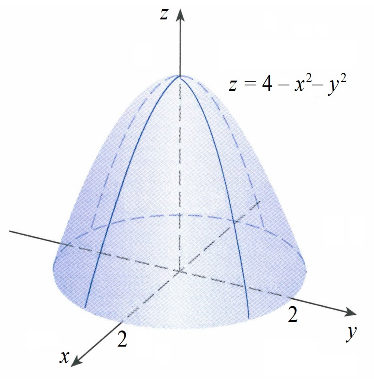
$$dxdydz \quad \text{to} \quad r\, dzdrd\theta.$$



Volume element in cylindrical coordinates: $dV = r\, dz\, dr\, d\theta$

Example 15.7.3.

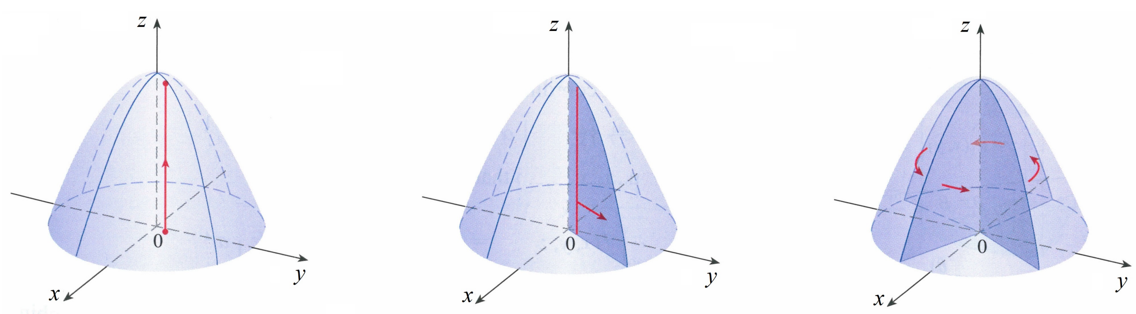
Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane.



Proof. Observe that E is symmetric about the z-axis, we use cylindrical coordinates. Moreover, the paraboloid $z - 4 - x^2 - y^2 = 4 - (x^2 + y^2)$ is easily expressed in cylindrical coordinates as $z = 4 - r^2$.

The paraboloid intersects the xy-plane in the circle $r^2 = 4$or, equivalently, $r = 2$. We have the projection of E onto the xy-]plane is the dist $r \le 2$. The region E is

$$\left\{(r, \theta, z) \;\middle|\; 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ 0 \le z \le 4 - r^2\right\}.$$



z varies from 0 to $4 = r^2$ while r and θ are constant.

r varies from 0 to 2 while θ is constant.

θ varies from 0 to 2π.

and the triple integral is

$$\begin{aligned}\iiint_E x^2\,dV &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(r\cos\theta)^2\,r\,dzdrd\theta\\ &= \int_0^{2\pi}\int_0^2 (r^3\cos^2\theta)(4-r^2)\,drd\theta\\ &= \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^2(4r^3-r^5)\,dr\\ &= \frac{1}{2}\Big[\theta+\frac{1}{2}\sin 2\theta\Big]_0^{2\pi}\Big[r^4-\frac{1}{6}r^6\Big]_0^2\\ &= \frac{1}{2}(2\pi)(16-\frac{32}{3}) = \frac{16}{3}\pi.\end{aligned}$$

□

Example 15.7.4. A solid E lies within the cylinder $x^2+y^2=1$ to the right of the xz-plane, below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density at any point is paoportional to its distance from the axis of the cylinder. Find the mass of E.

Proof.

In cylindrical coordinates, the cylinder is $r=1$ and the paraboloid is $z=1-r^2$. The solid is

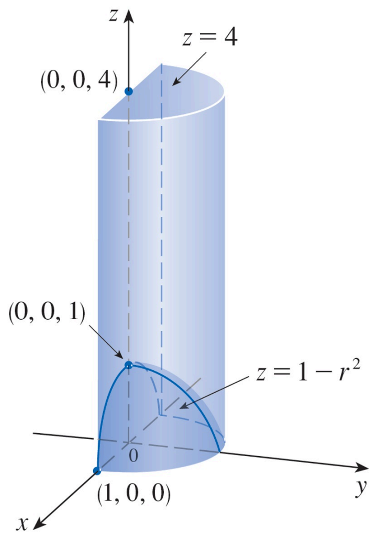
$$E=\{(r,\theta,z)\mid 0\le\theta\le\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}.$$

The density function is

$$f(x,y,z)=k\sqrt{x^2+y^2}=kr.$$

Thus, the mass is

$$\begin{aligned}M &= \iiint_E f(x,y,z)\,dV=\int_0^{\pi}\int_0^1\int_{1-r^2}^4 kr\cdot r\,dzdrd\theta\\ &= \int_0^{\pi}\int_0^1 kr^2\big[4-(1-r^2)\big]\,drd\theta=\frac{6\pi k}{5}.\end{aligned}$$



□

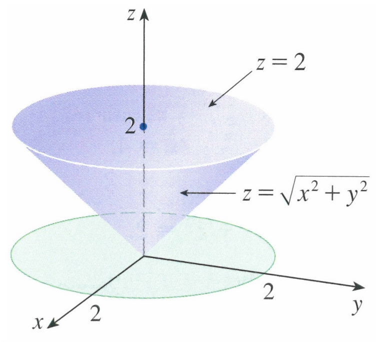
Example 15.7.5. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx$.

Proof.

$$\begin{aligned}E &= \{(x,y,z)\mid -2\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\ \sqrt{x^2+y^2}\le z\le 2\}\\ &= \int(r,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le r\le 2,\ r\le z\le 2\}.\end{aligned}$$

The triple integral is

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx$$
$$=\iiint_E (x^2+y^2)\,dV$$
$$=\int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r\,dzdrd\theta = \frac{16}{5}\pi.$$



□

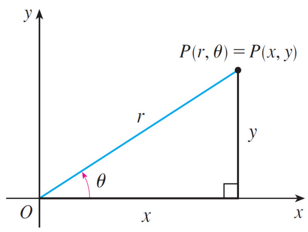
Homework 15.7. 9, 11, 17, 19, 23, 26, 31

15.8 Triple Integrals in Spherical Coordinates

Recall:

In two dimensions,

Cartensian Coordinates	$\longleftrightarrow$	Polar Coordinates
(x, y)	$\longleftrightarrow$	(r, θ)



Question: How about in three dimensions?

In the previous section, we learned the cylindrical coordinate,

$$(x, y, z) \longrightarrow (r, \theta, z).$$

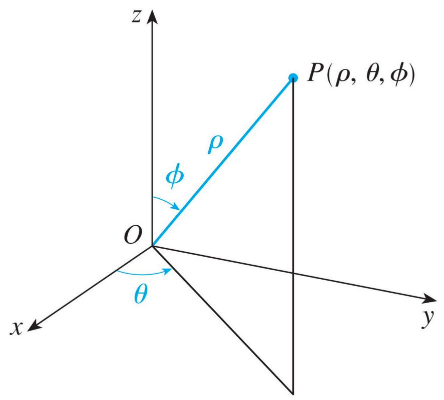
Question: Is there any other coordinate system?

❏ Spherical Coordinates

Let P be a point in space, Q be the projection of P onto xy-plane and $\overline{OQ}$ be the projection of $\overline{OP}$ onto xy-plane.

Denote

- Let $\rho = |\overline{OP}|$ be the distance from the origin to P.
- Let θ be the angle between the positive x-axis and the line segment $\overline{OQ}$. (the same angle as in cylindrical coordinateds)
- Let ϕ be the angle between the positive z-axis and the line segment $\overline{OP}$.

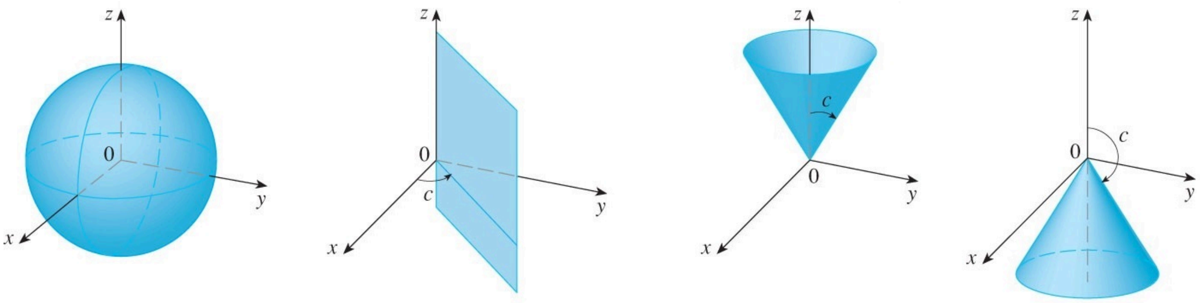


The spherical coordinates of a point

Note. In the above variables, we set

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Remark. The spherical coordinate system is useful in problems where there is symmetry about a point, and the origin is placed at the point. For example,



$0 < c < \pi/2$ $\quad$ $\pi/2 < c < \pi$

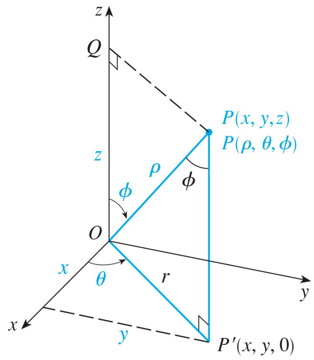
$\rho = c$, a sphere $\quad$ $\theta = c$, a half-plane $\quad$ $\phi = c$, a half-cone

■ Relationship between Rectangular and Spherical Coordinates

$$(x, y, z) \quad \longleftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} z = \rho \cos \phi \\ r = \rho \sin \phi \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \end{cases}$$

$$x = \rho \sin \phi \cos \theta,\ y = \rho \sin \phi \sin \theta,\ z = \rho \cos \phi,\ \rho^2 = x^2 + y^2 + z^2.$$



Example 15.8.1. The point $\overset{\rho}{2}, \overset{\theta}{\frac{\pi}{4}}, \overset{\phi}{\frac{\pi}{3}}$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

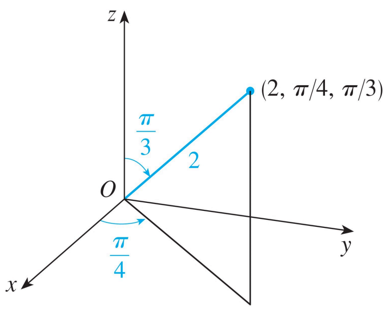
Proof.

Consider $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2} = 4$.

$$\cos \phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \quad \Longrightarrow \quad \phi = \frac{2\pi}{3}$$

$$\cos \theta = \frac{x}{\rho \sin \phi} = 0 \quad \Longrightarrow \quad \theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}$$

Check that $\sin \theta = \dfrac{y}{\rho \sin \phi} = 1$. Therefore,



$$(\rho, \theta, \phi) = (4, \frac{\pi}{2}, \frac{2\pi}{3}).$$

□

❑ Triple Integrals with Spherical Coordinates

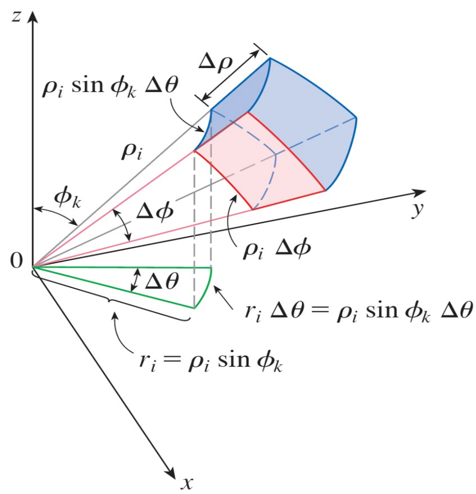
Let E be a spherical wedge. Then

$$E = \left\{(\rho, \theta, \phi) \;\middle|\; a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\right\}$$

where $a \ge 0$, $0 \le \beta - \alpha \le 2\pi$, $0 \le d - c \le \pi$. Evaluate $\displaystyle\iiint_E f(x, y, z)\, dV$.

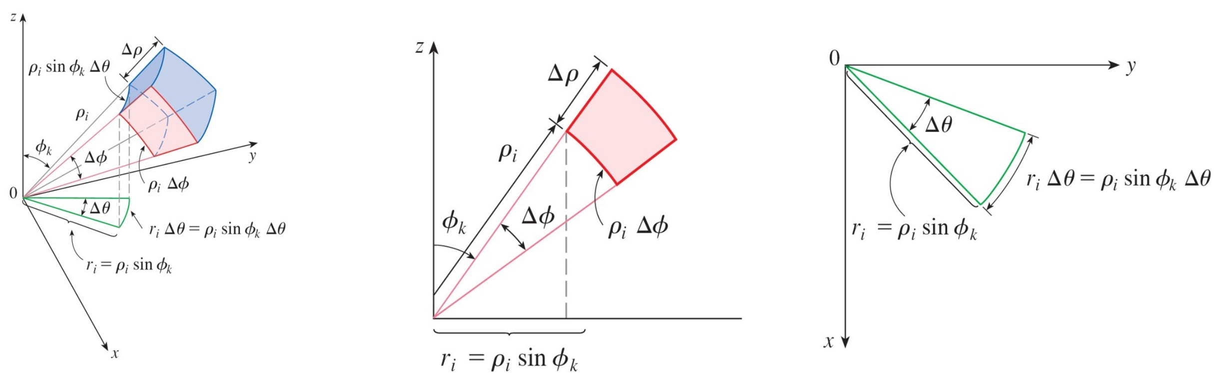
- Divide E into small sub-spherical wedges by

$$a = \rho_0 < \rho_1 < \cdots < \rho_\ell = b, \quad \Delta\rho = \frac{b-a}{\ell}$$
$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_m = \beta, \quad \Delta\theta = \frac{\beta-\alpha}{m}$$
$$c = \phi_0 < \phi_1 < \cdots < \phi_n = d, \quad \Delta\phi = \frac{d-c}{n}$$



- Consider the smaller spherical wedge E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cone $\phi = \phi_k$. Then E_{ijk} is approximately a rectangular box with dimensions $\triangle\rho$, $\rho_i\triangle\phi$ (arc of a circle with radius, ρ_i, angle $\triangle\phi$) and $\rho_i \sin\phi_k\triangle\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\triangle\theta$). We have

$$\triangle V_{ijk} \approx (\triangle\rho)(\rho_i\triangle\phi)(\rho_i \sin\phi_k\triangle\theta) = \rho_i^2 \sin\phi_k\triangle\rho\triangle\theta\triangle\phi.$$



(a) A spherical wedge (b) Side view (c) Top view

By the Mean Value Theorem,

$$\triangle V_{ijk} = \bar{\rho}_i^2 \sin\bar{\phi}_k\triangle\phi\triangle\theta\triangle\phi$$

for some $(\bar{\rho}_i, \bar{\theta}_j, \bar{\phi}_k)$ in E_{ijk} . Then

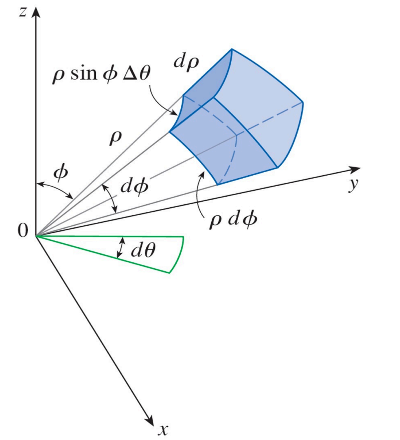
$$\begin{aligned}\iiint_E f(x, y, z)\, dV &= \lim_{\ell,\, m,\, n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \\ &= \int_c^d \int_\alpha^\beta \int_a^b f\left(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\right)\rho^2 \sin\phi\ d\rho d\theta d\phi.\end{aligned}$$

Note.

Convert a triple integral from rectangular coordinates to spherical coordinates by

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi.$$

The approximate limt of integration for x, y and z are replaced with respect to ρ, θ and ϕ. The infinitesimal volume dV is converted from "$dxdydz$" to "$\rho^2 \sin\phi\ d\rho d\theta d\phi$".



Volume element in spherical coordinates: $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

Note. The integration formula can be extedede to more general spherical region such as

$$E = \left\{(\rho, \theta, \phi) \mid \alpha \le \theta\beta,\ c \le \phi \le d,\ g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\right\}.$$

The triple integral formula becomes

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f\left(\rho \sin\phi \cos\theta, r \sin\phi \sin\theta, \rho \cos\phi\right) \rho^2 \sin\phi\ d\rho d\theta d\phi.$$

Example 15.8.2. Evaluate $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball

$$B = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\right\}.$$

Proof. Using spherical coordinates to express

$$B = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\right\}$$

and $x^2 + y^2 + z^2 = \rho^2$. The triple integral is

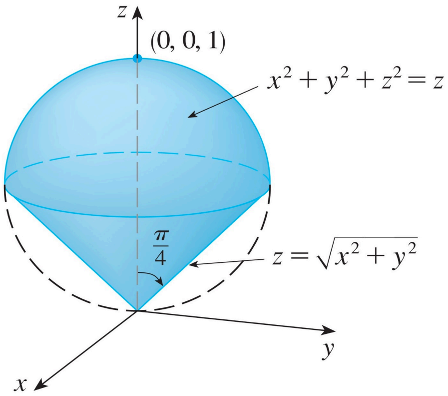
$$\begin{aligned}
\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \cdot \rho^2 \sin\phi\ d\rho d\theta d\phi \\
&= \left(\int_0^\pi \sin\phi\ d\phi\right)\left(\int_0^{2\pi} 1\ d\theta\right)\left(\int_0^1 \rho^2 e^{\rho^3}\ d\rho\right) \\
&= \frac{4}{3}\pi(e - 1).
\end{aligned}$$

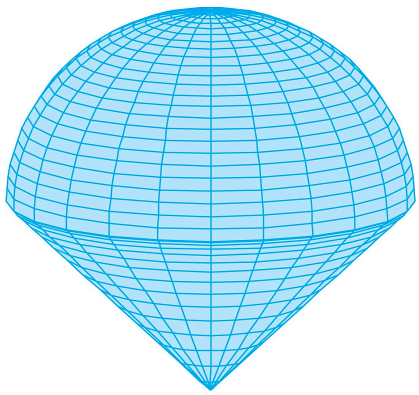
□

Note. It is difficult to evaluate the trple integral by

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\ dzdydx.$$

Example 15.8.3. Use spherical coordinates find the volume of the solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.





Proof. The sphereical coordinates of the above sphere has equation $\rho^2 = \rho = \cos\phi$ and the $\rho = \cos\phi$.

The spherical coordinate of the below cone has equatin

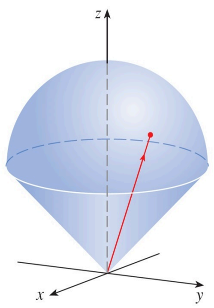
$$\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} = \rho\sin\phi.$$

We have $\cos\phi = \sin\phi$ and this implies $\phi = \frac{\pi}{4}$. Hence the solid E in spherical coordinate is

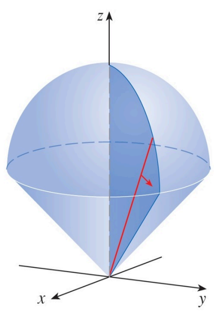
$$E = \left\{(\rho, \theta, \phi) \,\middle|\, 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le \cos\phi\right\}.$$

The volume of the solid is

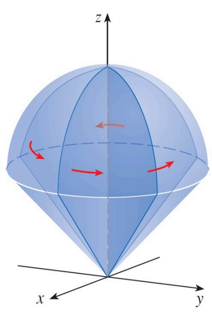
$$\begin{aligned} V(E) = \iiint_E 1\, dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta \\ &= \frac{2\pi}{3} \int_0^{4\pi} \sin\phi\cos^3\phi \, d\phi = \frac{\pi}{8} \end{aligned}$$



ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π.

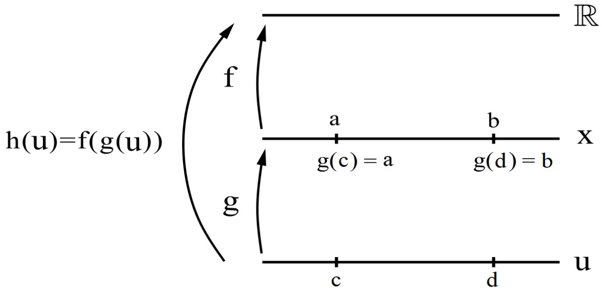
□

Homework 15.8. 17, 20, 22, 26, 29, 32, 38, 44

15.9 Change of Variables in Multiple Integrals

Recall: In one dimensional calculus,

$$y = f(x), \quad x = g(u), \quad a = g(c), \quad b = g(d).$$



To compute $\int_a^b f(x)\,dx$.

Suppose that $g : [c,d] \to [a,b]$ is a one-to-one and onto function. For example $x = x(u) = g(u) = 2u$. Consider $y = f\big(g(u)\big)$.

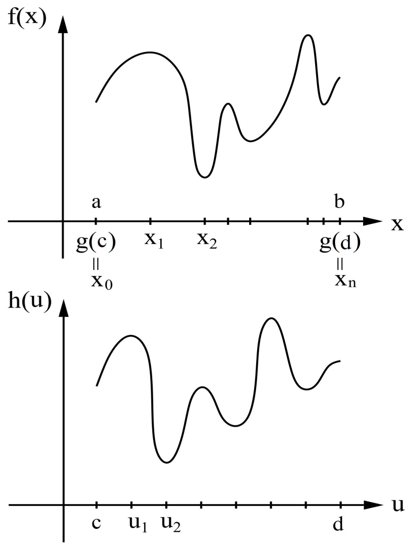
Substitution Rule

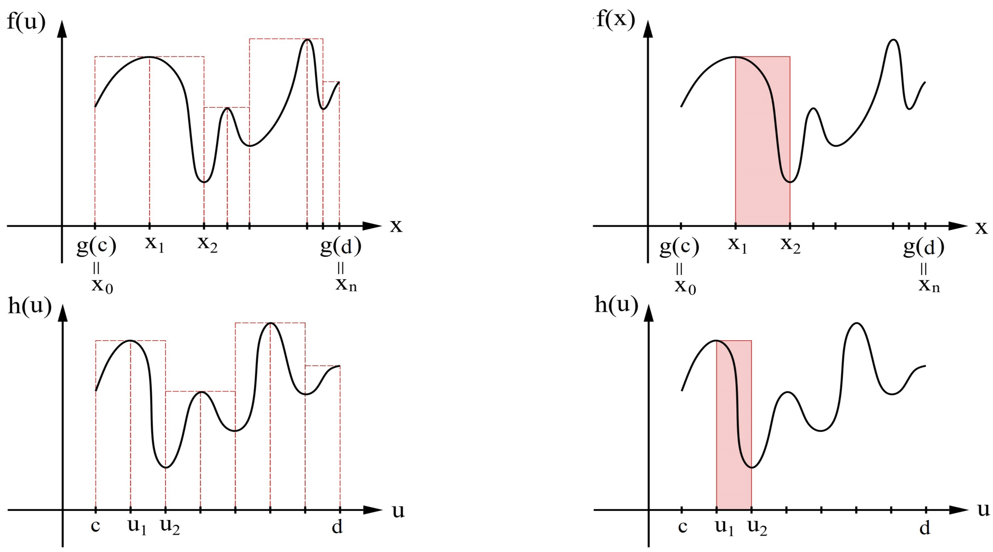
$$\int_a^b f(x)\,dx = \int_c^d f\big(g(u)\big)g'(u)\,du \quad \text{or} \quad \int_a^b f(x)\,dx = \int_c^d f\big(x(u)\big)\,\frac{dx}{du}\,du.$$

Remark.

The role of $\frac{dx}{du} = g'(u)$ is the multiple between the infinitesimal unit vector $du \to dx$ (imagine that $dx = \frac{dx}{du}\,du$.)

Look at the figures and we want to find the area of the region below the graph of f and above the x-axis over $[a,b]$ in the xy-plane. There exists a corresponding region in the uy-plane. We try to understand the relations between these two regions and areas.





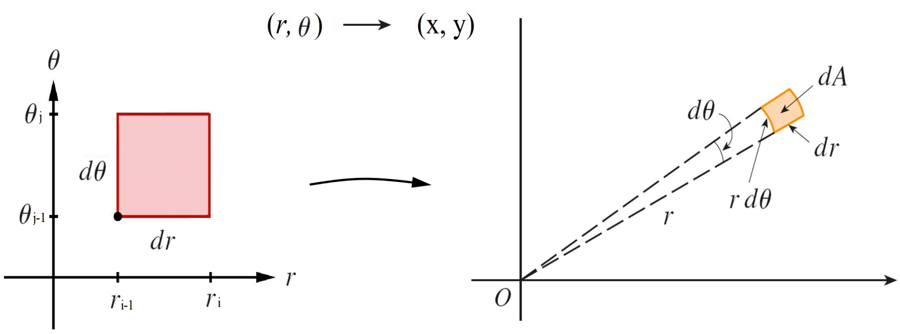
To consider the width of the base in the partition of $[a, b]$ and the one in the partition of $[c, d]$, we have $\dfrac{dx}{du} \approx \dfrac{\triangle x}{\triangle u}$. That is the approximation of the rate of the sizes of bases with respect to $\triangle x$ and $\triangle u$. Therefore the areas

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\triangle x_i = \sum_{i=1}^{n} h(u_i)\cdot \underbrace{\frac{\triangle x_i}{\triangle u_i}}_{\approx \frac{dx}{du}} \cdot \triangle u_i \approx \int_c^d h(u)\,\frac{dx}{du}\,du = \int_c^d f\big(g(u)\big)g'(u)\,du.$$

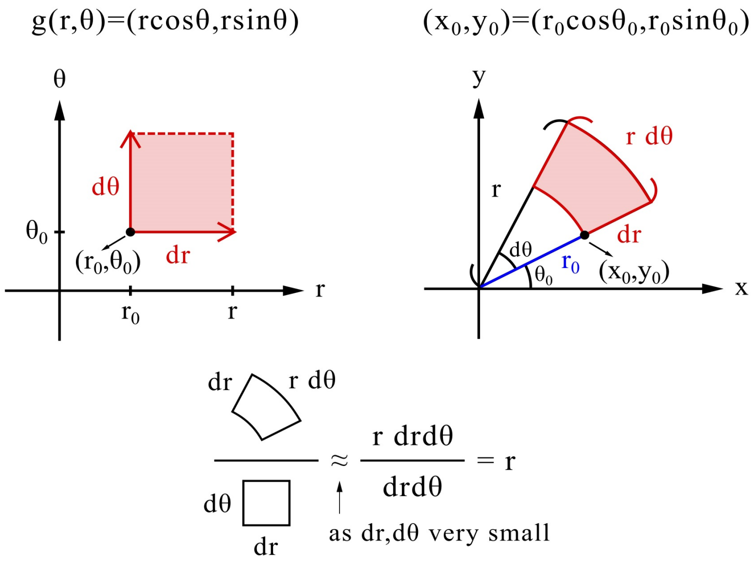
■ Change of Variables in Two Dimensions

Let $z = f(x, y)$. We consider the transformation from Cartesian coordinate to polar coordinates (see Section 15.3)

$$\begin{array}{ccc} \text{rectangular coordinates} & \longleftrightarrow & \text{polar coordinates} \\ (x, y) & \longleftrightarrow & (r, \theta) \end{array}$$



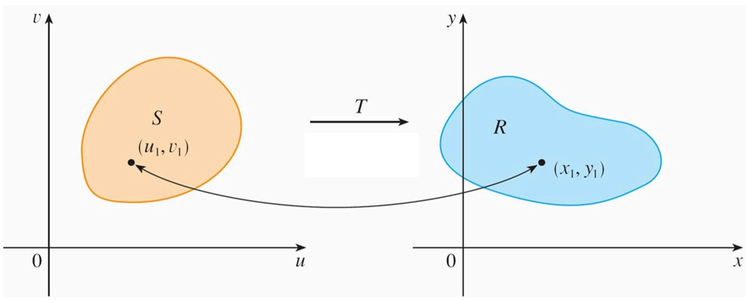
We have



and therefore the double integral

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, drd\theta.$$

In general, $T : C^1$ transformation from uv-plane to xy-plane



Note. We want to evaluate an double integral over R by evaluating a corresponding double integral over S.

Remark. There may have problems if T is not one-to-one.

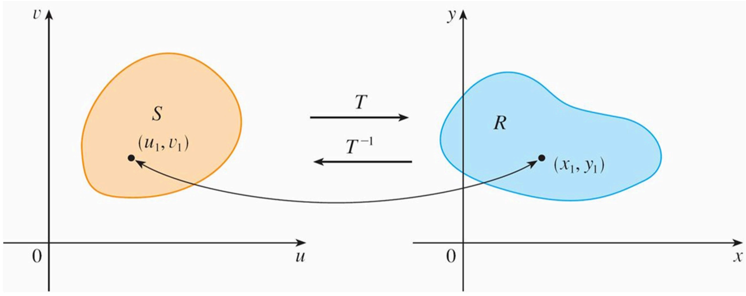
Definition 15.9.1. We say that a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is "*one-to-one*" if $(u_1, v_1) \neq (u_2, v_2)$ then

$$T(u_1, v_1) \neq T(u_2, v_2)$$

An equivalent definition is that if $T(u_1, v_1) = T(u_2, v_1)$ then $(u_1, v_1) = (u_2, v_2)$.

Remark. If T is a one-to-one transformation, then it has an inverse transformation T^{-1} from its image in xy-plane to its domain uv-plane and it may be possible to solve $x = g(u, v)$, $y = h(u, v)$ for u and v in terms of x and y:

$$(u, v) = T^{-1}(x, y), \quad u = G(x, y) \quad \text{and} \quad v = H(x, y).$$

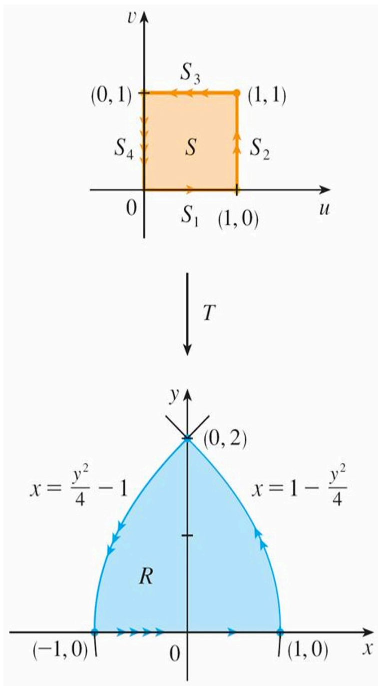


Example 15.9.2. A transformation is defined by

$$x = u^2 - v^2, \quad y = 2uv.$$

Find the image of the square $S = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$.

Proof. Note that the transformation maps the boundearies of S to the boundaries of its image. Hence, we try to find the image of the sides of S. Consider the four sides of S.



- $S_1 : (0 \le u \le 1,\ v = 0)$. Then $x = u^2$, $y = 0$ and we have

$$T(S_1) = \{(x, y) \mid 0 \le x \le 1,\ y = 0\}.$$

- $S_2 : (u = 1,\ 0 \le v \le 1)$. Then $x = 1 - v^2$, $y = 2v$ and we have $x = 1 - \frac{y^2}{4}$ and

$$T(S_2) = \left\{(x, y) \mid x = 1 - \frac{y^2}{4},\ 0 \le x \le 1\right\}.$$

- $S_3 : (0 \le u \le 1,\ v = 1)$. Then $x = \frac{y^2}{4} - 1$, $-1 \le x \le 0$ and we have

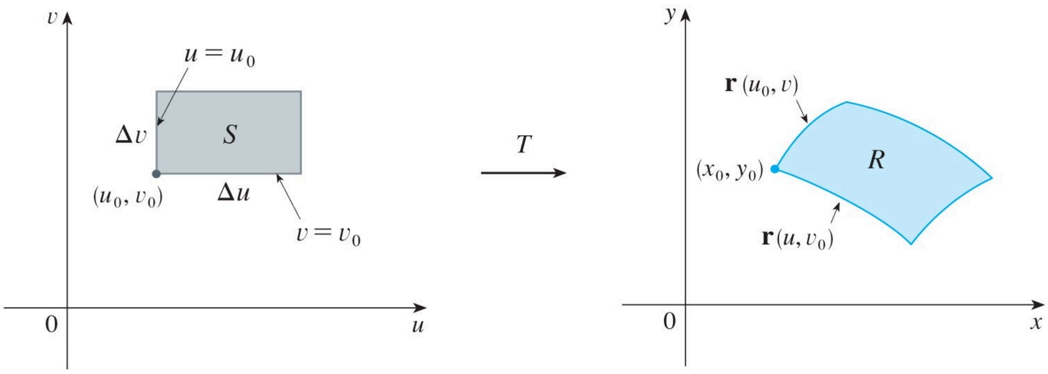
$$T(S_3) = \left\{(x, y) \mid x = \frac{y^2}{4} - 1,\ -1 \le x \le 0\right\}.$$

- $S_4 : (u = 0,\ 0 \le v \le 1)$. Then $x = -v^2$, $y = 0$ and we have

$$T(S_4) = \{(x, y) \mid -1 \le x \le 0,\ y = 0\}.$$

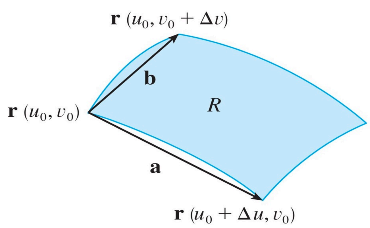
The image of S is the region R bounded by the four curves. □

Question: How does a change of variables affect a double integral?

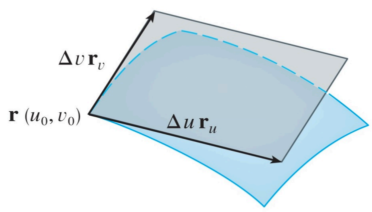


- The image of S is a region R in xy-plane. Suppose $(x_0, y_0) = T(u_0, v_0)$ is a boundary point of R. Let $\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$ be the position vector $\begin{bmatrix} x = g(u, v) \\ y = h(u, v) \end{bmatrix}$.
- Consider the boundary curve of R, $\mathbf{r}(u, v_0)$ and its tangent vector at (x_0, y_0) is
$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}.$$
Similarly, the bounde curve $\mathbf{r}(u_0, v)$ with the tangent vector at (x_0, y_0) is
$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}.$$
- To approximate $R = T(S)$ by a parallelogram determined by the secant vectors.

$$\begin{aligned} \mathbf{a} &= \mathbf{r}(u_0 + \triangle u, v_0) - \mathbf{r}(u_0, v_0) \approx \triangle u \mathbf{r}_u \\ \mathbf{b} &= \mathbf{r}(u_0, v_0 + \triangle v) - \mathbf{r}(u_0, v_0) \approx \triangle v \mathbf{r}_v \end{aligned}$$



Similarly, R can be approximated by the parallelogram determined by the vectors $\triangle u \mathbf{r}_u$ and $\triangle v \mathbf{r}_v$.



Hence, the area of R can be approximated by the area of the parallelogram
$$\left|(\triangle u \mathbf{r}_u) \times (\triangle v \mathbf{r}_v)\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \triangle u \triangle v.$$

The area of R
$$A(R) \approx \text{Area of the parallelogram} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \triangle u \triangle v = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \triangle u \triangle v.$$

Note. If regarding $\mathbf{r}_u$ and $\mathbf{r}_v$ as vector in three dimensions

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + 0\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + 0\mathbf{k}$$

and we have

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\mathbf{k}$$

Definition 15.9.3. The "*Jacobian*" of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

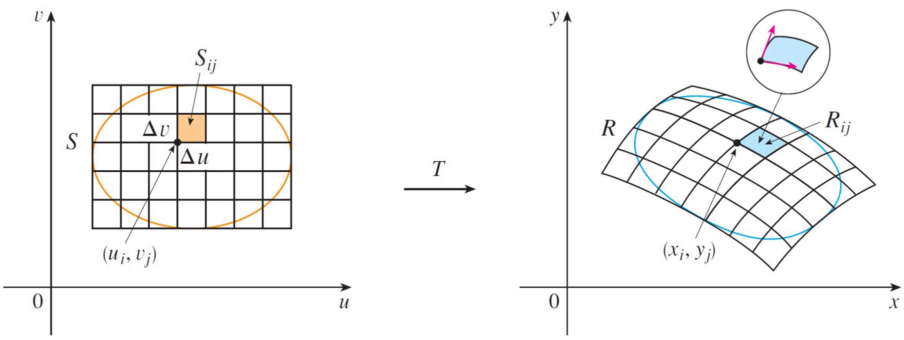
Remark. An approximation to the area $\triangle A$ of R is

$$\triangle A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \triangle u \triangle v$$

where the Jacobian is evaluated at (u_0, v_0).

■ The Riemann Sum

Now, we consider the double Riemann sum. Divide R in xy-plane into some subregion R_{ij} with area $\triangle A$.



$$\begin{aligned} \iint_R f(x, y)\, dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\triangle A \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\big(g(u_i, v_j), h(u_i, v_j)\big)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|\triangle u \triangle v \end{aligned}$$

where the Jacobian is evaluated at (u_i, v_j).

Let $m, n \to \infty$, the double Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(g(u_i, v_j), h(u_i, v_j)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\triangle u\triangle v$ converges to

$$\iint_S f\left(g(u,v), h(u,b)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

■ **Change of Variables in a Double Integral**

Theorem 15.9.4. *Suppose that*

(i) the transformation T is a C^1 map;

(ii) the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero;

(iii) T is a one-to-one and onto map from S in uv-plane to R in xy-plane except perhas on the boundary of S;

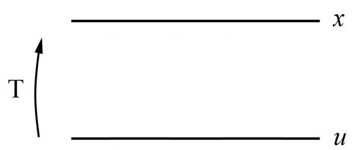
(iv) f is continuous on R and that R and S are Type I or Type II plane regions.

Then

$$\iint_R f(x,y)\, dA = \iint_S f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

■ **Compare with the Change of Variables between 1D and 2D**

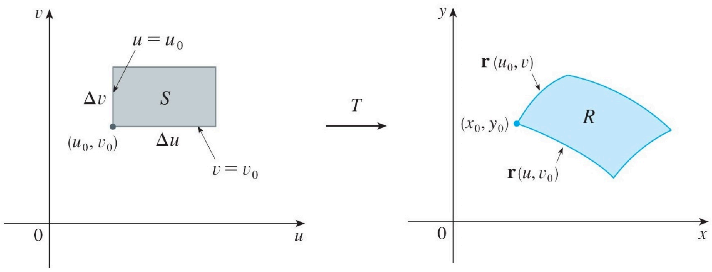
1D:



$x = \mathrm{T}(u) = x(u)$

$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$

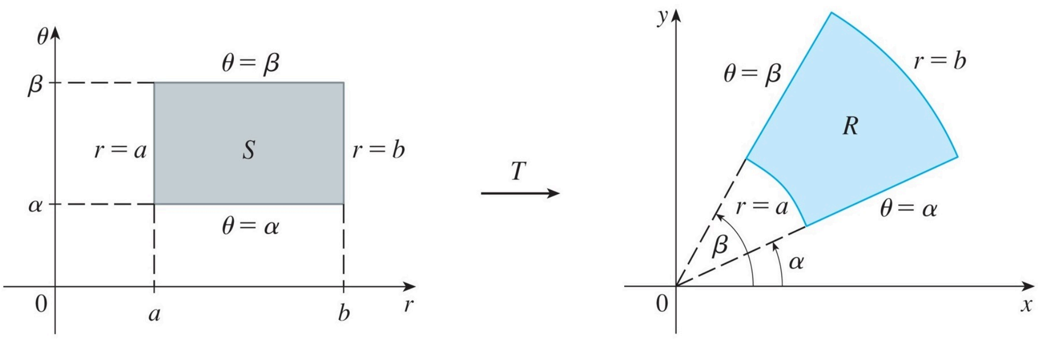
2D:



$(x, y) = \mathrm{T}(u, v) = \left(x(u,v), y(u,v)\right)$

$\mathrm{dA} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$

■ **Polar Coordinate Transformation**



The polar coordinate transformation

$$x = g(r,\theta) = r\cos\theta, \quad y = h(r,\theta) = r\sin\theta$$

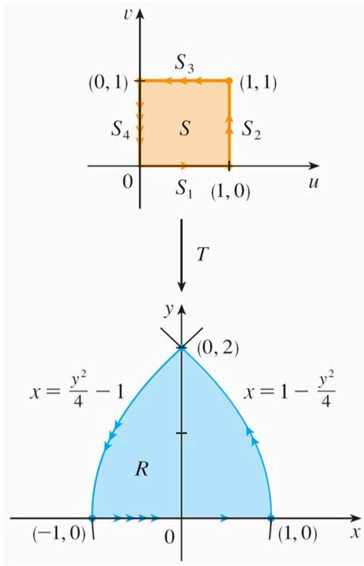
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Therefore,

$$\begin{aligned} \iint_R f(x,y)\,dA &= \iint_S f\big(r\cos\theta, r\sin\theta\big)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr d\theta \\ &= \int_\alpha^\beta \int_a^b f\big(r\cos\theta, r\sin\theta\big)\, r\, dr d\theta. \end{aligned}$$

Example 15.9.5. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Proof.



The transformation $S : [0,1]\times[0,1] \to R$ by $(u,v) \to (x,y) = (u^2 - v^2, 2uv)$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2) > 0.$$

The double integral

$$\iint_R f(x,y)\,dA = \iint_R y\,dA = \int_0^1 \int_0^1 2uv{\cdot}4(u^2{+}v^2)\,dudv = 2.$$

□

Note. For the original integral $\iint_R y\,dA$, the region R is awkward. So we use change of variables.

Example 15.9.6. Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

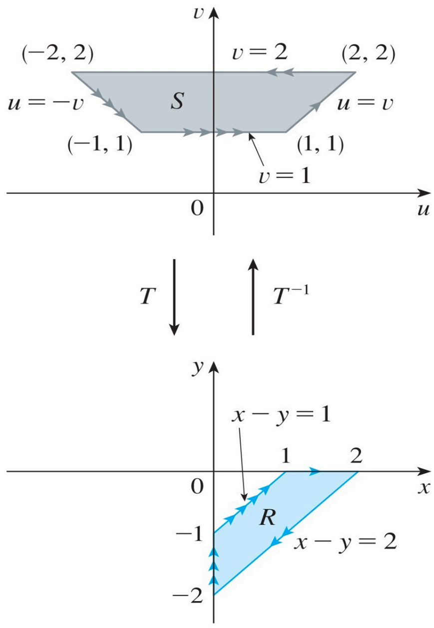
Proof.

Let $u = x+y$ and $v = x-y$. Then $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \neq 0.$$

The region R is bounded by $y = 0$, $x - y = 2$, $x = 0$ and $x - y = 1$. Hence, the corresponding region S in uv-plane is $u = v$, $v = 2$, $u = -v$ and $v = 1$. Then

$$S = \{(u,v) \mid 1 \le v \le 2,\ -v \le u \le v\}.$$



The double integral

$$\begin{aligned}\iint_R e^{-\frac{x+y}{x-y}}\, dA &= \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \left|(-\frac{1}{2})\right| \, dudv = \frac{1}{2}\int_1^2 v e^{\frac{u}{v}}\Big|_{u=-v}^{u=v} dv \\ &= \frac{1}{2}\int_1^2 e - e^{-1}\, dv = \frac{3}{4}(e - e^{-1}).\end{aligned}$$

□

❏ Triple Integrals

Let T be a transformation from a region S in uvw-space onto a region R in xyz-space. Then

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w).$$

The Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

■ Change of Variables in a Triple Integrals

$$\iiint_R f(x,y,z)\, dV = \iiint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw.$$

■ Spherical Coordinate Transformation

Let

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = -\rho^2 \sin\phi.$$

The triple integralis

$$\iiint_R f(x, y, z)\, dV = \iiint_S f\big(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\big)\Big| -\rho^2 \sin\phi \Big|\, d\rho d\theta d\phi.$$

Homework 15.9. 4, 8, 10, 13, 16, 19, 21, 25, 27, 30