Advanced Calculus (II)

Lecture Note 2022 Spring

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Chapter 0

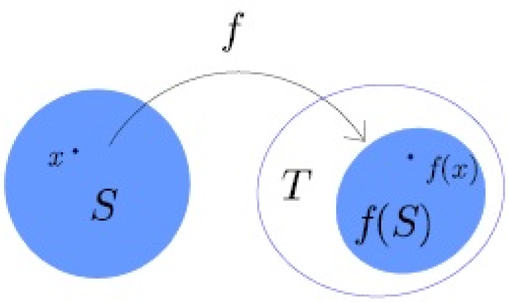
Introduction - Sets and Functions

0.1 Preliminaries

❏ Sets

❏ Functions

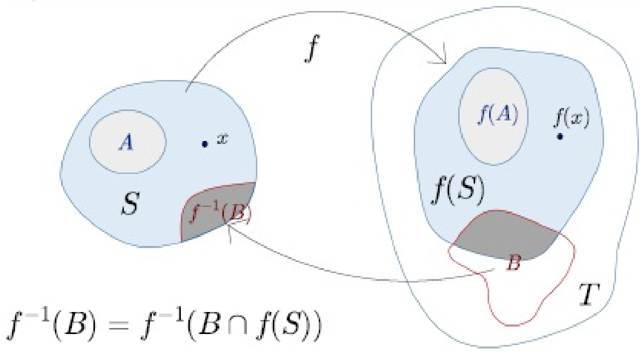
$$f : S\,(\text{domain}) \longrightarrow T\,(\text{codomain})$$



If f is an ono-to-one function, then the "*inverse function of* f", $f^{-1} : f(S) \longrightarrow S$ exists.

In general, f^{-1} may not exist. But we can still define the "*pre-image*" of f, $f^{-1}(B)$.

Definition 0.1.1. Let $f : S \to T$ be a function and $A \subseteq S$. We call $f(A) = \{f(x) \mid x \in A\}$ "*the image of A under f*". For $B \subseteq T$, we call the set $f^{-1}(B) = \{x \in S \mid f(x) \in B\}$ "*the pre-image of B under f*".



f
A
x
S
$f^{-1}(B)$
$f(A)$
$f(x)$
$f(S)$
B
T
$f^{-1}(B) = f^{-1}(B \cap f(S))$

Chapter 1

The Real Line and Euclidean Space

1.1 Motivation and Some Ideas

Question: We seem to be much familiar with the real numbers. Why do we want to investigate the real number system?

- We live in $\mathbb{R}^3$ (Really? What's $\mathbb{R}$?)
- We are too familiar with $\mathbb{R}$ to describe it.
- Many results of Calculus are based on some properties of $\mathbb{R}$. For example,

Limit		Continuity		Differentiation		Integration
Least Upper Bound Property	$\Rightarrow$	I.V.T. Extreme Value Theorem	$\Rightarrow$	Rolle's Theorem M.V.T.	$\Rightarrow$	F.T.C.

- $\mathbb{R}$ has "*Least Upper Bound Property*" but $\mathbb{Q}$ does not. For example

$$S = \{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \cdots\} \subset \mathbb{Q}$$

  There is no number in $\mathbb{Q}$ such that the number is the least upper bound of S.

Question: Does $\mathbb{R}$ really have "*Least Upper Bound Property (L.U.B.P)*"?
In fact, $\mathbb{R}$ is defined by an "*ordered field with least upper bound property*"(賦序體)

Question: Does this set really exist?

Question: How many this kind of set are there?

Question: How much do we recognize $\mathbb{R}$ or how about a set without L.U.B.P.?

❑ Five Axioms

■ **Euclidean Geometry (B.C 325 ∼ B.C. 265)** The Euclidean geometry is based on five axioms:*

(1) To draw a straight line from any point to any point.

(2) To produce (extend) a finite straight line continuously in a straight line.

(3) To describe a circle with any centre and distrance (radius).

(4) That all right angles are equal to one another.

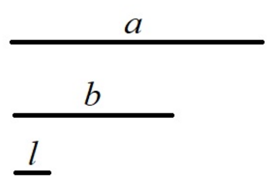
(5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two right angles, the two straight line, if produced indefinitely, meet on that side on which the angles are less than two right angles.

■**Non-Euclidean Geometry**

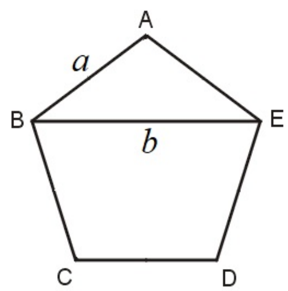
❑ Origin of irrational numbers

■**The Age of Pythagoras (B.C 570. ∼ B.C. 495)** (萬物皆數)

For any two segments with lengths a and b, there exists another segment with length ℓ and two integers $m$and n such that $a = m\ell$ and $b = n\ell$. Hence, $\frac{a}{b} = \frac{m}{n} \in \mathbb{Q}$.



But people find that the ratio of the lengths of some segments may not be a rational number. For example, the lenghts of the side and a diagonal line in a regular pentagon as in the figure have no common factor. This causes the well-known first crisis in mathematics.



People knew that there are many numbers which are not rational. But they do not figure out the real numbere system until Dedekind (1831-1916) and Cantor(1845-1918).

Question: Is there any number a satisfying $a^2 - 2 = 0$?
Obviously, there is no rational number satisfying the above equation. How about any number

*the following descriptions are from wiki.

which is not a rational number?

❑ Observation

■Properties of rational number system $\mathbb{Q}$

For $a, b, c \in \mathbb{Q}$,

(i) Two operators on $\mathbb{Q}$: addition "+" and multiplication "×", and $\mathbb{Q}$ is closed under these two operators: $a + b \in \mathbb{Q}$ and $a \times b \in \mathbb{Q}$.

(ii) The commutative law unber "+" and "×": $a + b = b + a$ and $a \times b = b \times a$.

(iii) The associative law under "+" and "×": $(a+b)+c = a+(b+c)$ and $(a\times b)\times c = a\times(b\times c)$.

(iv) The distributive law: $a \times (b + c) = a \times b + a \times c$.

■ Some questions of real number system

Question: Does $\mathbb{R}$ have above properties? Does $\mathbb{R}$ have any hole?

Question: Is there a set of numbers which has above properties and $\mathbb{Q}$ is densely contained in this set?

We are interested in the "structure" of $\mathbb{R}$, rather than the "members" of $\mathbb{R}$. In Algebra, we emphasis on the "structure of a set" more than the "members of a set". In analysis, we take attention on what changes of functions rather than the values of functions.

Question: Do we really figure out the real number system well?

In high school algebra, we have learned some operations and computation of real numbers. For example,

$$\sqrt[3]{2} \times \sqrt[3]{3} = \sqrt[3]{6}. \tag{1.1}$$

Teachers told us that it is true. Is it really reasonable? The three symbols "$\sqrt[3]{2}$", "$\sqrt[3]{3}$" and "$\sqrt[3]{6}$" means the "numbers" which satisfy $x^3 - 2 = 0$, $x^3 - 3 = 0$ and $x^3 - 6 = 0$ respectively.

Question: Do these "numbers", "$\sqrt[3]{2}$", "$\sqrt[3]{3}$" and "$\sqrt[3]{6}$" really exist?
Question: If $\sqrt[3]{2}$ really exists, it should be between 1.25992 and 1.25993. Is the argument really true? (Ordered Field).
Question: What does the product of two irraional numbers mean?
Question: Why is the equation (1.1) true under the definition of multiplication? How to explain it?
Question: What is the multiplication on irrational number system? Why is the area of a rectangle equal to the product of length and width (if they are irrational)?
Question: Is the equation $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3-2-1$ true? (Commutative law, distribution law, associative law)

❑ <u>Students' Difficulties:</u>

To understand how to construct a real number system, a second-year college student may have some difficulties:

(i) Unfamiliar with set theory, field theory, number theory, convergence, equivalent classes and other abstrct language.

(ii) Abuse notation: students usually do not notice that the symbols "+" in $f+g$ and $f(x)+g(x)$ are different. But the symbols "×" in $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ and $2 \times 3 = 6$ are easily to distinguished.

(iii) We are too familiar with the computation on $\mathbb{R}$. It is difficult to forget that some facts (for example $1 < 2$) are not so trivial.

(iv) We should remind ourselves to focus on the structure of real number system rather than how to obtain the correct answers.

我們在對 $(\mathbb{Q}, +, \cdot, \leq)$ 有一定程度了解的情形下，試圖將「數系」描述清楚。已經發現許多的「數」並不屬於有理數，如 $\sqrt{n}, n \in \mathbb{N}$。是否將那些非有理數的「數」加入到有理數系，如 $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \cdots)$，即可成為所有的數系。

事實顯然並非如此，人們無法得已該加入哪些以及多少無理數才能形成整個數系。我們暫時先稱呼整個數系為 $\mathcal{F}$，$\mathbb{Q} \subset \mathcal{F}$，並期望上面帶有加法 $\oplus$ 與乘法 $\odot$，稱為 $(\mathcal{F}, \oplus, \odot)$。而其運算法則與有理數中的運算法則一致，仍有交換律、分配律及結合律。且當拿兩個有理數來做運算時，其結果當與原本在有理數中的運算一致。即

$$a + b = a \oplus b \quad \text{and} \quad a \cdot b = a \odot b \quad \text{for every } a, b \in \mathbb{Q}.$$

此外，有理數中有「大小」、「距離」，則 $\mathcal{F}$ 當定一「順序」以形成大小關係。我們還需解決實數系中有無「洞」的問題，此問題需透過等價於「Completeness」的設定來達成。

1.2 Ordered Fields and the Number Systems

❑ <u>Fields</u>

In order to prevent that students may abuse and misunderstand the familiar symbols "+" and "·", we temporarily use "⊕" and "⊙" to denote the two binary operations on fileds. After careful explaination and understanding, we will still use the usual symbols "+" and "·".

Definition 1.2.1. (First Version) A set $\mathcal{F}$ is said to be a "*field*" if there exist two binary operations $\oplus$ and $\odot$ such that

(a) (Closedness) For $x, y \in \mathcal{F}$, $x \oplus y \in \mathcal{F}$ and $x \odot y \in \mathcal{F}$.

(b) (Commutative law of addition) $x \oplus y = y \oplus x$ for all $x, y \in \mathcal{F}$.

(c) (Associative law of addition) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in \mathcal{F}$.

(d) There exists an element $e \in \mathcal{F}$ such that $x \oplus e = x$ for all $x \in \mathcal{F}$. ("*additive identity*")

(e) For every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $x \oplus y = e$. The element y is usually denoted by $-x$ and is called the additive inverse of x.

(f) $x \odot y = y \odot x$ for all $x, y \in \mathcal{F}$.(Commutative law of multiplication)

(g) $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in \mathcal{F}$. (Associative law of multiplication)

(h) There exists an element $i \in \mathcal{F}$ such that $x \odot i = x$ for all $x \in \mathcal{F}$. ("*multiplicative identity*")

(i) For every $x \in \mathcal{F}$ where x is not the additive identity($x \neq e$), there exists $y \in \mathcal{F}$ such that $x \odot y = i$. The element y is usually denoted by x^{-1} and is called the multiplicative inverse of x.

(j) $x \odot (y \oplus z) = \big(x \odot y\big) \oplus \big(x \odot z\big)$ for all $x, y, z \in \mathcal{F}$ (Distributive law)

(k) $e \neq i$.

We can easily observe that the rational number system $\mathbb{Q}$ with the two usual binary operations: addition "+" and multiplication "$\cdot$" satisfies all the above conditions. The additive identity is 0 and the multiplicative identity is 1. Hence, $(\mathbb{Q}, +, \cdot)$ is a filed.

If students have no misunderstanding with the two binary operations, from now on, we replace the notation $\oplus$ and $\odot$ by "+" and "$\cdot$" respectively. Also, the additive and multiplicative identities are denoted by "0" and "1" respectively. Therefore, we rewrite the defition of a filed as follows.

Definition 1.2.2. A set $\mathcal{F}$ is said to be a "*field*" if there exist two binary operations + and $\cdot$ such that

(a) (Closedness) For $x, y \in \mathcal{F}$, $x + y \in \mathcal{F}$ and $x \cdot y \in \mathcal{F}$.

(b) (Commutative law of addition) $x + y = y + x$ for all $x, y \in \mathcal{F}$.

(c) (Associative law of addition) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathcal{F}$.

(d) There exists an element $0 \in \mathcal{F}$ such that $x + 0 = x$ for all $x \in \mathcal{F}$. ("*additive identity*")

(e) For every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $x + y = 0$. The element y is usually denoted by $-x$ and is called the additive inverse of x.

(f) $x \cdot y = y \cdot x$ for all $x, y \in \mathcal{F}$. (Commutative law of multiplication)

(g) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathcal{F}$. (Associative law of multiplication)

(h) There exists an element $1 \in \mathcal{F}$ such that $x \cdot 1 = x$ for all $x \in \mathcal{F}$. ("*multiplicative identity*")

(i) For every $x \in \mathcal{F}$ where x is not the additive identity($x \neq 0$), there exists $y \in \mathcal{F}$ such that $x \cdot y = 1$. The element y is usually denoted by x^{-1} and is called the multiplicative inverse of x.

(j) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{F}$ (Distributive law)

(k) $0 \neq 1$.

Remark. The additive identity 0 and multiplicative identity 1 in a field $(\mathcal{F}, +, \cdot)$ are unique. If a set satisfies conditions (a)-(j) and $0 = 1$, then the contain only one element.

❑ Partially Ordered Sets

Definition 1.2.3. Let P be a set. A "*partial order*" over P is a binary relation $\preccurlyeq$ which is reflexive, anti-symmetric and transitive. That is,

(a) $x \preccurlyeq x$ for all $x \in P$ (reflexive).

(b) If $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$ (anti-symmetric).

(c) If $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ (transitive).

A set with a partial order is called a "*partially ordered set*" and is usually denoted by $(P, \preccurlyeq)$.

Example 1.2.4. Let S be a set and 2^S be the power set of S; that is,

$$P = 2^S = \{A \mid A \subset S\} = \text{the collection of all subsets of } S.$$

Consider the binary relation $\subseteq$. Then

(a) $A \subseteq A$ (reflexivity).

(b) If $A \subseteq B$ and $B \subseteq A$, then $A = B$ (anti-symmetry)

(c) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (transitivity).

Hence, $(P, \subseteq)$ is a partially ordered set.

Note that for a partially ordered set P, not any two elements in P have relation between them. For example, let $S = \{1, 2\}$ and $P = 2^S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. There is no inclusive relation between the two element $\{1\}$ and $\{2\}$.

Definition 1.2.5. Let $(P, \preccurlyeq)$ be a partially ordered set. Two elements $x, y \in P$ are said to be "*comparable*" if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Example 1.2.6. Let $S = \{1, 2\}$ and $P = 2^S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. In the partially ordered set $(P, \subseteq)$, $\{1\}$ and $\{1, 2\}$ are comparable. But, $\{1\}$ and $\{2\}$ are not comparable.

Definition 1.2.7. A partial order under which every pair of elements is comparable is called a "*total order*" or "*linear order*".

Definition 1.2.8. An "*ordered filed*" is a totally ordered $(\mathcal{F}, +, \cdot, \preccurlyeq)$ satisfying that

(a) If $x \preccurlyeq y$, then $x + z \preccurlyeq y + z$ for all $z \in \mathcal{F}$.

(b) If $0 \preccurlyeq x$ and $0 \preccurlyeq y$, then $0 \preccurlyeq x \cdot y$.

From now on, the total order $\preccurlyeq$ of an ordered field will be denoted by $\leq$.

Definition 1.2.9. In an ordered field $(\mathcal{F}, +, \cdot, \leq)$, the binary relations $<$, $\geq$ and $>$ are defined by

(a) $x < y$ if $x \le y$ and $x \ne y$.

(b) $x \ge y$ if $y \le x$.

(c) $x > y$ if $y < x$.

Definition 1.2.10. The "*magnitude*" or the "*absolute value*" of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note. There are several results mentioned in the lecture note. Students are suggested to read them by yourselves.

❑ The natural numbers, the integers and the rational numbers

■ **Preparation** 在開始建構實數系統前，我們希望學生們先有一些心理準備。

1. 我們所要建構的是一個與預期中的實數系具有相同「結構」的物件。例如: $\{1, 2, 3, 4, \cdots\}$、$\{2, 4, 6, 8, \cdots\}$ 或 $\{a,\ a+a,\ a+a+a,\ a+a+a+a, \cdots\}$ 從集合論的角度是不同的物件，但就代數結構來說它們是一樣的，都是由一個生成元經重覆迭代產生出的物件，因此應被視為是同一物件。
   我們要建立的實數系統是
   (i) 一個包含有自然數、整數、有理數結構在內的代數物件
   (ii) 有理數密集分布於該物件中。為了解釋何為密集，此處需考慮到「順序」(大小)關係，因此涉及到 "$\le$" 的關係.
   (iii) 物件中的每個成員彼此 comparable，且可以做加法與乘法的運算。
   (iv) 在有順序的結構下，我們希望此物件是沒有「洞」的存在。(有理數系統是有洞的)。此處涉及到實數的完備性、最大下界性質等。

2. 希望學生暫時忘記熟悉的自然數、整數、有理數系統。我們試圖重新由結構上定義自然數、整數及有理數系，以此為出發點考慮是否存在能滿足上述結構的「實數系」。

Definition 1.2.11. Let $(\mathcal{F}, +, \cdot, \le)$ be an ordered field.

(a) The "*natural number system*", denoted by $\mathbb{N}$, is the collection of all the number 1, $\underbrace{1+1}_{2}$, $\underbrace{1+1+1}_{3}$, $1+1+1+1, \cdots$. We write $2 \equiv 1+1$, $3 \equiv 1+1+1$, $\cdots$, $n \equiv \underbrace{1+1+\cdots+1}_{n}$. Therefore,

$\mathbb{N} = \{1, 2, 3, \cdots\}$.

Since $1, 2, 3, \cdots \in \mathcal{F}$, their additive inverses $-1, -2, -3, \cdots$ also in $\mathcal{F}$.

(b) The "*integer number system*", denoted by $\mathbb{Z}$, is the set $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

(c) For every non-zero element $0 \neq n \in \mathcal{F}$, the multiplicative inverse n^{-1} exists and is usually denoted by $\dfrac{1}{n}$. We also use $\dfrac{m}{n}$ to denote $m \cdot n^{-1}$. The "*rational number system*", denoted by $\mathbb{Q}$, is the collection of all numbers of the form $\dfrac{q}{p}$ with $p, q \in \mathbb{Z}$ and $p \neq 0$. That is,

$$\mathbb{Q} = \left\{ x \in \mathcal{F} \;\middle|\; x = \frac{q}{p},\ p, q \in \mathbb{Z} \text{ and } p \neq 0 \right\}.$$

Definition 1.2.12. An ordered field $(\mathcal{F}, +, \cdot, \leq)$ is said to have the "*Archimedean property*" if for every $x \in \mathcal{F}$, there exists $n \in \mathbb{Z}$ such that $x < n$.

Theorem 1.2.13. *$\mathbb{Q}$ has the Archimedean property.*

Proof. If $x \leq 0$, it is clear by choosing $n = 1$ since $x \leq 0 < 1$ (transitivity of $\leq$). If $0 < x = \frac{q}{p}$ with $p, q \in \mathbb{N}$, let $n = q + 1$. Then,

$$x = \frac{q}{p} \leq q < q + 1 = n.$$

The above relations $\leq$ and $<$ are from the hypotheses of ordered field and the fact $0 < 1$. □

Definition 1.2.14. A "*well-ordered*" relation on a set S is a total order on S with the property that every non-empty subset of S has a least (smallest) element in this ordering.

Theorem 1.2.15. *(Peano axiom)(Principle of mathematical induction)*
If S is a subset of $\mathbb{N} \cup \{0\}$ (or $\mathbb{N}$) such that $0 \in S$ (or $1 \in S$) and $k + 1 \in S$ if $k \in S$, then $S = \mathbb{N} \cup \{0\}$ (or $S = \mathbb{N}$).

Proposition 1.2.16. *If $S \subset \mathbb{N}$ and $S \neq \emptyset$, then S has a smallest element; that is there exists $s_0 \in S$ such that $s_0 \leq x$ for every $x \in S$.*

Proof. If $1 \in S$, then 1 is the smallest element in S.
Now, we consider that case that $1 \notin S$. Assume that S does not contain a smallest element. Define

$$T = \mathbb{N} \backslash S \quad \text{and} \quad T_0 = \left\{ n \;\middle|\; \{1, 2, 3, \cdots, n\} \in T \right\}.$$

Since $1 \notin S$, $1 \in T$ and $1 \in T_0$. For $k \in T_0$, be definition of T, $1, 2, \cdots, k \in T$. Therefore, $1, 2, \cdots, k \notin S$.

If $k + 1 \in S$, then $k + 1$ is the smallest element in S. It contradicts that assumption that S has no smallest element. Hence, $k+1 \notin S$ and then $k+1 \in T$. This implies that $1, 2, \cdots, k, k+1 \in T$. We have $k + 1 \in T_0$ by the definition of T_0.

By the Peano axiom, $T_0 = \mathbb{N}$. Then $T = \mathbb{N}$ and $S = \emptyset$. We obtain a contradiction. □

Proposition 1.2.17. *If $r_1, r_2 \in \mathbb{Q}$ and $r_1 < r_2$, then there exists $r \in \mathbb{Q}$ such that $r_1 < r < r_2$.*

❑ Sequence and Limits

Definition 1.2.18. A "*sequence*" in a set S is a function $f : \mathbb{N} \to S$. The values of f are called the "*terms*" of the sequence. We usually denoted a sequence by $\{f(n)\}_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$ with $x_n = f(n)$.

Definition 1.2.19. A sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field $(\mathcal{F}, +, \cdot, \leq)$ is said to "*converge*" to a limit $x \in \mathcal{F}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \quad \text{whenever } n \geq N.$$

Denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Lemma 1.2.20. *(Sandwich) If $\lim_{n\to\infty} x_n = L$, $\lim_{n\to\infty} y_n = L$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence such that $x_n \leq z_n \leq y_n$, then*

$$\lim_{n\to\infty} z_n = L.$$

Proposition 1.2.21. *If $a \leq x_n \leq b$ and $\lim_{n\to\infty} x_n = x$, then $a \leq x \leq b$.*

Proposition 1.2.22. *(Uniqueness of Limit) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in an ordered field, and $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$*

Definition 1.2.23. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in an ordered field $\mathcal{F}$.

(a) $\{x_n\}_{n=1}^{\infty}$ is said to be "*bounded*" if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

(b) $\{x_n\}_{n=1}^{\infty}$ is said to be "*bounded from above*" if there exists $B \in \mathcal{F}$, called an "*upper bound*" of the sequence, such that $x_n \leq B$ for all $n \in \mathbb{N}$.

(c) $\{x_n\}_{n=1}^{\infty}$ is said to be "*bounded from below*" if there exists $A \in \mathcal{F}$, called an "*lower bound*" of the sequence, such that $A \leq x_n$ for all $n \in \mathbb{N}$.

Proposition 1.2.24. *A convergent sequence is bounded.*

❏ Monotone Sequence Property

Definition 1.2.25. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in an ordered field $\mathcal{F}$. We say that

(a) $\{x_n\}_{n=1}^{\infty}$ is "*increasing*" (or "*nondecreasing*") if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is said "*strictly increasing*" if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

(b) $\{x_n\}_{n=1}^{\infty}$ is "*decreasing*" (or "*nonincreasing*") if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. It is said "*strictly decreasing*" if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

(c) a sequence is called (strictly) "*monotone*" if it is either (strictly) increasing or (strictly) decreasing.

Definition 1.2.26. An ordered field $\mathcal{F}$ is said to satisfy the "*(strictly) monotone sequence property*" if every bounded (strictly) monotone sequence converges to a limit in $\mathcal{F}$.

Remark. An equivalent definition of the monotone sequence property is that every monotone increasing sequence bounded above converges; that is, if each sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ satisfying

(i) $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$,

(ii) there exists $M \in \mathcal{F}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$.

is convergent, then we say $\mathcal{F}$ satisfies the monotone sequence property.

Example 1.2.27. $(\mathbb{Q}, +, \cdot, \leq)$ is an ordered field. But it does not satisfy the monotone sequence property.

Theorem 1.2.28. *An ordered field satisfying the monotone sequence property has the Archimedean property; that is, if $\mathcal{F}$ is an ordered field satisfying the monotone sequence property, then for all $x \in \mathcal{F}$, there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. Assume that there exists an ordered field $(\mathcal{F}, +, \cdot, \leq)$ and $x \in \mathcal{F}$ such that $x \geq n$ for all $n \in \mathbb{N}$.

Let $x_n = n$, then the sequence $\{x_n\}$ is increasing and x is an upper bound of $\{x_n\}$. By the monotone sequence property, there exists $y \in \mathcal{F}$ such that $\lim\limits_{n\to\infty} x_n = y$. Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|n - y| = |x_n - y| < \frac{1}{4}.$$

We have

$$N + 1 < y + \frac{1}{4} = y - \frac{1}{4} + \frac{1}{2} < N + \frac{1}{2}.$$

Thus, we obtain a contradiction. □

❑ Completeness

Definition 1.2.29. An ordered field $\mathcal{F}$ is said to be "**complete**" if it satisfies the monotone sequence property.

Remark. Let $\mathcal{F}$ be an ordered field, the following statments are equivalent.

(a) $\mathcal{F}$ is complete.

(b) $\mathcal{F}$ has the monotone sequence property.

(c) $\mathcal{F}$ has the least upper bound property.

Theorem 1.2.30. *There is a "unique" complete ordered field, called the "real number system $\mathbb{R}$".*[†]

1.3 Construction of Real Number System

In this section, we introduce two methods to construct real number system which were established by Dedekind and Cantor. The ingredients of these two methods are similar by constructing an extension of rational number system.

[†]$\mathbb{R}$ is defined by an ordered field with least upper bound property. (Rudin)

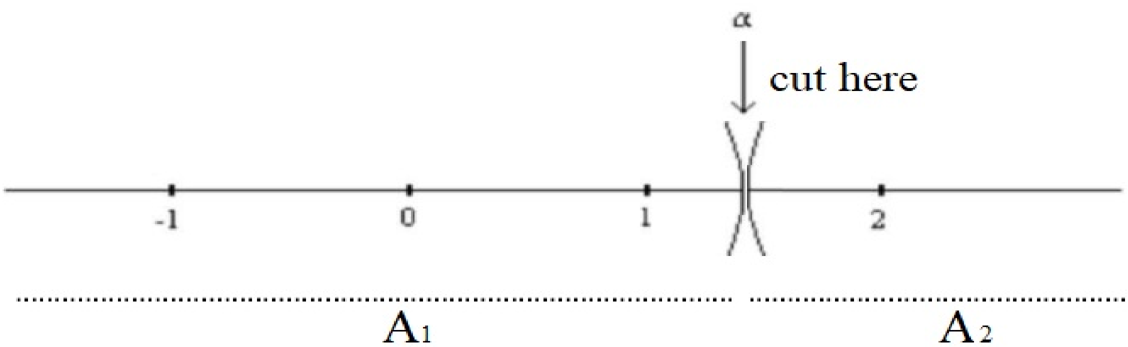
So far, we have known that $(\mathbb{Q}, +, \cdot, \leq)$ is an ordered field but has no monotone sequence property.

我們想要了解數線上的所有點及其結構。目前知道的是有理數是有賦序體，~~(且密集的分布在整個~~
首先，我們要為數線上每個點命名，給予一個適當的「符號」，此符號需用已知的有理數來構造，且對於每個點是獨一無二的一一對應。

❑ <u>Dedekind Cut</u>

■ **<u>Heuristical idea:</u>** For a point (temporarily called α) in the real line, it separates $\mathbb{Q}$ into two nonempty parts

$$A_1 = \{x \in \mathbb{Q} \mid x < \alpha\} \quad \text{and} \quad A_2 = \{x \in \mathbb{Q} \mid x > \alpha\}.$$



If $\alpha \in \mathbb{Q}$, we can put α in any one of A_1 or A_2. Then we can name the cut α as (A_1, A_2). Note that

(i) $A_1, A_2 \neq \emptyset$,

(ii) $A_1 \cup A_2 = \mathbb{Q}$,

(iii) For $x \in A_1$ and $y \in A_2$, $x < y$. (any two numbers in $\mathbb{Q}$ are comparable)

If $\alpha \in \mathbb{Q}$, we also use the same notation (A_1, A_2) to name the cut α. Note that we cannot give an explicit definition to A_1 and A_2 since $\alpha \notin \mathbb{Q}$. Hueristically, we know that as long as we separate $\mathbb{Q}$ into two nonempty parts as above, every separation would be corresponding to a unique point in the number line. Hence, we can use such notation (A_1, A_2) satisfying $(i), (ii), (iii)$ to name every point in the number line.

There are four situations:

(1) A_1 contains a maximum and A_2 contains a minimum. (Impossible!)

(2) A_1 contains a maximum and A_2 contains no minimum. (rational number)

(3) A_1 contains no maximum and A_2 contains a minimum. (rational number)

(4) A_1 contains no maximum and A_2 contains no minimum. (irrational number)

Define

$$\mathbb{R} = \{(A_1, A_2) \mid \text{all possible } A_1 \text{ and } A_2 \text{ satisfying } (i), (ii), (iii)\}.$$

Now, we have to define binary operations "+" and "·" as well as additive identity"0", multiplicative identity "1", additive inverse, multiplicative inverse and an "order" on $\mathbb{R}$. We give their defintions here and suggent students check that they are well-defined. Be careful that the operations + and · can only apply on rational numbers. We should use them to establish new opertaions $\oplus$ and $\odot$ which can apply on $\mathbb{R}$.

1. (Addition "$\oplus$"): For $(A_1, A_2), (B_1, B_2) \in \mathbb{R}$, define $(A_1, A_2) \oplus (B_1, B_2) = (C_1, C_2)$ where $C_1 = \{a + b \mid a \in A_1, b \in B_1\}$ and $C_2 = \mathbb{Q} \backslash C_1$.

2. (Additive identity): $\mathbf{0} = (A_1, A_2)$ where $A_1 = \{a \in \mathbb{Q} \mid a \leq 0\}$ and $A_2 = \{a \in \mathbb{Q} \mid a > 0\}$.

3. (Additive inverse): $-(A_1, A_2) = (B_1, B_2)$ where $B_1 = \{b \in \mathbb{Q} \mid -b \in A_2\}$ and $B_1 = \mathbb{Q} \backslash B_2$.

4. (Multiplication "$\odot$"): For $(A_1, A_2), (B_1, B_2) \in \mathbb{R}$, define $(A_1, A_2) \odot (B_1, B_2) = (C_1, C_2)$ where C_2 is defined below and $C_1 = \mathbb{Q} \backslash C_1$.

   i. if $0 \in A_1$ and $0 \in B_1$, then $C_2 = \{a \cdot b \mid a \in A_2, b \in B_2\}$.
   ii. if $0 \in A_1$ and $0 \notin B_1$, then $(C_1, C_2) = -\big[(A_1, A_2) \odot [-(B_1, B_2)]\big]$
   iii. if $0 \notin A_1$ and $0 \in B_1$, then $(C_1, C_2) = -\big[[-(A_1, A_2)] \odot (B_1, B_2)\big]$
   iv. if $0 \notin A_1$ and $0 \notin B_1$, then $(C_1, C_2) = [-(A_1, A_2)] \odot [-(B_1, B_2)]$

5. (Multiplicative identity): $\mathbf{1} = (A_1, A_2)$ where $A_1 = \{a \in \mathbb{Q} \mid a \leq 1\}$ and $A_2 = \{a \in \mathbb{Q} \mid a > 1\}$.

6. (Multiplicative inverse): For $0 \neq (A_1, A_2)$, define $(A_1, A_2)^{-1} = (B_1, B_2)$ where

   i. if $0 \in A_1$, then $B_1 = \{b \in \mathbb{Q} \mid b^{-1} \in A_2\} \cup \mathbb{Q}^-$
   ii. if $0 \notin A_1$, then $B_1 = \{b \in \mathbb{Q} \mid b^{-1} \in A_2\} \cap \mathbb{Q}^-$.

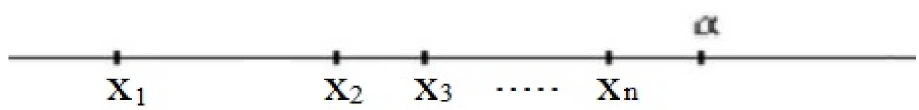
7. (Order relation): Define $(A_1, A_2) \preccurlyeq (B_1, B_2)$ if $B_2 \subseteq A_2$.

Theorem 1.3.1. $(\mathbb{R}, \oplus, \odot, \preccurlyeq)$ *is an ordered field and has Monotone Sequence Property.*

Proof. Skip □

❑ Cantor's Construction (Sketch)

Heuristically, for a point (called α) in the number line, we use an increasing sequence $\{x_n\}$ of rational numbers which converges to α. Then we use this sequence $\{x_n\}$ to name the point α.



■ **Problems:**

(1) The method to denote a point in number line is not well-defined. There has infinitely many increasing sequences which convege to a single point. Hence, every point is named by infinitely many sequences.

(2) If $\alpha \in \mathbb{Q}$, the convergence is easy. If $\alpha \notin \mathbb{Q}$, what is the convergence?

■ **Definition of $\mathbb{R}$**

Let S be the collection of all increasing and bounded above sequences of rational numbers

$$S = \left\{ \{x_n\}_{n=1}^{\infty} \;\middle|\; x_n \in \mathbb{Q} \text{ and } \{x_n\} \text{ is increasing and bounded above.} \right\}.$$

Define an equivalence relation $\sim$ on S that $\{x_n\} \sim \{y_n\}$ if the set of all rational upper bounds of $\{x_n\}$ is equal to the set of all rational upper bounds of $\{y_n\}$. Denote every equivalent class by $\left[\{x_n\}\right]$.
(概念上是數列遞增到同一點視為是同的等價類，定義上不能這麼設計，因為不知道什麼是無理數。因此改用具有相同的有理數上界集合來定義同一等價類的遞增數列。)

Define $\mathbb{R}$ as the collection of all equivalent classes under the relation $\sim$.

$$\mathbb{R} = S \big/ \sim = \left\{ \left[\{x_n\}\right] \;\middle|\; \{x_n\} \in S \right\}.$$

■ **Definition of binary operations, identities, inverses and order relation**

We will skip the details of those objects. Students can find them in the lecture note.

1. (Binary Operations and order relation): Define "$\oplus$", "$\odot$", and "$\preccurlyeq$" on $\mathbb{R}$
2. (Identities):
   - Define [0] by the class $\left[\{x_n\}\right]$ with the set of upper bound $\{q \in \mathbb{Q} \mid q \geq 0\}$.
   - Define [1] by the class $\left[\{x_n\}\right]$ with the set of upper bound $\{q \in \mathbb{Q} \mid q \geq 1\}$.

   Check that [0] is an additive identity on $\mathbb{R}$ and [1] is a multiplicative identity on $\mathbb{R}$.
3. (Inverses): Definie the additive inverse of $\left[\{x_n\}\right]$ and the multiplicative inverse of $\left[\{x_n\}\right]$ for $\left[\{x_n\}\right] \neq [0]$.

■ **Check**

We should check the following arguments.

1. Check that $(\mathbb{R}, \oplus, \odot, \preccurlyeq)$ forms a ordered field.
2. Check that the element [1] forms subsets in $\mathbb{R}$ by the following the steps of the construction of natural numbers, integers, rational numbers. We denote them by $\mathbb{N}^*$, $\mathbb{Z}^*$ and $\mathbb{Q}^*$ respectively.
3. Check that $(\mathbb{Q}^*, \oplus, \odot, \preccurlyeq)$ is isomorphic to $(\mathbb{Q}, +, \cdot, \leq)$. Hence, they have the same algebraic structure.

4. Check that $(\mathbb{R}, \oplus, \odot, \preccurlyeq)$ has Monotone Sequence Property.
5. Check that all ordered field with Monotone Sequence Property have the same structure. This implies that there has a unique complete ordered field.

❑ Some conclusions of $\mathbb{R}$

Definition 1.3.2. $\mathbb{R}$ is defined by an ordered field with monotone sequence property. There are some equivalent statements as follows.

1. $\mathbb{R}$ is defined by a complete ordered field; or
2. $\mathbb{R}$ is defined by an ordered field with the least upper bound preperty.

Remark. (1) A complete ordered field is unique under isomorphism.

(2) $\mathbb{R}$ has the Archimedean propety. That is, for all $r \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $r < n$.

❑ Density of $\mathbb{Q}$

Question: What is the distribution of $\mathbb{Q}$ in $\mathbb{R}$? What is the role of $\mathbb{Q}$ with respect to $\mathbb{R}$?

Definition 1.3.3. Let $S \subseteq T \subseteq \mathbb{R}$. We say that S is dense in T if for every $t \in T$ and $\varepsilon > 0$, there exists $s \in S$ such that

$$|s - t| < \varepsilon.$$

Remark. $\mathbb{N}$ and $\mathbb{Z}$ are not dense in $\mathbb{R}$ or in $\mathbb{Q}$.

Proposition 1.3.4. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Given $r \in \mathbb{R}$ and $\varepsilon > 0$. Since $\mathbb{R}$ has the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$ (or $\frac{1}{N} < \varepsilon$).

Claim: $\left\{ \frac{k}{N} \mid k \in \mathbb{Z} \right\} \cap (r - \varepsilon, r + \varepsilon) \neq \emptyset$.

Proof of Claim: If the claim is false, there exists $\ell \in \mathbb{Z}$ such that $\frac{\ell}{N} < r - \varepsilon$ and $\frac{\ell + 1}{N} > r + \varepsilon$. Then

$$\frac{1}{N} = \frac{\ell + 1}{N} - \frac{\ell}{N} > (r + \varepsilon) - (r - \varepsilon) = 2\varepsilon. \quad (Contradition!)$$

Hence, there exists $k_0 \in \mathbb{Z}$ such that $\left| \frac{k_0}{N} - r \right| < \varepsilon$. □

Remark. An equivalent statement of Proposition 1.3.4 is that if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.

❑ Appendix

In the end of this section, we review the convergence of sequence in $\mathbb{R}$ and give the definition of extended real number system.

■ Review the convergence and divergence

Definition 1.3.5. (1) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $\{x\}_{n=1}^{\infty}$ converges if there is $x \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - x| < \varepsilon.$$

We say that $\{x_n\}_{n=1}^{\infty}$ "*converges to x*" and denote $\lim_{n\to\infty} x_n = x$.

(2) If $\{x_n\}$ does not converge, we say that $\{x_n\}_{n=1}^{\infty}$ "*diverges*".

(3) We say that $\{x_n\}_{n=1}^{\infty}$ "*diverges to infintie*" if for every $M > 0$, there is $N = N(M) \in \mathbb{N}$ such that for all $n \geq N$,

$$x_n > M.$$

Denoted by $\lim_{n\to\infty} x_n = \infty$. Also, $\{x_n\}_{n=1}^{\infty}$ is said to "*diverge to* $-\infty$" if $\{-x_n\}_{n=1}^{\infty}$ diverges to ∞ and to be denoted by $\lim_{n\to\infty} x_n = -\infty$.

■ **Extended real number system**

Definition 1.3.6. The extended real number system, denoted by $\mathbb{R}^*$, is define by

$$\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

1.4 Countability

Heuristically, let $f : A \to B$ be a function. If f is one-to-one and onto, then the "size" of A is equal to the "size" of B. Hence, if we want to compare the sizes of two sets, a reasonable method is to consider whether we can establish an one-to-one correspondence from A to B.

Definition 1.4.1. Let A and B be two sets.

(1) We say that A can be put into 1-1 correspondence with B if and only if there exists a 1-1 and onto map f from A to B.
A and B are called "*equinumerous*" and denoted by $A \sim B$.[‡]

(2) We say that A is "*denumerable*" or "*countably infinite*" if A can be put into 1-1 correspondence with $\mathbb{N}$. That is, there exists a map $f : \mathbb{N} \to A$ which is 1-1 and onto.

(3) A set is called "*countable*" if it is either finite or countably infinite.

Example 1.4.2. (1) $\mathbb{N}$ is countable. Define $f(x) = x$ on $\mathbb{N}$. Then f is 1-1 and onto.

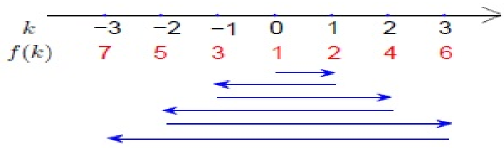
(2) $\mathbb{N}\backslash\{1\} = \{2, 3, 4, \cdots\}$ is countable. Define $f : \mathbb{N}\backslash\{1\} \to \mathbb{N}$ by $f(x) = x - 1$.

(3) $\mathbb{Z}$ is countable. Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x \in \mathbb{Z}^+ \\ -2x + 1 & \text{if } x \in \mathbb{Z}^-. \end{cases}$

Therefore, f is 1-1 and onto. Hence, $\mathbb{Z}$ is countable.

(4) The set $\mathbb{N} \times \mathbb{N} = \{(a, b) \mid a, b \in \mathbb{N}\}$ is countable. (Exercise)

[‡] A and B have the same cardinality.



Theorem 1.4.3. *(1) Any nonemtpy subset of* $\mathbb{N}$ *is countable.*

(2) Any nonempty subset of a countable set is countable.

Proof. (1) Let S be a nonempty subset of $\mathbb{N}$. If S is finite, then S is countable. Hence, we may assume that S is infinite.

Since $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, S contains a smallest element, say a_1.

Let $S_1 := S\backslash\{a_1\}$. Since the number of size of S is infinite, S_1 is also nonempty and its size is also infinite. Then S_1 contains a smallest element, say a_2.

Again, let $S_2 := S_1\backslash\{a_2\} = S\backslash\{a_1, a_2\}$. It is nonempty and its size is also infinite, and hence S_2 contains a smallest element, say a_3. Continue this process, we can choose $a_1, a_2, a_3, \cdots \in S$ with $a_1 < a_2 < a_3 < \cdots$ and $S_k = S\backslash\{a_1, a_2, \cdots, a_k\}$.

<u>*Claim:*</u> $S = \{a_1, a_2, a_3, \cdots\}$.
Proof of Claim: Clearly, $\{a_1, a_2, a_3, \cdots\} \subseteq S$. Assume that there is a number $p \in S\backslash\{a_1, a_2, \cdots\}$. Then there exists $k \in \mathbb{N}$ such that $a_k < p < a_{k+1}$. Hence, p is the smallest number of $S\backslash\{a_1, a_2, \cdots, a_k\}$.

By the choice of a_i, $p = a_{k+1} \in \{a_1, a_2, \cdots\}$. It contradicts the assumption and hence $S \subseteq \{a_1, a_2, \cdots\}$. Then $S = \{a_1, a_2, \cdots\}$ and the claim is proved.

Define $f : \mathbb{N} \to S$ by $f(n) = a_n$. Then f is 1-1 and onto. Thus, S is countable.

(2) (Exercise)

□

Corollary 1.4.4. *A nonempty set S is countable if and only if there exists an injection (*1-1 *function)* $f : S \to \mathbb{N}$.

Proof. (⇒) If S is finite, say $S = \{a_1, a_2, \cdots, a_n\}$. we define $f(a_k) = k$. Clearly, f is an injection. If S is countably infinite, by definition, there exists a bijection $f : S \to \mathbb{N}$.

(⇐) We may assume that S is infinit. Let $f : S \to N$ be an injection. Then $f(S) \subseteq \mathbb{N}$ is countable. Hence, there exists an 1-1 and onto function $g : f(S) \to \mathbb{N}$.

Define $h := g \circ f : S \to \mathbb{N}$ is 1-1 and ont and S is countable.

$$S \underset{1-1,\text{ onto}}{\overset{f}{\longrightarrow}} f(S) \underset{1-1,\text{ onto}}{\overset{g}{\longrightarrow}} \mathbb{N}.$$

□

Remark. If we want to prove the countability of a set S, this corollary says that we only need to find a 1-1 function $f : S \to \mathbb{N}$. The function is not necessary to be surjective.

Theorem 1.4.5. *Let S_1 and S_2 be two countable sets. Then $S_1 \cup S_2$ is countable.*

Proof. (Exercise) □

Corollary 1.4.6. *The union of finite countable sets is countable. That is, if $S_1, S_2, \cdots, S_n$ are countable, then $\bigcup_{k=1}^{n} S_k$ is countable.*

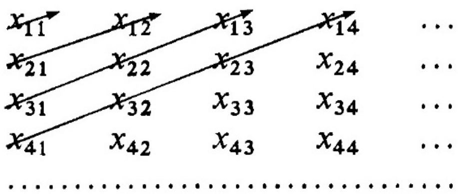
Proof. (Exercise) □

Theorem 1.4.7. *If $S_1, S_2, S_3, \cdots$ are countable sets, then $\bigcup_{k=1}^{\infty} S_k$ is countable.*

Proof. Since S_k is countable for every $k = 1, 2, 3, \cdots$, we may write $S_k = \{x_{k1}, x_{k2}, x_{k3}, \cdots\}$ for $k = 1, 2, 3 \cdots$. Then

$$\bigcup_{k=1}^{\infty} S_k = \left\{ \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{44} & x_{25} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{44} & x_{35} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\}.$$

Define $f(x_{nm}) = \frac{[1+(n+m-2)](n+m-2)}{2} + m = \frac{(n+m-1)(n+m-2)}{2} + m$. Then f is a injection and hence $\bigcup_{k=1}^{\infty} S_k$ is countable by Corollary 1.4.4.



Check that f is 1-1. If $f(x_{n_1 m_1}) = f(x_{n_2 m_2})$, then

$$\frac{(n_1 + m_1 - 1)(n_1 + m_1 - 2)}{2} + m_1 = \frac{(n_2 + m_2 - 1)(n_2 + m_2 - 2)}{2} + m_2.$$

It is easy to check that $m_1 = m_2$ and $n_1 = n_2$.

□

Corollary 1.4.8. *(1) $\mathbb{Z}$ is countable.*

(2) $\mathbb{Q}$ is countable.

Proof. (1) $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.

(2) Let $p \in \mathbb{N}$. Define $\mathbb{Q}_p := \left\{ \frac{q}{p} \mid q \in \mathbb{Z} \right\}$. Then $\mathbb{Q}_p$ is countable for all $p \in \mathbb{N}$. Moreover, $\mathbb{Q} = \bigcup_{p \in \mathbb{N}} \mathbb{Q}_p$ is an union of countable family of countable sets. Hence, $\mathbb{Q}$ is countable.

(**Method 2:**) Since $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$, it sufficies to show that $\mathbb{Q}^+$ is countable. Let $r \in \mathbb{Q}^+$, then $r = \frac{q}{p}$ for some $p, q \in \mathbb{N}$ and $g.c.d(p, q) = 1$. Define $f(r) = 2^p 3^q$. Then $f : \mathbb{Q}^+ \to \mathbb{N}$ is 1-1. By Corollary 1.4.4, $\mathbb{Q}^+$ is countable. This implies that $\mathbb{Q}$ is also countable.

□

Theorem 1.4.9. *Let A and B be countable, then $A \times B$ is countable.*

Proof. Since A and B are countable, say $A = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$, we have

$$A \times B = \big\{(a_n, b_m) \mid n, m \in \mathbb{N}\big\}.$$

Define $f : A \times B \to \mathbb{N}$ by $f\big((a_n, b_m)\big) = 2^n 3^m$. Then f is 1-1 and hence $A \times B$ is countable.

□

■ **$\mathbb{R}$ is uncountable**

Theorem 1.4.10. *$\mathbb{R}$ is uncountable.*

Proof. It suffices to prove thta the interval $(0, 1)$ is uncountable.

Assume that $(0, 1)$ is countable. Then we may arrange the numbers of $(0, 1)$ in a sequence $x_1, x_2, x_3, \cdots, x_n, \cdots$. Since $0 < x_n < 1$ for $n = 1, 2, 3, \cdots$, every x_n has a unique decimal expansion.

$$\begin{aligned} x_1 &= 0.d_{11}\, d_{12}\, d_{13}\, d_{14} \cdots \\ x_2 &= 0.d_{21}\, d_{22}\, d_{23}\, d_{24} \cdots \\ x_3 &= 0.d_{31}\, d_{32}\, d_{33}\, d_{34} \cdots \\ x_4 &= 0.d_{41}\, d_{42}\, d_{43}\, d_{44} \cdots \\ &\vdots \end{aligned}$$

where $d_{ij} \in \{0, 1, 2, \cdots, 9\}$ for all $i, j \in \mathbb{N}$.

We use the diagonal terms to find a number $x \in (0, 1)$ by the following way. Choose

$$x = 0.d_1\, d_2\, d_3\, d_4 \cdots \quad \text{where } d_k = \begin{cases} 1 & \text{if } d_{kk} \neq 1 \\ 2 & \text{if } d_{kk} = 1 \end{cases} \quad \text{for all } k \in \mathbb{N}.$$

Then $d_k \neq d_{kk}$ for all $k \in \mathbb{N}$ and $x \in (0, 1)$. But $x \neq x_n$ for every $n \in \mathbb{N}$. This says that there exists a number $x \in (0, 1)$ which is not counted and we obtain a contradition.

□

Corollary 1.4.11. *(1) $\mathbb{R} \backslash \mathbb{Q}$ is uncountable.*

(2) If A is uncountable. Let $B \subset A$ be countable. Then $A \backslash B$ is uncountable.

Proof. (Exercise) □

Remark. Not all uncountable sets have the same cardinality as $\mathbb{R}$. Let $S \neq \emptyset$. The power set of S is the set of all subsets of S, usually denoted by $\mathcal{P}(S)$ or 2^S. The fact is that the cardinalities of S and 2^S are not the same. (There is no 1-1 correspondence between S and 2^S.)

1.5 Least Upper Bounds and Greatest Lower Bounds

Definition 1.5.1. Let $\emptyset \neq S \subseteq \mathbb{R}$.

(1) A number $M \in \mathbb{R}$ is said to be an "*upper bound for S*" if $x \leq M$ for all $x \in S$. We say that S is "*bounded from above*".

(2) A number $m \in \mathbb{R}$ is said to be a "*lower bound for S*" if $x \geq m$ for all $x \in S$. We say that S is "*bounded from below*".

(3) S is said to be bounded if S is both bounded from above and from below.

(4) A number $b \in \mathbb{R}$ is called a "*least upper bound for S*" if

  (i) b is an upper bound for S, and
  (ii) if M is an upper bound for S, then $M \geq b$.

(5) A number $a \in \mathbb{R}$ is called a "*greatest lower bound for S*" if

  (i) a is a lower bound for S, and
  (ii) if m is a lower bound for S, then $m \leq a$.

Notation: We denote

(1) the least upper bound for S by "$\sup S$", called "*supremum of S*";

(2) the greatest lower bound for S by "$\inf S$", called "*infremum of S*".

Remark. Let $S \subseteq \mathbb{R}$ be a set.

(1) If S is not bounded above, then $\sup S = \infty$ and if S is not bounded below, then $\inf S = -\infty$.

(2) Suppose that $b = \sup S < \infty$ if and only if

  (i) $b \geq x$ for all $x \in S$;
  (ii) For any $\varepsilon > 0$, there exists $x \in S$ such that $x > b - \varepsilon$.

(3) $\sup S$ or $\inf S$ need not to be a member of S.

(4) If $\emptyset \neq A \subseteq B$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

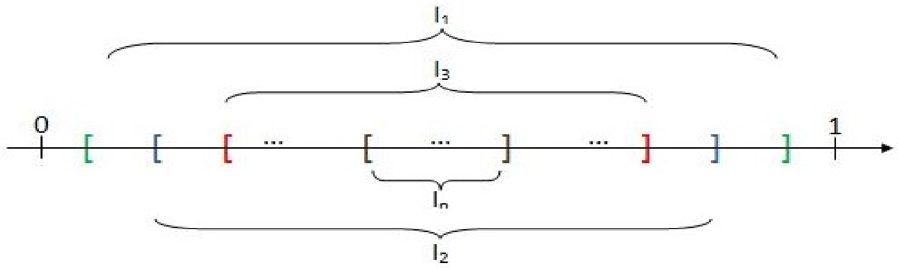
(5) Since $\emptyset$ is a subset of any set, we define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Proposition 1.5.2. *Suppose that* $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$. *Then* $\inf B \leq \inf A \leq \sup A \leq \sup B$.

Proof. (Exercise) □

Definition 1.5.3. (a) We say that $I \subseteq \mathbb{R}$ is an interval if for every $a, b \in I$ and $a < x < b$ then $x \in I$.

(b) The interval $(a, b) = \{x \mid a < x < b\}$ is called "*an open interval*" in $\mathbb{R}$ and $[a, b] = \{x \mid a \leq x \leq b\}$ is called "*an closed interval*" in $\mathbb{R}$.



Theorem 1.5.4. *(Nested interval theorem) Suppose that* $I_n = \{x \mid a_n \le x \le b_n\}$ *is a sequence of closed intervals such that* $I_{n+1} \subseteq I_n$ *for* $n = 1, 2, \cdots$. *If* $\lim_{n\to\infty}(b_n - a_n) = 0$ *then there exists one and only one number* x_0 *which is in every* I_n.

Proof. If $a_{n_0} = b_{n_0}$ for some $n_0 \in \mathbb{N}$, then $a_n = b_n$ for all $n \ge n_0$ since $I_{n+1} \subseteq I_n$. We have $I_n = \{a_{n_0}\}$ for all $n \ge n_0$. Let $x_0 = a_{n_0}$ and the theorem is proved.

We may assume that $a_n < b_n$ for all $n \in \mathbb{N}$. Since $I_{n+1} \subseteq I_n$, we have $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}$ is an increasing sequence which is bounded above by b_1. By Monotone Sequence Property, there exists x_0 such that $a_n \nearrow x_0$ as $n \to \infty$. Also, $\{b_n\}$ is a decreasing sequence. By Monotone Sequence Property again, there exists y_0 such that $b_n \searrow y_0$ as $n \to \infty$.

Since $0 = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = y_0 - x_0$, we have $x_0 = y_0$. Moreover, $a_n \le x_0 = y_0 \le b_n$ for all $n \in \mathbb{N}$. We have $x_0 \in I_n$ for all $n \in \mathbb{N}$.

Now, to prove that this point is unique. Assume that there exists $x_1 \in I_n$ for all $n \in \mathbb{N}$ and $x_0 \ne x_1$. Let $\varepsilon = |x_0 - x_1|$. There exist $N_0 \in \mathbb{N}$ such that

$$a_n > x_0 - \frac{\varepsilon}{2} \quad \text{and} \quad b_n < x_0 + \frac{\varepsilon}{2}$$

for all $n \ge N_0$. Then, either $x_1 < a_n$ or $x_1 > b_n$ for all $n \ge N_0$. It contradicts the hypothesis $x_1 \in I_{N_0}$. Therefore, $x_0 = x_1$ and the point is unique. □

Question: Is the closedness of the intervals in the nested intervals theorem necessary?(Exercise)

Theorem 1.5.5. *(1) (Least Upper Bound Property) If a set* $\emptyset \ne S \subset \mathbb{R}$ *has an upper bound, then it has a least upper bound.*

(2) (Greatest Lower Bound Property) If a set $\emptyset \ne S \subset \mathbb{R}$ *has a lower bound, then it has a greatest lower bound.*

Proof. (1) Let M be an upper bound for S. If $M \in S$, then M itself is the least upper bound for S. Hence, we may assume that $M \notin S$.

Let $b_1 = M$ and choose a_1 as any point in S. Consider $\frac{a_1 + b_1}{2}$.

(i) If $\frac{a_1 + b_1}{2}$ is greater than every point in S. We define $a_2 = a_1$ and $b_2 = \frac{a_1 + b_1}{2}$.

(ii) If there exists a point in S which is greater than $\dfrac{a_1+b_1}{2}$, we define $b_2 = b_1$ and choose a point $a_2 \in S$ and $a_2 \geq \dfrac{a_1+b_1}{2}$.

In both cases, $a_1, a_2 \in S$, and b_1 and b_2 are upper bounds for S. Also, we have $[a_2, b_2] \subset [a_1, b_1]$ and $b_2 - a_2 \leq \dfrac{b_1 - a_1}{2}$.

Continue this procedure, we can choose sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \in S$ and $b_n \notin S$ for every $n \in \mathbb{N}$ and

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a_2, b_2] \subset [a_1, b_1]$$

and

$$b_{n+1} - a_{n+1} \leq \frac{b_n - a_n}{2} \leq \frac{b_{n-1} - a_{n-1}}{2^2} \leq \cdots \leq \frac{b_1 - a_1}{2^n}.$$

By the nested interval theorem, there exists a unique $x_0 \in [a_n, b_n]$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0$.

(To prove that x_0 is an upper bound for S.) For $x \in S$, we have $x \leq b_n$ for every $n \in \mathbb{N}$. Therefore, $x \leq \lim_{n\to\infty} b_n = x_0$. This implies that x_0 is an upper bound for S.

(To prove that x_0 is the least upper bound for S.) Assume that x_0 is not the least upper bound for S. There exists y_0 which is an upper bound for S and $y_0 < x_0$. Let $\varepsilon = \dfrac{x_0 - y_0}{2}$. Then $y_0 < x_0 - \varepsilon < x_0$.

Since $a_n \leq x_0$ and $\lim_{n\to\infty} a_n = x_0$, there exists $N \in \mathbb{N}$ such that $|a_n - x_0| < \varepsilon$ for every $n \geq N$. Thus, $a_n > x_0 - \varepsilon > y_0$ for every $n \geq N$. It contradicts the assumption that y_0 is an upper bound for S.

(2) The second statement follows by the first one to the set $S' = \{-x \mid x \in S\}$.

□

Theorem 1.5.6. *Let $(\mathcal{F}, +, \cdot, \leq)$ be an order field with the least upper bound property. (That is, if $\emptyset \neq S \subseteq \mathcal{F}$ has an upper bound, then it has a least upper bound). Then $\mathcal{F}$ is complete.*

Proof. It suffices to prove that $\mathcal{F}$ has monotone sequence property. Let $\{x_n\}_{n=1}^{\infty}$ be an increasing sequence with an upper bound M. Then the set $S = \{x_1, x_2, \cdots, x_n, \cdots\}$ is bounded above by M.

Since $\mathcal{F}$ has least upper bound property, there exists a least upper bound for S, say s. Hence, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > s - \varepsilon$.

Since $\{x_n\}$ is an increasing sequence, $s - \varepsilon < x_{n_0} \leq x_n \leq s$ for every $n \geq n_0$. This implies that $\lim_{n\to\infty} x_n = s$. Since $\{x_n\}$ is an arbitrary increasing sequence, $\mathcal{F}$ is complete. □

Remark. Suppose that $(\mathcal{F}, +, \cdot, \leq)$ is an ordered field. Then $\mathcal{F}$ has monotone sequence property if and only if it has least upper bound property.

1.6 Cauchy Sequences

Motivation: Suppose that $\lim_{n\to\infty} x_n = x$. Then for a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$. This implies two facts:

(1) At most $N - 1$ terms of the sequence are outside the interval $(x - \varepsilon, x + \varepsilon)$.

(2) For $m, n \geq N$, $|x_m - x_n| < 2\varepsilon$.

Heuristically, if a sequence $\{x_n\}_{n=1}^{\infty}$ converges, then any two terms x_n and x_m are arbitrarily close to each other by taking m, n sufficiently large.

Question: How about the converse?

Definition 1.6.1. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ is "*Cauchy*" if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \varepsilon \quad \text{whenever} \;\; m, n > N.$$

Proposition 1.6.2. *Every convergent sequence is Cauchy.*

Proof. (Exercise) □

Proposition 1.6.3. *Every Cauchy sequence is bounded.*

Proof. (Exercise.) □

Lemma 1.6.4. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. If there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ converges, say $\lim_{k\to\infty} x_{n_k} = x_0$, then $\{x_n\}$ converges to x_0.*

Proof. (Exercise) □

■ **Bolzano-Weierstrass Theorem**

Observation: Every convergent sequence is bounded. But not every bounded sequence is convegent. A divergent, unbounded and monotonic sequence must not contain a convergent subsequence.
Question: Does a bounded and divergent sequence contain a convergent subsequence? Under what hypotheses of a sequence that must contain a convergent subsequence?

Theorem 1.6.5. *(Bolzano-Weierstrass Theorem) Every bounded sequence (in $\mathbb{R}$) has convergent subsequence.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence.

(**Method 1:**) We sketch the method by following steps

(i) $\{x_n\}_{n=1}^{\infty}$ contains a monotonic subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

(ii) $\{x_{n_k}\}_{k=1}^{\infty}$ is also bounded.

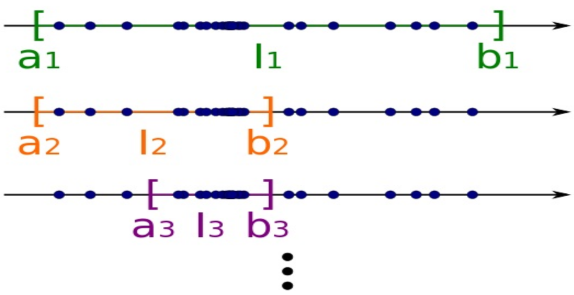
(iii) By using monotone sequence property, $\{x_{n_k}\}$ converges.

(**Method 2:**) Since $\{x_n\}$ is bounded, there exist $a_1, b_1 \in \mathbb{R}$ such that $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$. If $a_1 = b_1$, then $x_n = a_1$ for every $n \in \mathbb{N}$. Hence, $\{x_n\}$ itself converges to a_1.

We may assume that $a_1 < b_1$. Let $I_1 = \{x \mid a_1 \leq x \leq b_1\}$ and I_1 contains infinitly many terms of $\{x_n\}$. Divide I_1 into two equal length subintervals by the midpoint $\dfrac{a_1 + b_1}{2}$. At least one of the two subintervals contains infinitely many terms of $\{x_n\}$, say I_2.

Again, divide I_2 into two equal length subintervals by the midpoint of the endpoints of I_2. At leats, one of these two subintervals contains infinitely many term of $\{x_n\}$, say I_3.

Continue this proceduce, we have $I_{k+1} \subset I_k$ for every $k = 1, 2, \cdots$ and each interval I_k contains infinitely many terms of $\{x_n\}$. Let $I_k = [a_k, b_k]$ with length $b_k - a_k = \dfrac{b_1 - a_1}{2^{k-1}} \to 0$ as $k \to \infty$. By the nested intervals theorem, there exists $x_0 \in I_k$ for all $k \in \mathbb{N}$. Hence, $\lim\limits_{k\to\infty} a_k = \lim\limits_{n\to\infty} b_k = x_0$.



Now, we construct a subsequence of $\{x_n\}$ which converges x_0. Choose $n_1 \in \mathbb{N}$ such that x_{n_1} to be any number of $\{x_n\}$ in I_1. Again, choose $n_2 > n_1$ such that x_{n_2} to be any number of $\{x_n\}$ in I_2. Continue this procedure, we can choose $n_k > n_{k-1} > \cdots > n_2 > n_1$ such that x_{n_k} to be any number of $\{x_n\}$ in I_k. Therefore, $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$.

Since $x_{n_k} \in I_k$, $a_k \leq x_{n_k} \leq b_k$ and $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} b_k = x_0$. Hence, $\lim\limits_{k\to\infty} x_{n_k} = x_0$.

□

Theorem 1.6.6. *Every Cauchy sequence in* $\mathbb{R}$ *is convergent.*

Proof. (Exercise) □

Theorem 1.6.7. *Let $\mathcal{F}$ be an ordered field with Archimedean property. If every Cauchy sequence in $\mathcal{F}$ converges, then $\mathcal{F}$ is complete.*

Proof. Let $\{x_n\}$ be an increasing sequence which is bounded by M for some $M \in \mathcal{F}$. We want to prove that $\{x_n\}$ is Cauchy. Then by hypothesis, $\{x_n\}$ converges in $\mathcal{F}$ and hence $\mathcal{F}$ is complete

Suppose that $\{x_n\}$ is not Cauchy. By the definition of Cauchy sequnece, there exists $0 < \varepsilon \in \mathcal{F}$, such that for every $N \in \mathbb{N}$, there exists $m, n \in \mathbb{N}$ such that

$$|x_m - x_n| > \varepsilon \quad \text{whenever } m, n \geq N.$$

For $N = 1$, choose $1 \le m_1 < n_1$ such that $|x_{m_1} - x_{n_1}| \ge \varepsilon$.
For $N = n_1 + 1$, choose $n_1 + 1 \le m_2 < n_2$ such that $|x_{m_2} - x_{n_2}| \ge \varepsilon$.
Continue this procedure, we can choose $\cdots < m_k < n_k < m_{k+1} < n_{k+1} < \cdots$ such that $|x_{n_k} - x_{m_k}| \ge \varepsilon$ for every $k \in \mathbb{N}$. Since $\{x_n\}$ is an increasing sequence bounded above by M, we have

$$x_{m_1} \le x_{n_1} \le x_{m_2} \le x_{n_2} \le \cdots \le x_{m_k} \le x_{n_k} \le x_{m_{k+1}} \le x_{n_{k+1}} \le \cdots \le M.$$

Hence, $\{x_{n_k}\}$ is an increasing sequenve boudnd above by M and

$$|x_{n_{k+1}} - x_{n_k}| \ge \varepsilon.$$

Since $\mathcal{F}$ has Archimedean property, for the element $\dfrac{M - x_{n_1}}{\varepsilon} \in \mathcal{F}$, there exists $L \in \mathbb{N}$ such that $\dfrac{M - x_{n_1}}{\varepsilon} < L$. Therefore,

$$x_{n_{L+1}} = (x_{n_{L+1}} - x_{n_L}) + (x_{n_L} - x_{n_{L-1}}) + \cdots + (x_{n_2} - x_{n_1} + + x_{n_1} > \varepsilon + \varepsilon + \cdots + \varepsilon + x_{n_1} = L\varepsilon + x_{n_1} > M.$$

It contradicts the hypothesis that $\{x_n\}$ is bounded above by M. Hence, $\{x_n\}$ is Cauchy. □

Remark. In an ordered field with Archimedean property,

Completeness $\Longleftrightarrow$ Cauchy completeness
(Every Cauchy sequence converges.)

Remark. So far, we have learned several statements on an ordered field or $\mathbb{R}$:

(1) Completeness

(2) Monotone Sequence Property

(3) Nested Interval Theorem

(4) Least Upper Bound Property

(5) Bolzano-Weierstrass Theorem

(6) Cauchy Criterion

We have proved that

$$(1) \overset{\text{def}}{\Longleftrightarrow} (2), \quad (2) \overset{1.5.4}{\Rightarrow} (3) \overset{1.5.5}{\Rightarrow} (4) \overset{1.5.6}{\Rightarrow} (2), \quad (3) \overset{1.6.5}{\Rightarrow} (5) \overset{1.6.6}{\Rightarrow} (6) \overset{1.6.7}{\Rightarrow} (2)$$

Any one of the above statements on an order field can imply other statements. Thus, we can use any one of the statements as the definition of $\mathbb{R}$. (For the statement (4), we should carefully to define the "interval").

(7) Archimedean Property

We have the result that

$$(2) \overset{1.2.28}{\Rightarrow} (7), \quad (7) + \text{Cauchy criterion} \overset{1.6.7}{\Rightarrow} (2).$$

Remark. The above statement (2)–(4) only describe the properties of $\mathbb{R}$ [(3) can work on $\mathbb{R}^n$] since the partial order is necessary. The statements (5) and (6) are well-defined on general metric spaces. Therefore, we will use Cauchy criterion as the definition of completeness in the future.

1.7 Cluster Points and Limit Inferior, Limit Superior

❑ Cluster Points

Motivation: In the previous section, we have learned some results of convergent sequences and divergent sequence (Bolzano-Weierstrass). For a sequence $\{x_n\}$,

(1) if $\lim_{n\to\infty} x_n = x$, for any given $\varepsilon > 0$, only finitely many terms of $\{x_n\}$ outside the interval $(x-\varepsilon, x+\varepsilon)$. Heuristically, all but finitely many terms are clustered near x.

(2) if $\lim_{n\to\infty} x_n \neq x$, there exists $\varepsilon > 0$ such that there are infinitely many terms of $\{x_n\}$ outside $(x-\varepsilon, x+\varepsilon)$. In spite of this, it is possible that there still exists infinitely many terms near x. In this case, a subsequence is also clustered near x.

(3) by Bolzano-Weierstrass Theorem, a bounded and divergent sequence will have two or more cluster points. (For example, $\{(-1)^n\}$).

In some situations, those points where sequences are clustered there still to be worthy to study.

Definition 1.7.1. A point x is called a "*cluster point*" of a sequenc $\{x_n\}_{n=1}^{\infty}$ if for every $\varepsilon > 0$, there are infinitely many numbers of $\{x_n\}$ within $(x-\varepsilon, x+\varepsilon)$. (Note that we count x_i and x_j separately even if $x_i = x_j$).

Remark. (1) The difference between the limit of $\{x_n\}$ and a cluster point of $\{x_n\}$ is that

i. if x is a limit, all tail of $\{x_n\}$ are clustered in neighborhoods of x;
ii. if x is a cluster point, only infinitely many terms of $\{x_n\}$ are clustered in neighborhood of x.

(2) A limit is also a cluster point, but a cluster point may not be a limit.

Proposition 1.7.2. *Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$. The following statements are equivalent.*

(1) x is a cluster ponit of $\{x_n\}$.

(2) for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n > N$ such that $|x_n - x| < \varepsilon$.

(3) there exists a subsequence $\{x_{n_k}\}$ converges to x.

Proof. "(1)⇒ (2)"
For any given $N \in \mathbb{N}$, there are only finitely many indies which are smaller than N. By the definition of a cluster point, for $\varepsilon > 0$ and $N \in \mathbb{N}$, there must have $n > N$ such that $x_n \in (x-\varepsilon, x+\varepsilon)$ and hence $|x_n - x| < \varepsilon$.

"(2) ⇒ (3)"
For $n_0 = 1$ and $\varepsilon_1 = 1$. There exists $n_1 > n_0 = 1$ such that $|x_{n_1} - x| < 1$.
Again, for $n_1 \in \mathbb{N}$ and $\varepsilon_2 = \frac{1}{2}$. There exists $n_2 > n_1$ such that $|x_{n_2} - x| < \frac{1}{2}$.
Continue this procedure, we can choose $\varepsilon_k = \frac{1}{k}$ and find $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$|x_{n_k} - x| < \frac{1}{k}.$$

Hence, $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$.

"(3) ⇒ (1)"
Since $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x, for $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\left|x_{n_k} - x\right| < \varepsilon \quad \text{whenever } k \geq K.$$

There are infinitely many terms of $\{x_{n_k}\}$ within $(x-\varepsilon, x+\varepsilon)$. Hence, there are also infinitely many terms of $\{x_n\}$ withnin $(x-\varepsilon, x+\varepsilon)$. Then x is a cluster point of $\{x_n\}$.

□

Proposition 1.7.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. The following statements are equivalent.*

(1) $\lim_{n\to\infty} x_n = x$.

(2) $\{x_n\}_{n=1}^{\infty}$ *is bounded and x is the only cluster point of* $\{x_n\}_{n=1}^{\infty}$.

(3) every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence which converges to x.

Proof. (Exercise) □

❑ Liminf and Limsup (下極限與上極限)

Motivation:

(1) In some cases, we only focus on the tail of a sequence $\{x_n\}$ but not the whole sequence.

(2) We may only focus on the behavior of a subsequence of $\{x_n\}$. We may want to understand whether the behaviors of the tail of a sequence is bounded by two numbers.

Hence, we can track the supremum and infimum of the tails of $\{x_n\}$. If

$$\sup\{x_n\}_{n=k}^{\infty} - \inf\{x_n\}_{n=k}^{\infty} \searrow 0 \quad \text{as } k \to \infty,$$

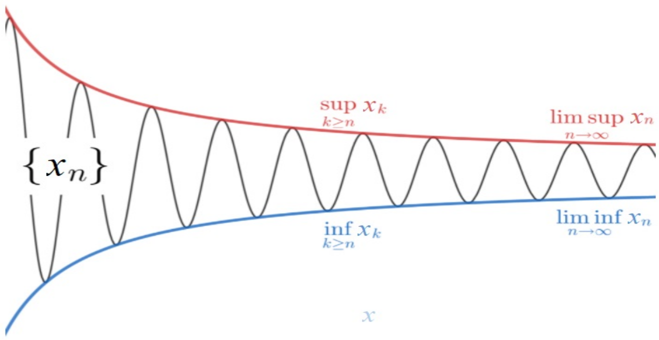
then the sequence coverges. Thus, let's observe the behavior of the sequence $\{a_k\}$ and $\{b_k\}$ where

$$a_k := \sup_{k\leq n<\infty}\{x_n\} \quad \text{and} \quad b_k := \inf_{k\leq n<\infty}\{x_n\}.$$

Definition 1.7.4. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

(1) The "*limit superior of* $\{x_n\}_{n=1}^{\infty}$" is the infimum of the seuqence $\left\{\sup_{k\leq n<\infty} x_n\right\}_{k=1}^{\infty}$. That is, $\inf_{k\in\mathbb{N}}\left(\sup_{k\leq n<\infty}\{x_n\}\right)$ or $\lim_{k\to\infty}\left(\sup_{k\leq n<\infty}\{x_n\}\right)$. Denoted by "$\limsup_{n\to\infty} x_n$" or "$\overline{\lim_{n\to\infty}}\, x_n$".

(2) The "*limit infimum of* $\{x_n\}_{n=1}^{\infty}$" is the supremum of the seuqence $\left\{\inf_{k\leq n<\infty} x_n\right\}_{k=1}^{\infty}$. That is, $\sup_{k\in\mathbb{N}}\left(\inf_{k\leq n<\infty}\{x_n\}\right)$ or $\lim_{k\to\infty}\left(\inf_{k\leq n<\infty}\{x_n\}\right)$. Denoted by "$\liminf_{n\to\infty} x_n$" or "$\underline{\lim}_{n\to\infty}\, x_n$".



Remark. The sequence $\{a_k\}$ is decreasing and $\{b_k\}$ is increasing. Hence, if $\pm\infty$ are allowed to be limits of sequences, $\lim_{k\to\infty} a_k$ and $\lim_{k\to\infty} b_n$ always exists. Moreover,

$$\lim_{k\to\infty}\Big(\sup_{k\leq n<\infty}\{x_n\}\Big) = \inf_{k\in\mathbb{N}}\Big(\sup_{k\leq n<\infty}\{x_n\}\Big) \quad \text{and} \quad \lim_{k\to\infty}\Big(\inf_{k\leq n<\infty}\{x_n\}\Big) = \sup_{k\in\mathbb{N}}\Big(\inf_{k\leq n<\infty}\{x_n\}\Big).$$

Example 1.7.5. (1) $x_n = (-1)^n$. Then $\sup_{n\geq k} x_n = 1$ and $\inf_{n\geq k} x_n = -1$ for every $k \in \mathbb{N}$. Then

$$\liminf_{n\to\infty} x_n = \inf_{k\in\mathbb{N}}\Big(\sup_{n\geq k} x_n\Big) = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \sup_{k\in\mathbb{N}}\Big(\inf_{n\geq k} x_n\Big) = -1.$$

(2) $x_n = \dfrac{(-1)^n}{n}$. Then $\sup_{n\geq k} x_n = \begin{cases} \dfrac{1}{k+1} & k \text{ is odd} \\ \dfrac{1}{k} & k \text{ is even}\end{cases}$ and $\inf_{n\geq k} x_n = \begin{cases} -\dfrac{1}{k} & k \text{ is odd} \\ -\dfrac{1}{k+1} & k \text{ is even}\end{cases}$ We have

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty}\Big(\sup_{n\geq k} x_n\Big) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} x_n = \lim_{k\to\infty}\Big(\inf_{n\geq k} x_n\Big) = 0.$$

Proposition 1.7.6. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

(1) $a = \liminf_{n\to\infty} x_n > -\infty$ if and only if

(i) for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $a - \varepsilon < x_n$, and

(ii) for $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists $n_0 \geq M$ such that $x_{n_0} < a + \varepsilon$.

(2) $b = \limsup_{n\to\infty} x_n < \infty$ if and only if

(i) for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $x_n < b + \varepsilon$, and

(ii) for $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists $n_0 \geq M$ such that $b - \varepsilon < x_{n_0}$.

Proof. It suffices to show (1) and the proof of (2) is similar.

($\Rightarrow$) Since $a = \liminf_{n\to\infty} x_n = \lim_{k\to\infty}(\inf_{n\geq k} x_n) > -\infty$, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $k \geq N$, $\left|\inf_{n\geq k} x_n - a\right| < \dfrac{\varepsilon}{2}$. Hence,

(i)

$$\inf_{n\geq k} x_n > a-\varepsilon \quad \text{whenever } k \geq N.$$

Then, for $n \geq N$, $x_n \geq \inf_{n\geq N} x_n > a-\varepsilon$ and (i) is proved.

(ii) Since $\inf_{n\geq N} x_n + \frac{\varepsilon}{2} < a+\varepsilon$, for given $M > 0$, we can choose $n_0 > \max(N, M)$ such that $x_{n_0} < \inf_{n\geq N} x_n + \frac{\varepsilon}{2} < a+\varepsilon$ and (ii) is proved.

($\Leftarrow$) Fix $\varepsilon > 0$, from (ii), for every $k \in \mathbb{N}$, $\inf_{n\geq k} x_n < a+\varepsilon$. Hence, $\liminf_{n\to\infty} x_n = \lim_{k\to\infty}\inf_{n\geq k} x_n \leq a+\varepsilon$.

Also, for $\varepsilon > 0$ and from (i), $\inf_{n\geq N} x_n \geq a-\varepsilon$. Hence,

$$\liminf_{n\to\infty} x_n = \sup_{k\in\mathbb{N}}\inf_{n\geq k} x_n \geq \inf_{n\geq N} x_n \geq a-\varepsilon.$$

We have for $\varepsilon > 0$,

$$a-\varepsilon \leq \liminf_{n\to\infty} x_n \leq a+\varepsilon.$$

Since ε is arbitrary positive number, $\liminf_{n\to\infty} x_n = a$. □

Remark. If $a = \liminf_{n\to\infty} x_n > -\infty$, then a is the smallest cluster point of $\{x_n\}$. If $b = \limsup_{n\to\infty} x_n < \infty$, then b is the largest cluster point of $\{x_n\}$.

Theorem 1.7.7. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

(1) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

(2) If $\{x_n\}$ is bounded above by M, then $\limsup_{n\to\infty} x_n \leq M$.

(3) If $\{x_n\}$ is bounded below by m, then $\liminf_{n\to\infty} x_n \geq m$.

(4) If $\limsup_{n\to\infty} x_n = \infty$, *then $\{x_n\}$ is not bounded above.*

(5) If $\liminf_{n\to\infty} x_n = -\infty$, *then $\{x_n\}$ is not bounded below.*

(6) If x is a cluster point of $\{x_n\}$, then $\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$.

(7) If $a = \liminf_{n\to\infty} x_n$ is finite, then a is a cluster point.

(8) If $b = \limsup_{n\to\infty} x_n$ is finite, then b is a cluster point.

(9) $\lim_{n\to\infty} x_n = x$ *if and only if* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Proof. (Exercise) □

Note. Let $S = \mathbb{Q}\cap[0,1]$. Then S is countable. Therefore, we can write $S = \{q_1, q_2, \cdots, q_n, \cdots\}$. For $p \in [0,1]$ and $\varepsilon > 0$, there are infinitely many points in S within $(p-\varepsilon, p+\varepsilon)$. Hence, p is a cluster point of S. We have the set of all cluster point of S is $[0,1]$.

1.8 Some Properties of $\mathbb{R}^n$

Definition 1.8.1. Euclidean n-space, denoted by $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers.

$$\begin{aligned} \mathbb{R}^n &= \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R} = \left\{(a_1,\cdots,a_n)\,\middle|\, a_i\in\mathbb{R} \text{ for } i=1,\cdots,n\right\} \\ &\cong \mathbb{R}\times\mathbb{R}^{n-1} = \left\{(a,\mathbf{b})\,\middle|\, a\in\mathbb{R},\ \mathbf{b}\in\mathbb{R}^{n-1}\right\} \\ &\cong \mathbb{R}^k\times\mathbb{R}^{n-k} = \left\{(\mathbf{a},\mathbf{b})\,\middle|\, \mathbf{a}\in\mathbb{R}^k,\ \mathbf{b}\in\mathbb{R}^{n-k}\right\}. \end{aligned}$$

Definition 1.8.2. We define a binary function (metric) $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$\|x-y\| = d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

where $x=(x_1,\cdots,x_n),\ y=(y_1,\cdots,y_n)\in\mathbb{R}^n$.

Remark. For $x,y,z\in\mathbb{R}^n$,

(1) $d(x,y)\geq 0$

(2) $d(x,y)=0$ if and only if $x=y$

(3) $d(x,y)=d(y,x)$

(4) $d(x,y)+d(y,z)\geq d(x,z)$ ("*triangle inequality*")

Remark. For $x,y\in\mathbb{R}^n$,

$$\max_{1\leq i\leq n}|x_i-y_i| \leq d(x,y) \leq \sum_{i=1}^{n}|x_i-y_i|.$$

Definition 1.8.3. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. We say that $\{x_k\}$ converges if there exists a point $L\in\mathbb{R}^n$ such that for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that if $n\geq N$,

$$d(x_n,L)<\varepsilon.$$

Denote by $\lim\limits_{n\to\infty} x_k = L$.

Lemma 1.8.4. *Let $x_k=(x_k^{(1)},\cdots,x_k^{(n)})$ and $L=(L_1,\cdots,L_k)$. Then*

$$\lim_{k\to\infty} x_k = L \quad \textit{if and only if} \quad \lim_{k\to\infty} x_k^{(i)} = L_i \quad \textit{for every } i=1,\cdots,k.$$

Proof. (Exercise) □

Definition 1.8.5. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. We say that $\{x_k\}$ is Cauchy if for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that if $m,n\geq N$, then

$$d(x_m,x_n)<\varepsilon.$$

Theorem 1.8.6. *Every Cauchy seuqence in $\mathbb{R}^n$ converges. Hence, $\mathbb{R}^n$ is (Cauchy) complete.*

Proof. (Exercise) □

Definition 1.8.7. We define a "closed interval" in $\mathbb{R}^n$ by

$$\left\{(x_1, \cdots, x_n) \,\middle|\, a_i \le x_i \le b_i \text{ for every } i = 1, \cdots, n\right\} = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

and an "open interval" in $\mathbb{R}^n$ by

$$\left\{(x_1, \cdots, x_n) \,\middle|\, a_i < x_i < b_i \text{ for every } i = 1, \cdots, n\right\} = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

Remark. Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $J = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be two intervals in $\mathbb{R}^n$. Then $I \subseteq J$ if and only if $[a_i, b_i] \subseteq [c_i, d_i]$ for every $i = 1, \cdots, n$.

Theorem 1.8.8. *(Nested Interval Theorem) Suppose taht* $I^{(k)} = [a_1^{(k)}, b_1^{(k)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}]$, $k = 1, 2, \cdots$ *is a sequence of closed intervals in* $\mathbb{R}^n$ *such that* $I^{(k+1)} \subseteq I^{(k)}$ *for every* $k = 1, 2, \cdots$. *If* $\lim_{k\to\infty}(b_i^{(k)} - a_i^{(k)}) = 0$ *for* $i = 1, \cdots, n$, *then there exists one and only one point* x_0 *which is in every interval* $I^{(k)}$.

Proof. (Exercise) □

Definition 1.8.9. We say that a set $S \subset \mathbb{R}^n$ is "*bounded*" if there exists $M > 0$ such that

$$S \subseteq [-M, M] \times \cdots \times [-M, M].$$

A sequence $\{x_k\}_{k=1}^{\infty}$ is "*bounded*" if the set $\{x_k \mid k = 1, 2, \cdots\}$ is bounded.

Remark. $S \subset \mathbb{R}^n$ is bounded if and only if therr exists $M > 0$ such that $\sup_{x \in S} d(x, \mathbf{0}) < M$ where $\mathbf{0} = (0, 0, \cdots, 0)$ is the origin in $\mathbb{R}^n$.

Theorem 1.8.10. *(Bolzano-Weierstrass Theorem) Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Proof. (Exercise) □

Chapter 2

Metric Spaces

2.1 Metrics and Topology

Motivation: 透過對於數列收斂概念的了解，只要有一個適當計算「距離」方式，我們可以將此想法推廣至平面或空間，甚至是討論 R^n 中點集的收斂問題。更進一步，若能在一個集合中定義適當的「距離」，我們亦可以探討集合的收斂現象，例如我們曾經學習過泰勒多項式收斂至函數的現象。

此外，在集合上定義「距離」，我們可以討論集合上函數的連續、微分、積分等問題。

❏ Metric on a Set

Definition 2.1.1. A metric space (M, d) is a set M associated with a function $d : M \times M \to \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ for every $x, y \in M$.

(2) $d(x, y) = 0$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$ for every $x, y \in M$.

(4) $d(x, y) + d(y, z) \geq d(x, z)$ for every $x, y, z \in M$ (*Triangle Inequality*)

A function d satisfies (1)−(4) is called a "*metric*" on M.

Example 2.1.2. (1) $M \subseteq \mathbb{R}^n$ and $d(x, y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then (M, d) is a metric space. (Check)

(2) $M \subseteq \mathbb{R}^n$ and $\bar{d}(x, y) = \max_{1\le i\le n} |x_i - y_i|$. Then $(M, \bar{d})$ is a metric space. (Check)

(3) M is any nonempty set and

$$\tilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \qquad (\textit{discrete metric})$$

Then $(M, \tilde{d})$ is a metric space. (Check)

(4) Let (M, d) be a metric space. Define $\rho(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. Then ρ is a (bounded) metric on M.

Example 2.1.3. Let $C\big([0, 1]\big)$ be the collection of all continuous function on $[0, 1]$. That is,

$$C\big([0, 1]\big) := \big\{f : [0, 1] \to \mathbb{R} \bigm| f \text{ is continuous.}\big\}.$$

Define $d(f, g) = \max_{x\in[0,1]} \big|f(x) - g(x)\big|$. Then d is a metric on $C\big([0, 1]\big)$.

Example 2.1.4. Let $M = C\big([0, 1]\big)$ and $d(f, g) = \Big[\displaystyle\int_0^1 |f(x) - g(x)|^2\, dx\Big]^{1/2}$. Then d is a metric on M.

Example 2.1.5. Let

$$M_{n\times m} := \big\{n \times m \text{ matrix with entries in } \mathbb{R}\big\}$$

and

$$d(A, B) = \sum_{\substack{1\le i\le n \\ 1\le j\le m}} \big|a_{ij} - b_{ij}\big|.$$

Then $\big(M_{n\times m}, d\big)$ is a metric space.

Remark. From Example2.1.2, a set M may have many metrics. In fact, different metric will give rise to different properties for M.

Definition 2.1.6. Let (M, d_1) and (M, d_2) be two metric spaces. We say that the two metrics d_1 and d_2 are equivalent if there exist two positive numbers $\alpha, \beta > 0$ such that

$$\alpha d_1(x, y) \le d_2(x, y) \le \beta d_1(x, y)$$

for every $x, y \in M$.

Note. Consider the metric spaces $(\mathbb{R}^n, d)$, $(\mathbb{R}^n, \bar{d})$ and $(\mathbb{R}^n, \tilde{d})$ where d, $\bar{d}$ and $\tilde{d}$ are defined in Example2.1.2.

$$\bar{d}(x, y) \le d(x, y) \le n\bar{d}(x, y) \quad \text{for every } x, y \in \mathbb{R}^n.$$

Hence, d and $\bar{d}$ are equivalent. However, d and $\tilde{d}$ are not equivalent. (Check)

Definition 2.1.7. Let (M, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in M. We say that $\{x_n\}$ *converges to* x if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \text{whenever } n \ge N.$$

Definition 2.1.8. Let (M,d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in M is said to be "*Cauchy*" if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{whenever } m, n \geq N.$$

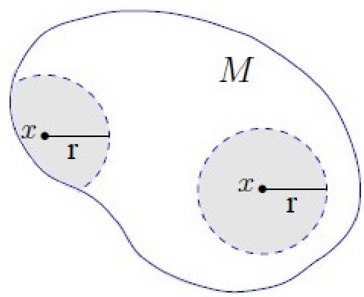
Proposition 2.1.9. *Let d_1 and d_2 be equivalent metric on M and $\{x_n\}_{n=1}^{\infty}$ be a sequence in M. Then $\{x_n\}_{n=1}^{\infty}$ converges to x in (M,d_1) if and only if $\{x_n\}_{n=1}^{\infty}$ converges to x in (M,d_2).*

Definition 2.1.10. A metric space (M,d) is said to be "*complete*" if every Cauchy sequence in M converges to a limit in M.

Remark. Let d be the discrete metric on a nonempty set M. Then M is complete.(exercise)

❑ Open Sets

Definition 2.1.11. Let (M,d) be a metric space. For $x \in M$ and $r > 0$, then r-ball (or r-disc) with center x and radius r is given by the set $\{y \in M \mid d(x,y) < r\}$. Denoted by $B(x,r)$ (or $D(x,r)$).



Remark. For $0 < r_1 < r_2$, $B(x,r_1) \subset B(x,r_2)$.

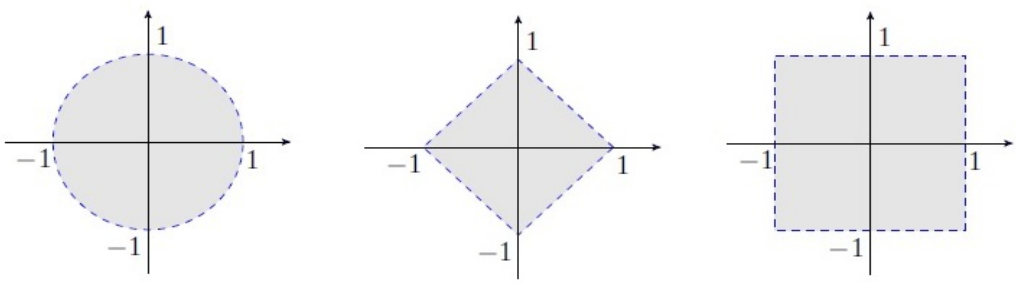
Example 2.1.12. Let $M = \mathbb{R}^2$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Consider the different metrics on M.

(1) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(2) $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

(3) $d_2(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$.

The following figure is the 1-ball $B(\mathbf{0},1)$ in $(\mathbb{R}^2, d)$, $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_2)$.

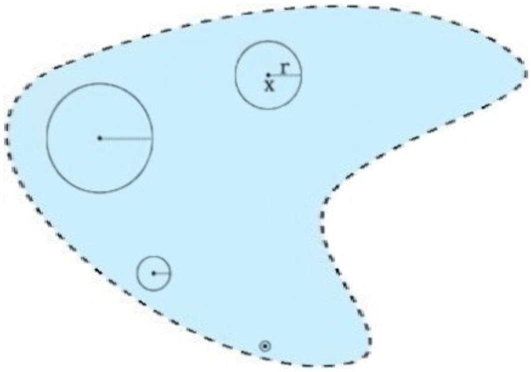


(4) $d_3 = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Then $B(\mathbf{0}, r) = \begin{cases} \{\mathbf{0}\} & \text{if } r \leq 1 \\ M & \text{if } r > 1. \end{cases}$

■ Open Sets

Definition 2.1.13. Let (M, d) be a metric space. We say that a set $U \subseteq M$ is "*open*" if for every $x \in U$ there exists $r > 0$ such that

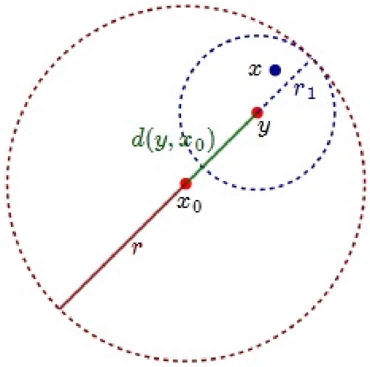
$$B(x, r) \subseteq U.$$



Example 2.1.14. Let (M, d) be a metric space. The r-ball $B(x_0, r)$ is open in (M, d).

Proof.

For $y \in B(x_0, r)$, we want to find $r_1 > 0$ such that $B(y, r_1) \subseteq B(x_0, r)$.



Since $y \in B(x_0, r)$, $d(x_0, y) < r$. Let $r_1 = r - d(x_0, y) > 0$. To show that the ball $B(y, r_1) \subset B(x_0, r)$. For $z \in B(y, r_1)$, $d(y, z) < r_1$. By the triangle inequality,

$$\begin{aligned} d(x_0, z) &\leq d(x_0, y) + d(y, z) \\ &< d(x_0, y) + r_1 \\ &= d(x_0, y) + r - d(x_0, y) = r. \end{aligned}$$

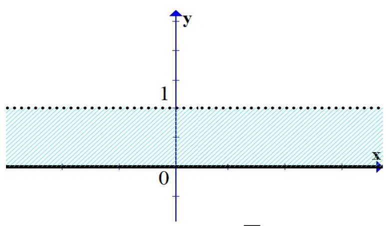
Thus $z \in B(x_0, r)$. Since z is an arbitrary point in $B(y, r_1)$, we have $B(y, r_1) \subseteq B(x_0, r)$. Moreover, since y is an arbitrary point in $B(x_0, r)$, the ball $B(x_0, r)$ is open. □

Proposition 2.1.15. *For* $M = \mathbb{R}^2$ *with metric* $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, *the set* $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 1\}$ *is not open in* $(\mathbb{R}^2, d)$.

Proof.

Let $u = (0, 0)$. To prove that no matter how small number $\varepsilon > 0$ is, the ball $B(u, \varepsilon) \not\subset A$.

For given $\varepsilon > 0$, the point $(0, -\frac{\varepsilon}{2}) \in B(u, \varepsilon)$ but $(0, -\frac{\varepsilon}{2}) \notin A$. Then $B(u, \varepsilon) \not\subset A$. There exists no ball with center u contained in A and hence A is not open in $(\mathbb{R}^2, d)$.



□

Example 2.1.16. Let $M = \mathbb{R}^2$, $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and $d_2(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$. Then the set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is open in (M, d_1) and open in (M, d_2).

Proof. (Exercise) □

Remark. Let d and ρ be equivalent metrics on M. Then U is open in (M,d) if and only if U is open in (M,ρ).

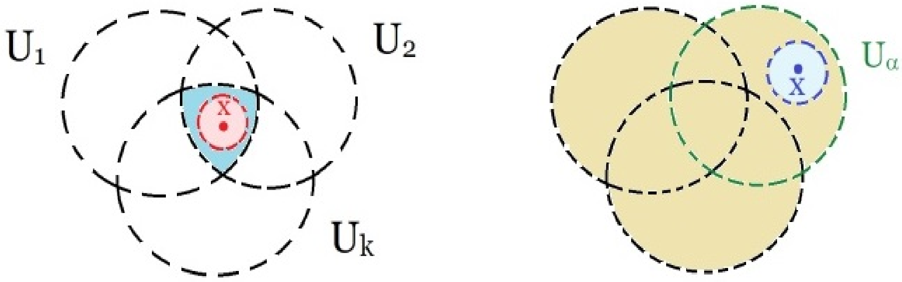
Proof. (Exercise) □

Proposition 2.1.17. *Let (M,d) be a metric space.*

(1) The intersection of finitely many open sets is open. That is, if $U_1, \cdots, U_n$ are open in (M,d), then $\bigcap_{i=1}^{n} U_i$ is open.

(2) The union of arbitrary family of open sets is open. That is, let $\mathcal{F} = \{U_\alpha \mid U_\alpha \text{is open in } M,\ \alpha \in I\}$ is a family of open sets, then $\bigcup_{\alpha\in I} U_\alpha$ is open.

(3) $\emptyset$ and M are open in M.



Proof. (Exercise) □

Corollary 2.1.18. *Let (M,d) be a metric space with discrete metric. Then every subset of M is open.*

Proof. Let $A \subseteq M$ and $a \in A$. Since $d(a, \frac{1}{2}) = \{a\} \in A$, the open ball $B(a, \frac{1}{2}) \subseteq A$. Hence, A is open. □

Remark. Infinite intersection of open sets may not be open. Consider $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then $\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not open in $\mathbb{R}$ with usual metric.

Example 2.1.19. Let $A \subseteq \mathbb{R}^n$ be an open set and $B \subseteq \mathbb{R}^n$ be any set. Then the set

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}$$

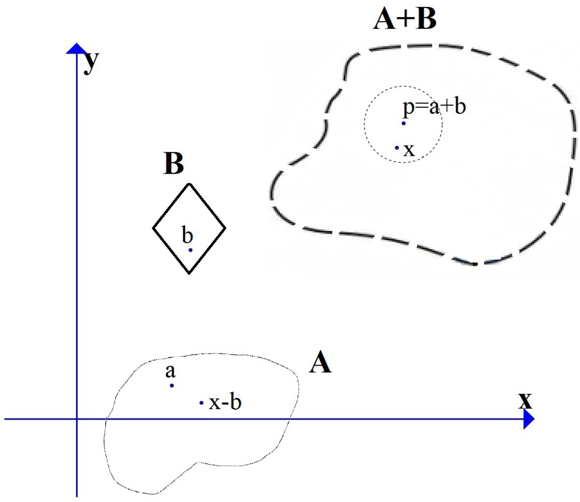
is open.

Proof.

For $p \in A+B$, there exists $a \in A$ and $b \in B$ such that $p = a+b$.

Since A is open and $a \in A$, there exists $r > 0$ such that $B(a,r) \subseteq A$. It sufficies to show that $B(p,r) \subseteq A+B$.

For $x \in B(p,r)$, $x-b \in B(a,r) \subseteq A$. Thus, there exists $a_1 \in A$ such that $x-b = a_1$. Then $x = a_1 + b \in A+B$. Hence, $B(p,r) \subseteq A+B$ and $A+B$ is open.



□

❑ Interior Points

Definition 2.1.20. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M.

(1) We call a point $x \in A$ an "*interior point*" of A if there exists $\delta > 0$ such that $B(x,\delta) \subseteq A$.

(2) The "*interior*" of A is the set of all interior point of A, and is denoted by "int(A)" or "$\mathring{A}$".

Example 2.1.21. Let $M = \mathbb{R}$ with the metric $d(x,y) = |x-y|$. $A = [0,1)$ and $B = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then $\mathring{A} = (0,1)$ and $\mathring{B} = \emptyset$.

Note. The interior of a set might be an empty set.

Theorem 2.1.22. *Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. Then $\mathring{A}$ is the largest open set contained in A. In other word, if $U \subseteq A$ is open (in M) then $U \subseteq \mathring{A}$.*

Proof. (i) By the definition of the interior of A, $\mathring{A} \subseteq A$ ($\mathring{A}$ is contained in A).

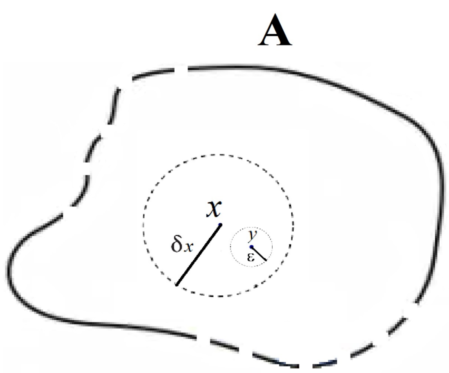
(ii) To prove that $\mathring{A}$ is open.

For $x \in \mathring{A}$, by the definition, there exists $\delta_x > 0$ such that $B(x,\delta_x) \subseteq A$. We will show that $\mathring{A} = \bigcup_{x \in \mathring{A}} B(x,\delta_x)$ and then $\mathring{A}$ is open.

"$\subseteq$": Clearly,

$$\mathring{A} = \bigcup_{x \in \mathring{A}} \{x\} \subseteq \bigcup_{x \in \mathring{A}} B(x,\delta_x).$$

"$\supseteq$": For $y \in \bigcup_{x \in \mathring{A}} B(x,\delta_x)$, there exists $x \in \mathring{A}$ such that $y \in B(x,\delta_x)$ and hence $d(x,y) < \delta_x$. Let $\varepsilon = \delta_x - d(x,y)$, then $B(y,\varepsilon) \subseteq B(x,\delta_x) \subseteq A$.



This implies that y is an interior point of A, that is, $y \in \mathring{A}$ and then $\bigcup_{x \in \mathring{A}} B(x, \delta_x) \subseteq \mathring{A}$. We have

$$\mathring{A} = \bigcup_{x \in \mathring{A}} B(x, \delta_x).$$

(iii) To prove that every open set contained in A is a subset of $\mathring{A}$.

Let $U \subseteq A$ be an open set. For $z \in U$, there exists $r > 0$ such that $B(z, r) \subseteq U \subseteq A$. Thus, $z \in \mathring{A}$. Since z is an arbitrary point in U, we have $U \subseteq \mathring{A}$.

□

Theorem 2.1.23. *Let (M, d) be a metric space. The set $A \subseteq M$ is open if and only if $A = \mathring{A}$.*

Proof. ($\Longrightarrow$) Clearly, $\mathring{A} \subseteq A$. Since A is open and $\mathring{A}$ is the largest open set contained in A, we have $A \subseteq \mathring{A}$ and hence $A = \mathring{A}$.

($\Longleftarrow$) This direction is trivial. □

Remark. If $A \subseteq M$ is open, then every point in A is an interior point of A.

Remark. (1) Let A and B be two sets in (M, d). Then

$$\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ.$$

In general, $\mathring{A} \cup \mathring{B} \subsetneq (A \cup B)^\circ$. For example $A = [0, 1)$ and $B = [1, 2]$.

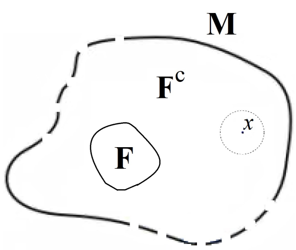
(2) $\{y \in M \mid d(x, y) < r\} \subseteq \operatorname{int}\left(\{y \in M \mid d(x, y) \leq r\}\right)$.

The relation is, in general, "$\subsetneq$". For exampe, let d be the discrete metric and $r = 1$. Consider $M = \{y \in M \mid d(x, y) \leq 1\}$. Then $\operatorname{int}\left(\{y \in M \mid d(x, y) \leq 1\}\right) = M$. But $\{y \in M \mid d(x, y) < 1\} = \{x\}$.

2.2 Closed Sets, the Closure of Sets, and the Boundary of Sets

❏ Closed Sets

Definition 2.2.1. Let (M, d) be a metric space. A set $F \subseteq M$ is said to be "*closed*" if $F^c = M \backslash F$ is open.

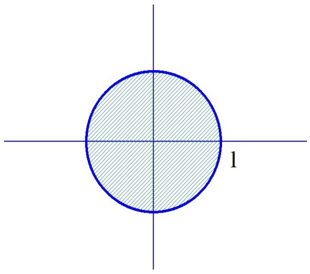


Remark. F is closed if and only if for every $x \in F^c$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subseteq F^c$.

Remark. The set $\{y \mid d(x, y) \leq r\}$ is close. (Exercise)

Exercise. (1) $[0, 1] \subset \mathbb{R}$ is closed and $\mathbb{R} \backslash [0, 1] = (-\infty, 0) \cup (1, \infty)$ is open

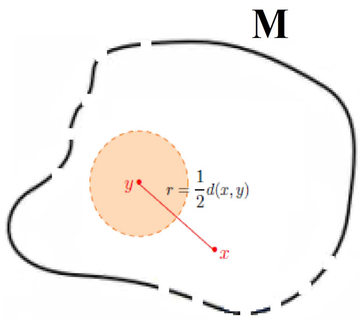
(2) $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is closed in $\mathbb{R}^2$.



Proposition 2.2.2. *Every point in a metric space is closed. That is, for a metric space (M, d) and a point $x \in M$, the set $\{x\}$ is closed in M.*

Proof. For $x \in M$, let $y \in M \backslash \{x\}$. (We want to find $r > 0$ such that $B(y, r) \subseteq M \backslash \{x\}$.)

Since $x \neq y$, $d(x, y) > 0$. Let $r = \frac{1}{2} d(x, y)$, then $d(x, y) > \frac{1}{2} d(x, y) = r$. Thus, $x \notin B(y, r)$. We have $B(y, r) \subseteq M \backslash \{x\}$. Therefore, $M \backslash \{x\}$ is open and $\{x\}$ is closed.



□

Proposition 2.2.3. *Let (M, d) be a metric space.*

(1) The union of finitely many closed sets is closed. That is, if $F_1, \cdots, F_n$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

(2) The intersection of arbitrary family of closed sets is closed. That is, let $\mathscr{F} = \{F_\alpha \mid F_\alpha \text{ is closed}, \alpha \in I\}$ be a family of closed sets. The intersection $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(3) $\emptyset$ and M are closed.

Proof. (1) Let $F_1, \cdots, F_n$ be closed. Then $F_1^c, \cdots, F_n^c$ are open. Since $\left(\bigcup_{i=1}^{n} F_i\right)^c = \left(\bigcap_{i=1}^{n} F^c\right)$ is the intersection of finitely many open sets and hence is open. Thus, $\bigcup_{i=1}^{n} F_i$ is closed.

(2) Let $\mathscr{F} = \{F_\alpha \mid F_\alpha \text{ is closed for} \alpha \in I\}$ is a family of closed sets. Then F_α^c is open for $\alpha \in I$. Then $\left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c$ is the union of a family of open sets and hence is open. Therefore, $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(3) Since $\emptyset^c = M$ and $M^c = \emptyset$ are open, $\emptyset$ and M are also closed.

□

Corollary 2.2.4. *Every set consists of finitely many points is closed.*

Proof. (Exercise) □

Remark. The union of infinitely many closed sets may not be closed. For example,

(i) $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is not closed since $\{0\} \in A^c$ but $\{0\}$ is not an interior point of A^c.

(ii) $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1)$ is open.

(iii) $\bigcup_{n=1}^{\infty}[0, 1 - \frac{1}{n}] = [0, 1)$ is not open and not closed.

❑ Accumulation Points, Limit Points and Isolated Points

Definition 2.2.5. Let (M, d) be a metric space and $A \subseteq M$.

(1) A point $x \in M$ is called an "*accumulation point*" of A if for every $\varepsilon > 0$, the open ball $B(x, \varepsilon)$ contains a point $y \in A$ and $y \neq x$. That is, for every $\varepsilon > 0$,

$$B(x, \varepsilon) \cap \left(A \backslash \{x\}\right) \neq \emptyset.$$

Example: $A = (0, 1)$, every point in $[0, 1]$ is an accumulation point of A.
Example: $A = (0, 1) \cup \{2\}$, every point in $[0, 1]$ is an accumulation point of A but $\{2\}$ is not an accumulation point of A.

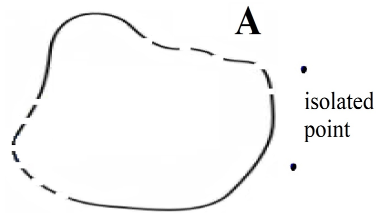
(2) A point $x \in M$ is called a "*limit point*" of A if for every $\varepsilon > 0$, the open ball $B(x, \varepsilon)$ contains a point in A. That is,

$$B(x, \varepsilon) \cap A \neq \emptyset.$$

Example: $A = (0, 1) \cup \{2\}$, every point in $[0, 1] \cup \{2\}$ is a limit point of A

(3) A point $x \in A$ is called an "*isolated point*" if there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \cap A = \{x\}.$$



(4) We denote the set of all accumulation points of A by A' and is called the "*derived set*" of A.

(5) We denote the set of all limit points of A by $\overline{A}$. The set will be called the "*closure*" of A later.

Remark. (1) An accumulation point of A may not be in A. For example, let $A = (0, 1)$, the point $\{0\}$ is an accumulation point.

(2) A set A consists of finitely many points has no accumulation point. That is, $A' = \emptyset$.
Question: How about infinitely many points?
Consider $\mathbb{Z}$ has no accumulation point and hence $\mathbb{Z}' = \emptyset$. On the other hand, let $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and $A' = \{0\}$.

(3) Accumulation points are also called "*cluster points*" in some books.

(4) $A' \subseteq \overline{A}$.

(5) $A \subseteq \overline{A}$.

(6) An isoloated point is a limit point but not an accumulation point.

Example 2.2.6. (1) $A = (0, 1) \cup \{2\}$. Then $A' = [0, 1]$ and $\overline{A} = [0, 1] \cup \{2\}$.

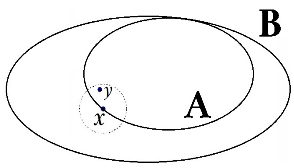
(2) $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ consists of infinitely many distinct points and is bounded. By Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, say $\lim_{k\to\infty} x_{n_k} = x_0$. Then $\{x_0\}$ is an accumulation point of $\{x_n\}_{n=1}^{\infty}$.

(3) Let (M, d) be a metric space with discrete metric d and $A \subseteq M$. Then $A' = \emptyset$.

Proposition 2.2.7. *If $A \subseteq B$, then*

(1) $A' \subseteq B'$.

(2) $\overline{A} \subseteq \overline{B}$.



Proof. (1) Let $x \in A'$. For $\varepsilon > 0$ there exists $y \neq x$ and $y \in A$ such that $y \in B(x, \varepsilon)$. Since $A \subseteq B$, we have $y \in B$. Hence, y is a point in B where $x \neq y$ and $y \in B(x, \varepsilon)$. That is, $x \in B'$. Since x is an arbitrary point in A', $A' \subseteq B'$.

(2) (Exercise)

□

Proposition 2.2.8. *Let $A \subseteq \mathbb{R}^n$, then $\mathring{A} \subseteq A'$.*

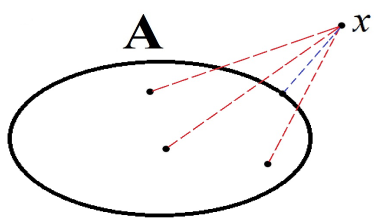
Proof. Let $x \in \mathring{A}$, there exists $\delta > 0$ such that $B(x, \delta) \subset A$. For given $\varepsilon > 0$, $B(x, \delta) \cap B(x, \varepsilon) = B\big(x, \min(\delta, \varepsilon)\big) \subseteq A$. Hence, there exist a point $y \neq x$, y$\in B\big(x, \min(\delta, \varepsilon)\big) \cap A$. This implies that x is an accumulation point in A. □

Note. The above proposition is true for the usual metric, but is false for the discrete metric.

Definition 2.2.9.

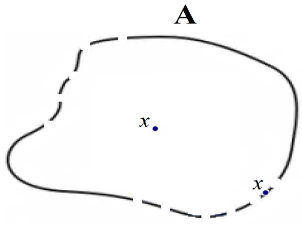
Let (M, d) be a metric space, $x \in M$ and $A \subseteq M$.
We define the distance from $\{x\}$ to A by

$$d(x, A) = \inf\big\{d(x, y) \mid y \in A\big\} = \inf_{y\in A} d(x, y).$$



Proposition 2.2.10.

Let (M, d) be a metric space and $A \subseteq M$. Then $x \in \overline{A}$ if and only if $d(x, A) = 0$.



Proof. ($\Longrightarrow$) Since $x \in \overline{A}$ is a limit point of A, for every $\varepsilon > 0$, $B(x, \varepsilon) \cap A \neq \emptyset$. Then there exists $z \in B(x, \varepsilon) \cap A$ and hence $d(x, z) < \varepsilon$. The distance

$$d(x, A) = \inf \left\{ d(x, y) \,\middle|\, y \in A \right\} \leq d(x, z) < \varepsilon.$$

Since ε is an arbitrary positive number, we have $d(x, A) = 0$.

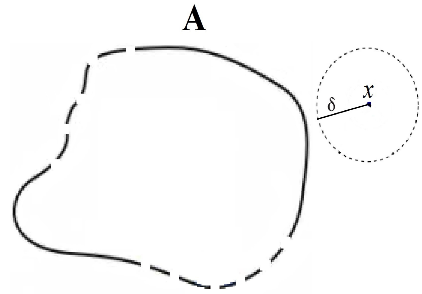
($\Longleftarrow$) If $d(x, A) = 0$, for given $\varepsilon > 0$, there exists $z \in A$ such that $d(x, z) < \varepsilon$. Thus, $z \in B(x, \varepsilon) \cap A$ and then $B(x, \varepsilon) \cap A \neq \emptyset$. This implies that $x \in \overline{A}$. □

Remark. Let (M, d) be a metric space and $x \in A'$. Then $d(x, A) = 0$. But the converse is false. For example, $A = (0, 1) \cup \{2\}$ and hence $d\big(\{2\}, A\big) = 0$. But $\{2\} \notin A' = [0, 1]$.

Theorem 2.2.11. *Let (M, d) be a metric space and $A \subseteq M$, then A is closed if and only if $A = \overline{A}$.*

Proof.

($\Longrightarrow$) Clearly, $A \subseteq \overline{A}$. On the other hand, since A is closed, for every $x \in A^c$, there exists $\delta > 0$ such that $B(x, \delta) \subseteq A^c$. That is, $B(x, \delta) \cap A = \emptyset$. Therefore, x is not a limit point of A, i.e. $x \notin \overline{A}$. Then $A^c \subseteq \big(\overline{A}\big)^c$ and thus $\overline{A} \subseteq A$. We have $A = \overline{A}$.



($\Longleftarrow$) Since $A = \overline{A}$, for every $x \in A^c$, x is not a limit point of A. Hence, there exists $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$. That is, $B(x, \delta) \subseteq A^c$. This implies that A^c is open and then A is closed. □

Theorem 2.2.12. *Let (M, d) be a metric space and $A \subseteq M$. Then $\overline{A} = A \cup A'$.*

Proof. ($\subseteq$) Let $x \in \overline{A}$. For every $\delta > 0$, $B(x, \delta) \cap A \neq \emptyset$. If $x \in \overline{A} \backslash A$, then $B(x, \delta) \cap \big(A \backslash \{x\}\big) \neq \emptyset$. We have $x \in A'$ and

$$\overline{A} = \left[\big(\overline{A} \backslash A\big) \cup A \right] \subseteq A' \cup A.$$

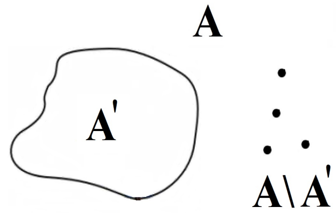
($\supseteq$) Since $A \subseteq \overline{A}$ and $A' \subseteq \overline{A}$, we have $A \cup A' \subseteq \overline{A}$. □

Corollary 2.2.13. *If $A \subseteq B$ and B is closed, then $\overline{A} \subseteq B$.*

Proof. Since $A \subseteq B$ and B is closed, we have $\overline{A} \subseteq \overline{B} = B$. □

Proposition 2.2.14. *Let (M,d) be a metric space and $A \subseteq M$. Then $A \backslash A'$ is the collection of all isolated points of A.*

Proof. Let B be the set of all isolated points of A. Clearly, $B \subseteq A\backslash A'$.
Let $x \in A\backslash A'$. Then x is not an accumulation point of A. There exists $\delta > 0$ such that $B(x,\delta) \cap A = \{x\}$. Hence x is an isolated point of A and $x \in B$. We have $A\backslash A' \subseteq B$. □



Theorem 2.2.15. *Let (M,d) be a metric space and $A \subseteq M$. Then A' is closed.*

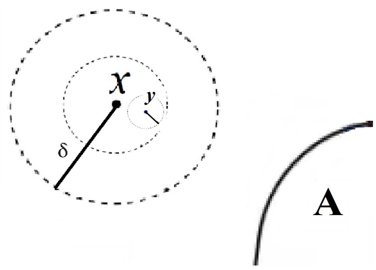
Proof. Let $x \notin A'$. There exists $\delta > 0$ such that $B(x,\delta) \cap \big(A\backslash\{x\}\big) = \emptyset$. (We want to prove $B(x,\frac{\delta}{2}) \subseteq \big(A'\big)^c$.)

Assume $B(x,\dfrac{\delta}{2}) \nsubseteq (A')^c$. Then $B(x,\dfrac{\delta}{2}) \cap A' \neq \emptyset$, say $y \in B(x,\dfrac{\delta}{2}) \cap A'$. Clearly, $x \neq y$ since $x \notin A'$.

Let $\varepsilon = \min\big(\dfrac{\delta}{2}, d(x,y)\big)$. Since $y \in A'$, we have $B(y,\varepsilon) \cap \big(A\backslash\{y\}\big) \neq \emptyset$. Thus, there exists $z \in B(y,\varepsilon) \cap \big(A\backslash\{y\}\big)$. Since $d(y,z) < \varepsilon = \min\big(\frac{\delta}{2}, d(x,y)\big)$, we obtain $z \neq x$ and

$$d(x,z) \leq d(x,y) + d(y,z) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Therefore, it gives a contradiction that $z \in B(x,\delta) \cap \big(A\backslash\{x\}\big)$. This implies that $B(x,\frac{\delta}{2}) \subseteq (A')^c$. We have $(A')^c$ is open and A' is closed.



□

Alternating Proof:

Let $x \notin A'$. Then there exists $\delta > 0$ such that $B(x,\delta) \cap \big(A\backslash\{x\}\big) = \emptyset$. Therefore, $A \subseteq \big(B(x,\delta)\backslash\{x\}\big)^c$.

Since $B(x,\delta)\backslash\{x\} = B(x,\delta)\cap\{x\}^c$ is open, $\big(B(x,\delta)\backslash\{x\}\big)^c$ is closed and thus $\overline{A} \subseteq \big(B(x,\delta)\backslash\{x\}\big)^c$. We have $\overline{A} \cap \big(B(x,\delta)\backslash\{x\}\big) = \emptyset$.

Also, since $\overline{A} = A \cup A'$, $A' \cap \big(B(x,\delta)\backslash\{x\}\big) = \emptyset$ and hence $B(x,\delta) \subseteq \big(A'\big)^c$. We have that x is an interior point of $\big(A'\big)^c$ and A' is closed.

❑ Closure

Definition 2.2.16. Let (M,d) be a metric space and $A \subseteq M$. The closure of A is the intersection of closed set containing A, and is denote by $\mathrm{cl}(A)$. In other words,

$$\mathrm{cl}(A) = \bigcap_{\substack{F:\text{closed}\\ A\subseteq F}} F.$$

Remark. $\mathrm{cl}(A)$ is the smallest closed set containing A.

Proposition 2.2.17. *Let (M, d) be a metric space and $A \subseteq M$.*

(1) $A \subseteq cl(A)$.

(2) A is closed if and only if $A = cl(A)$.

Proof. (1) The proof is direct by definition.

(2) ($\Longrightarrow$) Since $\mathrm{cl}(A)$ is the smallest set containing A and A is closed and $A \subseteq A$, $\mathrm{cl}(A) \subseteq A$. Also, $A \subseteq \mathrm{cl}(A)$ by definition of $\mathrm{cl}(A)$ and hence $A = \mathrm{cl}(A)$.

($\Longleftarrow$) Since $\mathrm{cl}(A)$ is closed and $A = \mathrm{cl}(A)$, we have A is closed.

□

Proposition 2.2.18. *Let (M, d) be a metric space. Then $cl(A) = \overline{A}$.*

Proof. ($\supseteq$) Since $\mathrm{cl}(A)$ is closed and $A \subseteq \mathrm{cl}(A)$, we have $\overline{A} \subseteq \mathrm{cl}(A)$.

($\subseteq$) Clearly, $A \subseteq \overline{A}$. (We want to prove that $\overline{A}$ is closed.)
For $x \notin \overline{A} = A \cup A'$, there exists $r > 0$ such that $B(x, r) \cap A = \emptyset$ and hence $A \subseteq \big(B(x, r)\big)^c$.

Since $\big(B(x, r)\big)^c$ is closed, by the definition of the closure, $\overline{A} \subseteq \big(B(x, r)\big)^c$. Then $B(x, r) \cap \overline{A} = \emptyset$ and thus $B(x, r) \subseteq \overline{A}^c$ and x is an interior point of $\overline{A}^c$. We obtain that $\overline{A}^c$ is open and $\overline{A}$ is closed.

Since $\mathrm{cl}(A)$ is the smallest closed set containing A, we have $\mathrm{cl}(A) \subseteq \overline{A}$. □

Remark. In a metric space (M, d) and $A \subseteq M$, since $\overline{A} = \mathrm{cl}(A)$, we also call "$\overline{A}$ the closure of A".

Example 2.2.19. Let $A = [0, 1) \cup \{2\}$. Then $\mathrm{cl}(A) = [0, 1] \cup \{2\}$.

Remark.

$$\overline{A} = A \cup A' = \text{the collection of all limit points of } A.$$

$$\begin{aligned} \mathrm{cl}(A) &= \text{the intersection of all closed sets containing } A. \\ &= \text{the smallest closed set containing } A. \end{aligned}$$

$$\overline{A} = \mathrm{cl}(A).$$

Proposition 2.2.20. *In a metric space (M, d), $x \in cl(A)$ if and only if $d(x, A) = \inf \big\{ d(x, y) \bigm| y \in A \big\} = 0$.*

Proof. (Exercise) □

Note. In a metric space (M, d), a subset $A \subseteq M$ is dense in $\overline{A}$.

❑ <u>Dense</u>

Definition 2.2.21. Let (M, d) be a metric space and $A \subseteq B \subseteq M$. We say that A is "*dense*" in B if

$$A \subseteq B \subseteq \overline{A}.$$

Remark. (1) If A is dense in B, then for every $x \in B$ and every $\varepsilon > 0$, $B(x, \varepsilon) \cap A \neq \emptyset$.

(2) A is dense in $\overline{A}$.

Example 2.2.22. $\mathbb{Q}$ is dense in $\mathbb{R}$.

❑ Boundary

Definition 2.2.23. Let (M, d) be a metric space and $A \subseteq M$. The "*boundary*" of A is the intersection of $\overline{A}$ and $\overline{A^c}$, and denoted by ∂A. Hence,

$$\partial A = \overline{A} \cap \overline{A^c}.$$

Remark. (1) ∂A is closed since $\partial A = \overline{A} \cap \overline{A^c}$ is an intersection of closed sets.

(2) $\partial A = \partial(A^c)$.

Proposition 2.2.24. *Let (M, d) be a metric space and $A \subseteq M$. Then $x \in \partial A$ if and only if for every $\varepsilon > 0$,*

$$B(x, \varepsilon) \cap A \neq \emptyset \quad \textit{and} \quad B(x, \varepsilon) \cap A^c \neq \emptyset.$$

Proof. By the definition of limit points, a point $x \in \partial A = \overline{A} \cap \overline{A^c}$ is on the boundary of A if and only if for every $\varepsilon > 0$

$$B(x, \varepsilon) \cap A \neq \emptyset \quad \text{and} \quad B(x, \varepsilon) \cap A^c \neq \emptyset.$$

□

Proposition 2.2.25. *Let (M, d) be a metric space and $A \subseteq M$. Then $\partial A = \overline{A} \backslash \mathring{A}$.*

Proof. ($\subseteq$) Let $x \in \partial A = \overline{A} \cap \overline{A^c}$. Since $x \in \overline{A^c}$, for every $\varepsilon > 0$, $B(x, \varepsilon) \cap A^c \neq \emptyset$. Then $B(x, \varepsilon) \nsubseteq A$ which implies that $x \notin \mathring{A}$. Thus, $x \in \overline{A} \backslash \mathring{A}$. We obtain $\partial A \subseteq \overline{A} \backslash \mathring{A}$.

($\supseteq$) Let $x \in \overline{A} \backslash \mathring{A}$. Then $x \notin \mathring{A}$. For every $\varepsilon > 0$, $B(x, \varepsilon) \nsubseteq A$. Therefore, $B(x, \varepsilon) \cap A^c \neq \emptyset$ which implies that $x \in \overline{A^c}$. We have $x \in \overline{A} \cap \overline{A^c} = \partial A$ and hence $\overline{A} \backslash \mathring{A} \subseteq \partial A$. □

Example 2.2.26. Let $M = \mathbb{R}$ be a space with metric $d(x, y) = |x - y|$ and $A = [0, 1] \cap \mathbb{Q}$. Then

$$A' = [0, 1], \quad \overline{A} = [0, 1], \quad \mathring{A} = \emptyset \quad \text{and} \quad \partial A = \overline{A} \backslash \mathring{A} = [0, 1].$$

Example 2.2.27. Let (M, d) be a metric space with discrete metric and $A \subseteq M$.

(1) A is open. (Every set is open.)

(2) A is closed. (A^c is open.)

(3) $A = \mathring{A}$, (A is open.)

(4) $A' = \emptyset$.

(5) $A = \overline{A}$. (A is closed.)

(6) $\partial A = \overline{A} \backslash \mathring{A} = \emptyset$.

Remark. $A \subseteq B \not\Rightarrow \partial A \subseteq \partial B$.

(i) Let $A = (0, 1)$ and $B = [0, 1]$. Then $\partial A = \{0, 1\} = \partial B$.

(ii) Let $A = \mathbb{Q} \cap [0, 1]$ and $B = [0, 1]$. Thne $\partial B = \{0, 1\} \subseteq [0, 1] = \partial A$.

(iii) Let $A = [1, 2]$ and $B = [0, 3]$. Then $\partial A = \{1, 2\}$, $\partial B = \{0, 3\}$ and $\partial A \cap \partial B = \emptyset$.

Remark. (1) $\partial A \not\subseteq A'$. For example, $A = \{0\}$, then $A' = \emptyset$ and $\partial A = \{0\}$.

(2) $\partial A \neq \partial(\mathring{A})$. For example, $A = [0, 1] \cup \{2\}$, then $\partial A = \{0, 1, 2\}$, $\mathring{A} = (0, 1)$ and $\partial \mathring{A} = \{0, 1\}$.

Proposition 2.2.28. *Let (M, d) be a metric space and $A, B \subseteq M$. Then*

(1) $\partial(A \cup B) \subseteq \partial A \cup \partial B$, and

(2) $\partial(A \cap B) \subseteq \partial A \cup \partial B$.

Proof. (1)

$$x \in \partial(A \cup B) \quad \text{if and only if} \quad \begin{aligned} &\text{for every } \varepsilon > 0,\ B(x, \varepsilon) \cap (A \cup B) \neq \emptyset \text{ and} \\ &B(x, \varepsilon) \cap (A \cup B)^c = B(x, \varepsilon) \cap (A^c \cap B^c) \neq \emptyset. \end{aligned}$$

Hence, either (i) $B(x, \varepsilon) \cap A \neq \emptyset$ and $B(x, \varepsilon) \cap A^c \neq \emptyset$, or (ii) $B(x, \varepsilon) \cap B \neq \emptyset$ and $B(x, \varepsilon) \cap B^c \neq \emptyset$. Case(i) implies $x \in \partial A$ and case (ii) implies $x \in \partial B$. Thus, $x \in \partial A \cup \partial B$ and $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

(2) By (1),

$$\partial(A \cap B) = \partial\left[(A \cap B)^c\right] = \partial(A^c \cup B^c) \subseteq \partial A^c \cup \partial B^c = \partial A \cup \partial B$$

□

2.3 Sequences and Completeness

❑ Sequence

Definition 2.3.1. Let (M, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in M.

(1) We say that $\{x_n\}_{n=1}^{\infty}$ "*converges to x*" if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \qquad \text{whenever } n \geq N.$$

Denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) $\{x_n\}_{n=1}^{\infty}$ is said to be "*Cauchy*" if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \qquad \text{whenever } m,\ n \geq N.$$

(3) $\{x_n\}_{n=1}^{\infty}$ is said to be "*bounded*" if there is a point $x_0 \in M$ and a number $R > 0$ such that

$$d(x_n, x_0) < R \qquad \text{for every } n \in \mathbb{N}.$$

(4) (M, d) is said to be "*complete*" if every Cauchy sequence in M converges to a limit in M.

Remark. (1) Every Cauchy sequence is bounded.

(2) Every convergent sequence is Cauchy.

(3) If a subsequence of a Cauchy sequence converges, then this sequence converges.

(4) Let $\left\{\mathbf{x_k}\right\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ where $\mathbf{x_k} = (x_k^{(1)}, \cdots, x_k^{(n)})$. Then $\left\{\mathbf{x_k}\right\}_{k=1}^{\infty}$ is convergent (Cauchy) in $\mathbb{R}^n$ if and only if $\left\{x_k^{(i)}\right\}_{k=1}^{\infty}$ is convergent (Cauchy) in $\mathbb{R}$ for $i = 1, 2, \cdots, n$.

$$\text{Componentwise convergenece} \quad \Longleftrightarrow \quad \text{Convergence}$$

(5) $\mathbb{R}^n$ is complete.

Proposition 2.3.2. *Let (M, d) be a metric space and $A \subseteq M$. Let $x \in M$ be a point. Then $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ such that $\lim_{n\to\infty} x_n = x$.*

Proof. ($\Longrightarrow$) Since $x \in \overline{A}$, for every $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \emptyset$. We can choose any point in $B(x, \frac{1}{n}) \cap A$ and denote this point x_n. (We will prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.)

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then, for $n \geq N$, we have $x_n \in B(x, \frac{1}{n})$ and hence

$$d(x_n, x) < \frac{1}{n} < \varepsilon.$$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

($\Longleftarrow$) Let $\{x_n\}_{n=1}^{\infty} \subseteq A$ converges to x. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ when $n \geq N$. Therefore, $x_N \in B(x, \varepsilon) \cap A$. Since ε is an arbitrary positive number, we have $x \in \overline{A}$.

□

Proposition 2.3.3. *Let (M, d) be a metric space and $A \subseteq M$. Let $y \in M$ be a point. Then $y \in A'$ if and only if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq A$ converges to y where $y_n \neq y$ for every $n \in \mathbb{N}$.*

Proof. (Exercise) □

Remark. A is closed if and only if

$$A = \mathrm{cl}(A) = \overline{A}$$

if and only if

$$A = \left\{x \in M \,\middle|\, \text{there exists } \{x_n\}_{n=1}^{\infty} \subseteq A \text{ such that } \lim_{n\to\infty} x_n = x\right\}.$$

Proposition 2.3.4. *Let (M, d) be a metric space and $A \subseteq M$. Then A is closed if and only if every convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ converges to a limit in A.*

Proof. ($\Longrightarrow$) Since A is closed, $A = \overline{A}$. Let $\{x_n\}_{n=1}^{\infty} \subseteq A$ be a convergent sequence, say $\lim_{n\to\infty} x_n = x$. (We want to prove $x \in A$.)

For given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. That is, $x_n \in B(x, \varepsilon) \cap A$. Hence, $B(x, \varepsilon) \cap A \neq \emptyset$ and this implies that $x \in \overline{A} = A$.

($\Longleftarrow$) To prove that A^c is open. (That is, for every $y \in A^c$, there exists $\delta_y > 0$ such that $B(y, \delta_y) \subseteq A^c$ which is equivalent to $B(y, \delta_y) \cap A = \emptyset$.)

Assume that A^c is not open. There exists $y \in A^c$ such that $B(y, \frac{1}{n}) \nsubseteq A$ for every $n \in \mathbb{N}$ and then $B(y, \frac{1}{n}) \cap A \neq \emptyset$. Choose $y_n \in B(y, \frac{1}{n}) \cap A$ for every $n \in \mathbb{N}$. Then $\{y_n\}$ is a sequence in A which conveges to $y \in A^c$. It contradicts the hypothesis that $\{y_n\}$ converges in A.

Therefore , there exists $N \in \mathbb{N}$ such that $B(y, \frac{1}{N}) \cap A = \emptyset$. Then $B(x, \frac{1}{N}) \subseteq A^c$. We have A^c is open and A is closed. □

Example 2.3.5. $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that $x_n \to \sqrt{2}$ as $n \to \infty$. Then $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$ but the limit is not in $\mathbb{Q}$.

Theorem 2.3.6. *Let (M, d) be a complete metric space and $N \subseteq M$ be a closed subset. Then (N, d) is complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in N. (To prove that $\{x_n\}_{n=1}^{\infty}$ converges in N.)

Since $\{x_n\}_{n=1}^{\infty} \subseteq N \subseteq M$ is Cauchy and (M, d) is complete, there exists $x_0 \in M$ such that $\lim_{n\to\infty} x_n = x_0$. Moreover, since N is closed, $x_0 \in N$. Hence, $\{x_n\}_{n=1}^{\infty}$ converges in N and (N, d) is complete. □

Theorem 2.3.7. *Let (M, d) be a metric space and A is dense in M. (That is, $A \subseteq M \subseteq \overline{A}$.) If every Cauchy sequence in A converges in M, then (M, d) is complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. (To prove that $\{x_n\}_{n=1}^{\infty}$ is convergent in M.)

Step1: (To construct a new sequence $\{y_n\}_{n=1}^{\infty}$ in A which is Cauchy by using the denseness of A.)

Since A is dense in M, for every $n \in \mathbb{N}$, $B(x_n, \frac{1}{n}) \cap A \neq \emptyset$, say $y_n \in B(x_n, \frac{1}{n}) \cap A$. Hence, $\{y_n\}_{n=1}^{\infty}$ is a sequence in A.

Now, to show that $\{y_n\}_{n=1}^{i} nfty$ is Cauchy. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, for given $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \frac{\varepsilon}{3} \quad \text{whenever } m, n \geq N.$$

Choose $N_1 \in \mathbb{N}$ such that $N_1 > N$ and $\frac{1}{N_1} < \frac{\varepsilon}{3}$. Then if $m, n \geq N_1$,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n) \\ &\leq \frac{1}{n} + d(x_m, x_n) + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{\varepsilon}{3} + \frac{1}{N} < \varepsilon. \end{aligned}$$

Hence, $\{y_n\}$ is Cauchy.

Step2: Since $\{y_n\}_{n=1}^{\infty}$ is Cauchy in A, by the hypothesis, $\{y_n\}_{n=1}^{\infty}$ converges in M. That is, there exists $x_0 \in M$ such that $\lim_{n\to\infty} y_n = x_0$.

Step3: (To prove taht $\lim_{n\to\infty} x_n = x_0$.)

Since $\lim_{n\to\infty} y_n = x_0$, for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $\frac{1}{K} < \frac{\varepsilon}{2}$ and

$$d(y_n, x_0) < \frac{\varepsilon}{2} \quad \text{whenever } n \geq K.$$

Therefore, if $n \geq K$,

$$d(x_n, x_0) \leq d(x_n, y_n) + d(y_n, x_0) < \frac{1}{n} + \frac{\varepsilon}{2} \leq \frac{1}{K} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and hence (M, d) is complete. □

■ **Cluster points**

Definition 2.3.8. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space. We say that x is a "*cluster point*" of $\{x_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converging to x.

Example 2.3.9. 1 and -1 are cluster points of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

Proposition 2.3.10. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in a metric space (M, d), then*

(1) x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $d(x_n, x) < \varepsilon$.

(2) $\lim_{n\to\infty} x_n = x$ if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.

Proof. (Exercsie) □

Theorem 2.3.11. *The collection of all cluster points of a sequence is clsoed.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subseteq M$ be a sequence and $A = \left\{x \;\middle|\; x \text{ is a cluster point of } \{x_n\}_{n=1}^{\infty}\right\}$. For $y \notin A$, (that is, y is not a cluster point of $\{x_n\}_{n=1}^{\infty}$), there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \cap \left\{x_n \;\middle|\; n \in \mathbb{N}\right\} = \emptyset$. For $z \in B(y, \varepsilon)$, we want to prove that $z \notin A$.

Choose $r = \frac{1}{2}\big(\varepsilon - d(y, z)\big)$. Then $B(z, r) \subseteq B(y, \varepsilon)$. Hence, $B(z, r) \cap \left\{x_n \;\middle|\; n \in \mathbb{N}\right\} = \emptyset$. This implies that z is not a cluster point of $\{x_n\}_{n=1}^{\infty}$. Therefore, $B(y, \varepsilon) \cap A = \emptyset$. Then A^c is open and A is closed. □

2.4 Compact Sets

❑ Idea:

Some important results and properties only applied on the closed interval $[a, b] \subset \mathbb{R}$. For example,

(1) Every sequence has a subsequence which converges to a limit in $[a, b]$.

(2) Every open cover has a finitely many subcover.

(3) closed and bounded.

(4) Extreme Value Theorem

(5) Uniform Continuity.

Question: In a metric space (M, d), is there any set which has some similar properties?

❑ Sequentially Compact

Definition 2.4.1. Let (M, d) be a metric space. A subset $K \subseteq M$ is called "*sequentially compact*" if every sequence in K has a subsequence which converges to a point in K.

Example 2.4.2. $[a, b] \subset \mathbb{R}$ is sequentially compact. (a, b) is not sequentially compact.

Proposition 2.4.3. *Any closed and bounded set in* $(\mathbb{R}, |\cdot|)$ *is sequentially compact.*

Proof. Let A be a closed and bounded set in $(\mathbb{R}, |\cdot|)$ and $\{x_n\}_{n=1}^{\infty} \subseteq A$. Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded.
By Bolzano-Weierstrass theorem, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to a point x_0. Moreover, since A is closed, $x_0 \in A$ and thus A is sequentially compact. □

Proposition 2.4.4. *Let* (M, d) *be a metric space and* $K \subseteq M$ *be sequentially compact. If* E *is an infinite subset of* K*, then* E *has an accumulation point in* K*. That is,* $E' \cap K \neq \emptyset$*.*

Proof. (Exercise) □

Remark. In general, a closed and bounded set in a metric space (M, d) may not be sequentially compact. For example,

(1) Consider the space $C\big([0,1]\big) = \big\{f \mid f : [0,1] \to \mathbb{R} \text{ is continuous}\big\}$ with metric

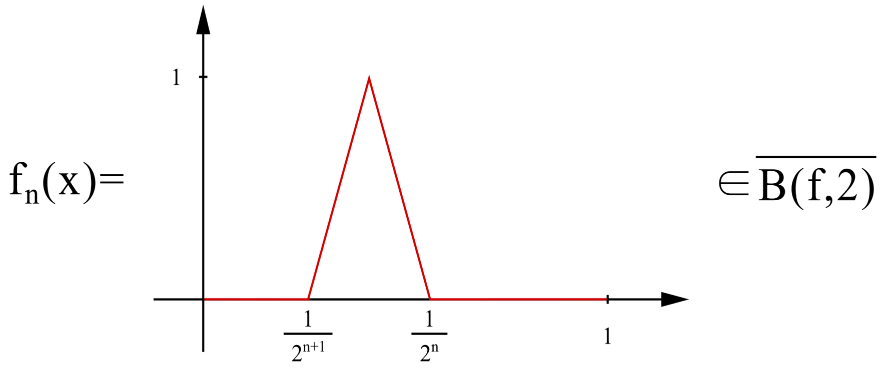
$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

Then $\Big(C\big([0,1]\big), d\Big)$ is a metric space. Let $f(x) \equiv 0$ on $[0, 1]$. Consider the set

$$B(f, 2) = \big\{g \mid g : [0,1] \to \mathbb{R} \text{ is continuous, } \max_{x \in [0,1]} |g(x) - f(x)| < 2\big\}.$$

Then $\overline{B(f, 2)}$ is a closed and bounded set in $C\big([0,1]\big)$.

Let $f_n(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2^{n+1}}] \\ 2^{n+2}(x - \frac{1}{2^{n+1}}) & \text{if } x \in [\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}] \\ 1 - 2^{n+2}(x - \frac{3}{2^{n+2}}) & \text{if } x \in [\frac{3}{2^{n+2}}, \frac{1}{2^n}] \\ 0 & \text{if } x \in [\frac{1}{2^n}, 1] \end{cases}$. Then $d(f_n, f_m) = \max_{x \in [0,1]} \big|f_m(x) - f_n(x)\big| = 1$ for every $m \neq n$. Therefore, $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\overline{B(f, 2)}$, but there exists no convergent subsequence since $\{f_n\}_{n=1}^{\infty}$ is not Cauchy. The set $\overline{B(f, 2)}$ is not sequentially compact.



(2) Let d be the discrete metric. The sequence $\{n\}_{n=1}^{\infty}$ in $(\mathbb{R}, d)$ is closed and bounded but there exists no convergent subsequence.

Proposition 2.4.5. *Let (M, d) be a metric space and $K \subseteq M$ be sequentially compact. Then K is closed and bounded.*

Proof. (**Closed**) To prove that K contains all its limit points. That is, if $\{x_n\}_{n=1}^{\infty} \subseteq K$ converges to x_0 then $x_0 \in K$.

Let $\{x_n\}_{n=1}^{\infty} \subseteq K$ be a sequence which converges to x_0. Since K is sequentially compact, $\{x_n\}_{n=1}^{\infty}$ contains a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to a point in K, say $\lim\limits_{k\to\infty} x_{n_k} = y_0 \in K$.

Since $\{x_n\}_{n=1}^{\infty}$ converges to x_0, we have

$$x_0 = \lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = y_0.$$

Hence $x_0 \in K$ and K is closed.

(**Bounded**) Assume that K is not bounded. Choose a point $x_1 \in K$. There exists a point $x_2 \notin B(x_1, 1)$. Again, there exists $x_3 \notin B\big(x_1, d(x_1, x_2) + 1\big)$ since K is not bounded. Continue this process, we can find a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that

$$d(x_1, x_n) > d(x_1, x_{n-1}) + 1 \quad \text{for } n = 2, 3, \cdots$$

Hence, $d(x_n, x_m) > 1$ for every $m \neq n$. The sequence $\{x_n\}_{n=1}^{\infty}$ cannot contain a convergent subsequence and this contradicts the sequentially compactness of K. We obtain that K is bounded. □

Remark. In a metric space (M, d),

$$\text{Sequentially Compact} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \text{Closed and Bounded}$$

In particular, in $(\mathbb{R}^n, \|\cdot\|)$, the converse "$\Longleftarrow$" holds.

Corollary 2.4.6. *If $K \subseteq \mathbb{R}$ is sequentially compact, then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Since $K \subseteq \mathbb{R}$ is sequentially compact, K is closed and bounded in $\mathbb{R}$. Hence, $-\infty < \inf K \leq \sup K < \infty$.

Let $\{x_n\}_{n=1}^{\infty} \subseteq K$ be a sequence which converges to $\inf K$. Since K is closed, $\inf K = \lim\limits_{n\to\infty} x_n \in K$. Similarly, $\sup K \in K$. □

Theorem 2.4.7. *Let (M,d) be a metric space and $K \subseteq M$ be sequentially compact. Then every Cauchy sequence in K is convergent in K.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subseteq K$ be a Cauchy sequence. Then $\{x_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit in K, say $\lim\limits_{k\to\infty} x_{n_k} = x \in K$.

Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, $\{x_n\}_{n=1}^{\infty}$also converges to x.

□

❑ Compact Sets

Definition 2.4.8. Let (M,d) be a metric space and $A \subseteq M$.

(1) We say that a collection of sets $\{U_\alpha \mid \alpha \in I\}$is a cover of A if $A \subseteq \bigcup\limits_{\alpha \in I} U_\alpha$.

(2) In particular, if all U_α's are open sets, we say that $\{U_\alpha\}_{\alpha \in I}$ is an "*open cover*" of A.

(3) Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of A. We say that $\{U_\alpha \mid \alpha \in J\}$ is a "*subcover*" of $\{U_\alpha \mid \alpha \in I\}$ if

  (i) $\{U_\alpha\}_{\alpha \in J} \subseteq \{U_\alpha\}_{\alpha \in I}$

  (ii) $\{U_\alpha\}$ is a cover of A,

(4) We say that $\{U_\alpha \mid \alpha \in I\}$ is a "*finite cover*" of A if

  (i) $\{U_\alpha \mid \alpha \in I\}$ is a cover of A

  (ii) the number of $\{U_\alpha \mid \alpha \in I\}$ is finite.

Definition 2.4.9. (Compact) Let (M,d) be a metric space. A set $K \subseteq M$ is called "*compact*" if every open cover of K contains a finite subcover.

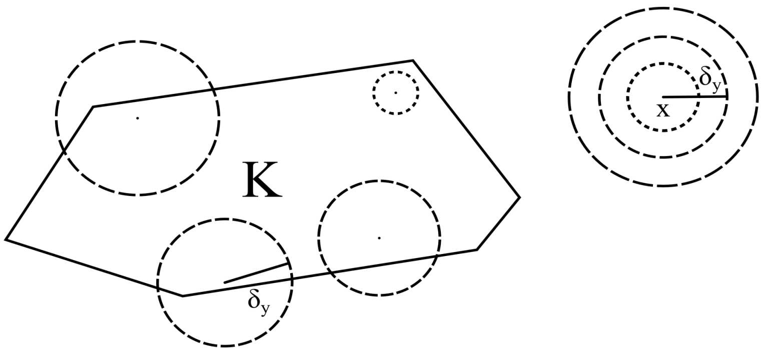
Example 2.4.10. $[a,b] \subset (\mathbb{R}, |\cdot|)$ is compact and $(a,b) \subset (\mathbb{R}, |\cdot|)$ is not compact.

Let $U_n = (a + \frac{1}{n}, b - \frac{1}{n})$. Then $(a,b) \subseteq \bigcup\limits_{n=1}^{\infty} U_n$. But there is no finite subcover of $\{U_n\}_{n=1}^{\infty}$ which covers (a,b).

Lemma 2.4.11. *Every compact set in a metric space is closed and bounded.*

Proof. (**Closed**) Let (M,d) be a metric space and $K \subseteq M$ be compact. Let $x \in K^c$. (We want to prove that there exists $r > 0$ such that $B(x,r) \subseteq K^c$.)

Since $x \in K^c$, for every $y \in K$, choose $0 < \delta_y < \frac{1}{2}d(x,y)$. Then $B(x,\delta_y) \cap B(y,\delta_y) = \emptyset$. Hence, $\{B(y,\delta_y)\}_{y \in K}$ is an open cover of K. That is, $K \subseteq \bigcup\limits_{y \in K} B(y,\delta_y)$.



Since K is compact, there exists a finite subcover of K, say $K \subseteq \bigcup_{i=1}^{N} B(y_i, \delta_{y_i})$. Let $= \min(\delta_{y_1}, \cdots, \delta_{y_N})$. Since $B(x, \delta_{y_i}) \cap B(y_i, \delta_{y_i})$ for every $i = 1, 2, \cdots, N$, we have

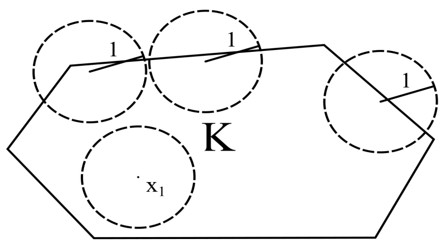
$$\underbrace{\Big(\bigcap_{i=1}^{N} B(x, \delta_{y_i})\Big)}_{=B(x,r)} \cap \underbrace{\Big(\bigcup_{i=1}^{N} B(y_i, \delta_{y_i})}_{\supseteq K} = \emptyset.$$

Therefore, $B(x, r) \cap K = \emptyset$ and then $B(x, r) \subseteq K^c$. This implies that x is an interior point of K^c. We have K^c is open and K is closed.

(**Bounded**) Since $K \subseteq \bigcup_{x \in K} B(x, 1)$, $\big\{B(x, 1)\big\}_{x \in K}$ is an open cover of K. The compactness of K implies that there exists $x_1, x_2, \cdots, x_N \in K$ such that $K \subseteq \bigcup_{i=1}^{N} B(x_i, 1)$. Let $M = \max_{1 \le i \le N} d(x_1, x_i) + 1$. Then for $y \in K$, $y \in B(x_i, 1)$ for some $i \in \{1, \cdots, N\}$. We have

$$d(x_1, y) \le d(x_1, x_i) + d(x_i, y) \le M.$$

Hence, $K \subseteq \bigcup_{i=1}^{N} \subseteq B(x_1, M)$ and K is bounded.



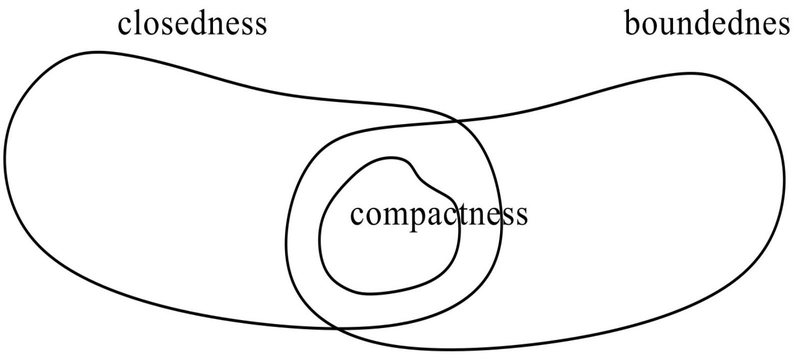
□

Remark. In a metric space,

Compactness $\Longrightarrow$ Closedness and Boundedness

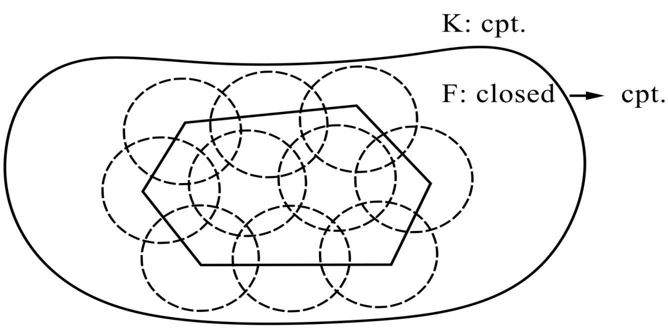
$\not\Longleftarrow$

Example 2.4.12. Consider the discrete metric d on $\mathbb{R}$. $\mathbb{R}$ is closed and bounded. Observe that $\bigcup_{x\in\mathbb{R}} B(x, \frac{1}{2}) \supseteq \mathbb{R}$. But $\{B(x, \frac{1}{2})\}_{x\in\mathbb{R}}$ cannot contain a finite subcover of $\mathbb{R}$. Hence, $\mathbb{R}$ is not compact.



Lemma 2.4.13. *Every closed subset in a compact set is compact.*

Proof. Let K be a compact set and $F \subseteq K$ be a closed subset of K. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of F. (We want to prove that $\{U_\alpha\}_{\alpha\in I}$ contains a finite subcover of F.)



Since F is closed , F^c is open. Define $V_\alpha = U_\alpha \cup F^c$ for all $\alpha \in I$. Then V_α is open for every $\alpha \in I$ since U_α and F^c are open. Consider

$$K \subseteq F \cup F^c \subseteq \Big(\bigcup_{\alpha\in I} U_\alpha\Big) \cup F^c = \bigcup_{\alpha\in I} \big(U_\alpha \cup F^c\big) = \bigcup_{\alpha\in I} V_\alpha.$$

Then $\{V_\alpha\}_{\alpha\in I}$ is an open cover of K. Since K is compact, $\{V_\alpha\}_{\alpha\in I}$ contains a finite subcover of K, say

$$F \subseteq K \subseteq \bigcup_{i=1}^{N} V_i = \Big(\bigcup_{i=1}^{N} U_i\Big) \cup F^c.$$

Hence, $F \subseteq \bigcup_{i=1}^{N} U_i$ and F is compact. □

Remark. In a general metric space, not all closed set is compact. For exmaple, $(\mathbb{R}, |\cdot|)$ is closed but not compact.

Even a closed and bounded set in a general metric space may not be compact. For example, $(\mathbb{R}, d)$ with discrete metric d.

■ **Heine-Borel Theorem**

Lemma 2.4.14. $[a, b] \subseteq \mathbb{R}$ *is compat.*

Proof. Assume that $[a, b]$ is not compact. There there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of $[a, b] := I_0$ which does not contain a finite subcover. Then at least one of $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ cannot be covered by finitely many elements in $\{U_\alpha\}_{\alpha\in I}$. We call such an interval $I_1 = [a_1, b_1]$. Then $I_1 \subseteq I_0$ and $|I_1| = \frac{1}{2}|I_0|$.

Again, at least one of $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$ cannot be covered by finitely many elements in $\{U_\alpha\}_{\alpha\in I}$. We call such an interval $I_2 = [a_2, b_2]$. Then $I_2 \subseteq I_1$ and $|I_2| = \frac{1}{2}|I_1|$.

Continue this process, we can choose

$$\cdots \subseteq I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_1 \subseteq I_0 \quad \text{and} \quad |I_{k+1}| = \frac{1}{2}|I_k|$$

such that each I_i connot be covered by finitely many elements of $\{U_\alpha\}_{\alpha\in I}$.

By Nested Interval Theorem, there exists $x_0 \in I_k$ for every $k \in \mathbb{N}$. Since $[a, b] \subseteq \bigcup_{\alpha\in I} U_\alpha$, there exists $\alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U_{\alpha_0}$. Also, $\lim_{k\to\infty} |I_k| = 0$. There exists $N \in \mathbb{N}$ such that if $k \geq N$, $|I_k| < \frac{\delta}{2}$. This implies that $I_k \subseteq (x-\delta, x+\delta) \subseteq U_{\alpha_0}$. It contradicts that each I_k cannot be covered by finitely many subcover. Hence, $[a, b]$ is compact. □

Remark. Every n-cell $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is compact.

Corollary 2.4.15. *Every closed and bounded subset in* $\mathbb{R}$ *(or* $\mathbb{R}^n$*) is compact.*

Proof. Let $K \subseteq \mathbb{R}$ be closed and bounded. There exists $M > 0$ such that $K \subseteq [-M, M]$. Since $[-M, M]$ is compact and K is a closed subset of $[-M, M]$, we have K is compact.

□

Theorem 2.4.16. *(Heine-Borel Theorem) Let S be a subset of* $\mathbb{R}^n$*. Then S is closed and bounded if and only if S is compact.*

Corollary 2.4.17. *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof. Since K is compact, it is closed. Also, F is closed and hence $F \cap K$ is closed. Then $F \cap K \subseteq K$ is a closed subset of K and it is also compact.

□

❑ Finite Intersection Property

Definition 2.4.18. Let (M,d) be a metric space, $A \subseteq M$ and $\{F_\alpha\}_{\alpha\in I}$ be a collection of subsets of M.

(1) We say that $\{F_\alpha\}_{\alpha\in I}$ have the "*finite intersection property*" if the intersection over any finite subcollection of $\{F_\alpha\}_{\alpha\in I}$ is nonempty. That is,

$$\bigcap_{\alpha\in J} F_\alpha \neq \emptyset \quad \text{for any finite subcollection } J \subseteq I.$$

(2) $\{F_\alpha\}_{\alpha\in I}$ is said to have the "*finite intersection property for A*" if the intersection over any finite subcollection of $\{F_\alpha\}_{\alpha\in I}$ with A is nonempty. That is,

$$\Big(\bigcap_{\alpha\in J} F_\alpha\Big) \cap A \neq \emptyset \quad \text{for any finite subcollection } J \subseteq I.$$

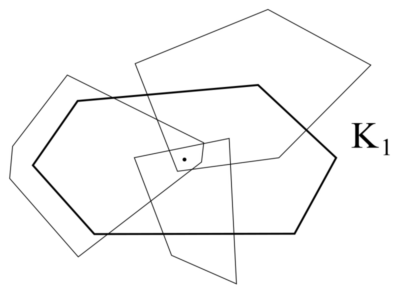
Theorem 2.4.19. *Let (M,d) be a metric space and $\{K_\alpha\}_{\alpha\in I}$ be a collection of compact sets in M. If $\{K_\alpha\}_{\alpha\in I}$ have the finite intersection property, then*

$$\bigcap_{\alpha\in I} K_\alpha \neq \emptyset.$$

Proof. Fix a member K_1 of $\{K_\alpha\}_{\alpha\in I}$. Assume that $K_1 \cap \Big(\bigcap_{\alpha\in I \alpha\neq 1} K_\alpha\Big) = \bigcap_{\alpha\in I} K_\alpha = \emptyset.$

Then

$$K_1 \subseteq \Big(\bigcap_{\substack{\alpha\in I\\ \alpha\neq 1}} K_\alpha\Big)^c = \bigcup_{\substack{\alpha\in I\\ \alpha\neq 1}} K_\alpha^c.$$



Since K_α is compact for $\alpha \in I$, it is closed and hence K_α^c is open. The collection $\{K_\alpha^c\}_{\substack{\alpha\in I\\ \alpha\neq 1}}$ is an open cover of K_1.

Since K_1 is compact, $\{K_\alpha\}_{\substack{\alpha\in I\\ \alpha\neq 1}}$ has a finite subcover of K_1, say

$$K_1 \subseteq \bigcup_{i=1}^{N} K_{\alpha_i}^c = \Big(\bigcap_{i=1}^{N} K_{\alpha_i}\Big)^c.$$

Hence, $K \cap \Big(\bigcap_{i=1}^{N} K_{\alpha_i}\Big) = \emptyset$. It contradicts that $\{K_\alpha\}_{\alpha\in I}$ has the finite intersection property. Hence,

$$\bigcap_{\alpha\in I} K_\alpha \neq \emptyset.$$

□

Remark. The result is false if not all $\{K_\alpha\}_{\alpha\in I}$ are compact. For example, $K_n = (0, \frac{1}{n})$. Then $\{K_n\}_{n=1}^\infty$ finite intersection property, but $\bigcap_{n=1}^\infty K_n = \emptyset$.

Theorem 2.4.20. *Let (M, d) be a metric space and $K \subseteq M$. Then K is compact if and only if every collection of closed sets with the finite intersection property for K has nonempty intersection with K. That is, for a collection of closed sets $\{F_\alpha\}_{\alpha\in I}$, if $K \cap \left(\bigcap_{\alpha\in J} F_\alpha\right) \neq \emptyset$ for any finite subcollection J then $K \cap \left(\bigcap_{\alpha\in I} F_\alpha\right) \neq \emptyset$.*

Proof. ($\Longrightarrow$) Define $K_\alpha := K \cap F_\alpha$. Then K_α is compact and $\{K_\alpha\}_{\alpha\in I}$ has finite intersection property. Then

$$\bigcap_{\alpha\in I} K_\alpha = K \cap \left(\bigcap_{\alpha\in I} F_\alpha\right) \neq \emptyset.$$

($\Longleftarrow$) Let $\left\{U_\alpha\right\}_{\alpha\in I}$ be an open cover of K. That is, $K \subseteq \bigcup_{\alpha\in I} U_\alpha$. Then

$$K \cap \left(\bigcup_{\alpha\in I} U_\alpha\right)^c = K \cap \left(\bigcap_{\alpha\in I} U_\alpha^c\right) = \emptyset.$$

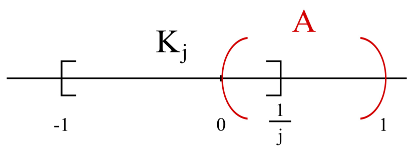
Since U_α is open for $\alpha \in I$, U_α^c is closed for $\alpha \in I$. By the hypothesis, the collection $\left\{U_\alpha^c\right\}_{\alpha\in I}$ does not have finite intersection property for K. That is, there exists a finite subcollection $\left\{U_\alpha^c\right\}_{\alpha\in J}$ such that

$$\emptyset = K \cap \left(\bigcap_{\alpha\in J} U_\alpha^c\right) = K \cap \left(\bigcup_{\alpha\in J} U_\alpha\right)^c.$$

Hence, $K \subseteq \bigcup_{\alpha\in J} U_\alpha$ and K is compact. □

Example 2.4.21. $K = (0, 1)$, $F_j = [-1, \frac{1}{j}]$. For any finite subcollection of $\left\{F_j\right\}_{j\in\mathbb{N}}$, say $\left\{F_{j_1}, F_{j_2}, \cdots, F_{j_N} \,\middle|\, j_1 < j_2 < \cdots < j_N\right\}$. Then

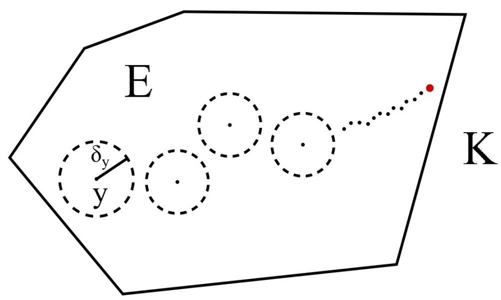
$$K \cap \left(\bigcap_{i=1}^N F_{j_i}\right) = [0, \frac{1}{j_n}] \neq \emptyset.$$



But $K \cap \left(\bigcap_{i=1}^\infty F_{j_i}\right) = (0, 1) \cap [-1, 0] = \emptyset$. Hence, K is not compact.

Theorem 2.4.22. *If E is an infinite subset of a compact set K, then E has an accumulation point in K. (That is, $E' \cap K \neq \emptyset$.)*

Proof.



Assume that $E' \cap K = \emptyset$. Then for every $y \in K$, $y \notin E'$, there exists $\delta_y > 0$ such that $K \cap \left(B(y, \delta_y)\backslash\{y\}\right) = \emptyset$. Therefore, $B(y, \delta_y)$ contains at most one element in E (namely, y if $y \in E$.)

Since E contains infinitely many elements, E cannot be covered by any finite collecction of $\{B(y,\delta_y)\}_{y\in K}$. On the other hand, since $K \subseteq \bigcup_{y\in K} B(y,\delta_y)$ and K is compact, there exists $y_1,\cdots,y_N \in K$ such that $\bigcup_{i=1}^{N} B(y_i,\delta_{y_i}) \supseteq K \supseteq E$. It contradicts the above argument that K cannot be covered by $\{B(y,\delta_y)\}_{y\in K}$. Hence, $E' \cap K \neq \emptyset$.

□

Theorem 2.4.23. *If $\{K_n\}_{n=1}^{\infty}$ is a sequence of nonempty compact sets in a metric space and $K_{n+1} \subseteq K_n$ for $n = 1,2,\cdots$, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.*

Proof. (Exercise) □

Corollary 2.4.24. *Let $\{U_k\}_{k=1}^{\infty}$ be a collection of open sets in a metric space (M,d) such that $U_k \subseteq U_{k+1}$ and U_k^c is compact for all $k \in \mathbb{N}$. Then $\bigcup_{k=1}^{\infty} U_k \neq M$.*

Proof. Since $U_k \subseteq U_{k+1}$ for all $k \in \mathbb{N}$, $U_{k+1}^c \subseteq U_k^c$ for all $k \in \mathbb{N}$. Hence, $\{U_k^c\}_{k=1}^{\infty}$ have finite intersection property. Since every U_k^c is compact, $\bigcap_{k=1}^{\infty} U_k^c \neq \emptyset$. Therefore,

$$\left(\bigcap_{k=1}^{\infty} U_k^c\right)^c = \bigcup_{k=1}^{\infty} U_k \neq M.$$

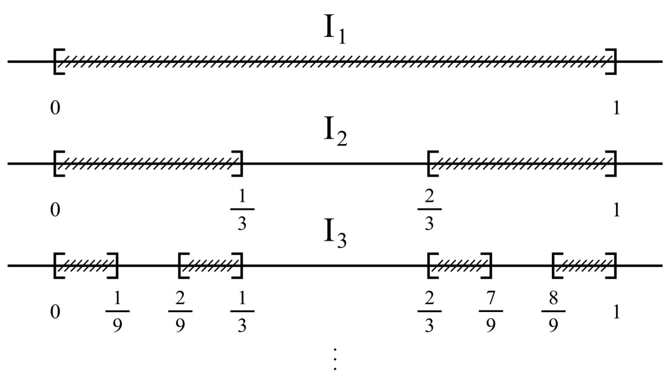
□

Example 2.4.25. Let $U_k = (-\infty, -\frac{1}{k}) \cup (\frac{1}{k}, \infty)$. Then $U_k \subseteq U_{k+1}$ and $U_k^c = [-\frac{1}{k}, \frac{1}{k}]$ is compact. Moreover,

$$\bigcup_{k=1}^{\infty} U_k = (-\infty, 0) \cup (0, \infty) \neq \mathbb{R}.$$

❑ Applications

1. **Cantor Set**



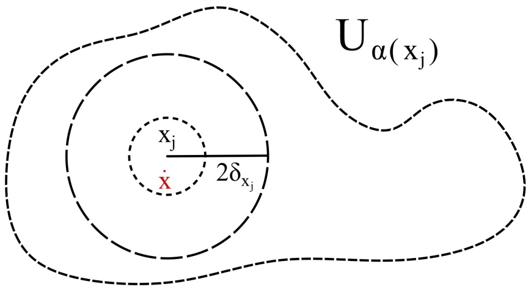
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$$

Each I_k is compact and any finite intersection of $\{I_k\}_{k=1}^{\infty}$ is nonempty. Then $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$. In fact, $C = \bigcap_{k=1}^{\infty} I_k$ is called "*Cantor set*" which is perfect, and uncountable.

2. **Lebesgue Covering Theorem**

Let K be compact and $\{U_\alpha\}_{\alpha \in I}$ be an open cover of K. Then there exists $r > 0$ such that for each $x \in K$, $B(x, r) \subseteq U_{\alpha(x)}$ for some $\alpha(x) \in I$.

Proof.
For $x \in K$, there exists $\alpha(x) \in I$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is open, there exists $\delta_x > 0$ such that $B(x, 2\delta_x) \subseteq U_{\alpha(x)}$. Then $\bigcup_{x \in K} B(x, \delta_x) \supseteq K$.
Since K is compact, there exists $x_1, \cdots, x_N \in K$ such that $\bigcup_{i=1}^{N} B(x_i, \delta_x) \supseteq K$.
Let $r = \min(\delta_{x_1}, \cdots, \delta_{x_N})$. For $x \in K$, there exists $1 \leq j \leq N$ such that $x \in B(x_j, \delta_{x_j})$. Hence,



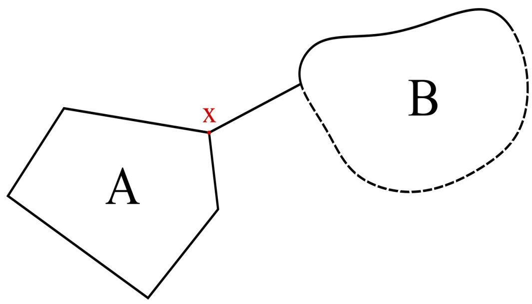
$$B(x, r) \subseteq B(x_j, 2\delta_{x_j}) \subseteq U_{\alpha(x_j)}.$$

□

Remark. The supremum of all such r is called the Legesgue number for the cover $\{U_\alpha\}_{\alpha \in I}$.

3. **Nearest Point Throrem** Let (M, d) be a metric space and $\emptyset \neq A \subseteq M$ be compact and $B \subseteq M$ there exists $x_0 \in A$ such that $d(A, B) = d(x_0, B)$.

Let $A_k = \left\{ x \in A \;\middle|\; d(x, B) \leq d(A, B) + \frac{1}{k} \right\}$. Then A_k is nonempty and closed. Also, A_k is compact since A is compact and $A_k \subseteq A$. Moreover, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. The collection $\{A_k\}_{k=1}^{\infty}$ has finite intersection property.
Hence $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$.
Let $x_0 \in \bigcap_{k=1}^{\infty} A_k \subseteq A$. Then



$$d(A, B) \leq d(x_0, B) \leq d(A, B) = \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

We have $d(A, B) = d(x_0, B)$.

Remark. (1) If A is compact and B is closed, then there exists $x_0 \in A$ and $y_0 \in B$ such that

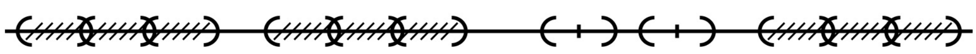
$$d(x_0, y_0) = d(A, B).$$

(2) The same results are true if replacing compactness of A by sequentially compactness of A.

❑ Totally Boundedness

Definition 2.4.26. Let (M, d) be a metric space. We say that a set $A \subseteq M$ is "*totally bounded*" if for every $r > 0$, there exists finitely many balls with radius r, say $\left\{B(x_i, r)\right\}_{i=1}^{N}$ where $x_1, \cdots, x_N \in M$ such that $A \subseteq \bigcup_{i=1}^{N} B(x_i, r)$.

Remark. (1) Every bounded set in $\mathbb{R}$ is totally bounded. (Exercise)



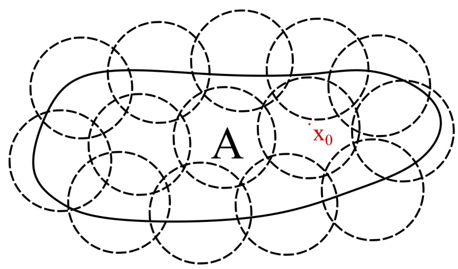
(2) Let (M, d) be a metric space with discrete metric. Then every set is bounded. But a set consisting of infinitely many points is NOT totally bounded.

Proof. Let $r = \frac{1}{2}$. Then $B(x, \frac{1}{2}) = \{x\}$. If the size of A is infinite, then A cannot be covered by finitely many balls with radius $\frac{1}{2}$. □

Proposition 2.4.27. *Every totally bounded set in a metric space is bounded.*

Proof.
Let (M, d) be a metric space and $A \subseteq M$ be totally bounded. For $r = 1$, there exists $x_0, \cdots, x_N \in M$ such that $A \subseteq \bigcup_{i=0}^{N} B(x_i, 1)$.
Let $L = \max_{0 \le i \le N} d(x_0, x_i) + 1$. For $x \in A$, there exists $k \in \{0, 1, \cdots, N\}$ such that $x \in B(x_k, 1)$ and

$$d(x_0, x) \le d(x_0, x_k) + d(x_k, x) \le \max_{0 \le i \le N} d(x_0, x_1) + 1 = L.$$



We have $A \subseteq B(x_0, L)$ and hence A is bounded. □

Remark. In a metric space,

totally bounded $\Longrightarrow$ bounded
$\not\Longleftarrow$ (discrete metric)

In $(\mathbb{R}^n, \|\cdot\|)$,

totally bounded $\Longleftrightarrow$ bounded

Proposition 2.4.28. *Every subset of a totally bounded set is totally bounded.*

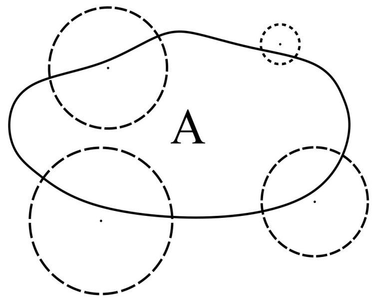
Proof. (Exercise) □

Proposition 2.4.29. *Let (M, d) be a metric space and $A \subseteq M$. Then A is totally bounded if and only if for every $r > 0$, there exists finitely many balls with centers in A and radius r, say $\left\{B(y_i, r)\right\}_{i=1}^{N}$ where $y_1, \cdots y_N \in A$, such that*

$$A \subseteq \bigcup_{i=1}^{N} B(y_i, r).$$

Proof.
($\Longleftarrow$) It is trival by definition.

($\Longrightarrow$) Since A is totally bounded, there exists $x_1, \cdots, x_N \in M$ such that $\bigcup_{i=1}^{N} B(x_i, \frac{r}{2}) \supseteq A$. W.L.O.G, we may assume $B(x_i, \frac{r}{2}) \cap A \neq \emptyset$ for every $i = 1, \cdots, N$.



Choose $y_i \in B(x_i, \frac{r}{2}) \cap A$. Then $B(x_i, \frac{r}{2}) \subseteq B(y_i, r)$. We have

$$A \subseteq \bigcup_{i=1}^{N} B(x_i, \frac{r}{2}) \subseteq \bigcup_{i=1}^{N} B(y_i, r).$$

□

Lemma 2.4.30. *(1) A compact set in a metric space is totally bounded.*

(2) A sequentially compact set in a metric space is totally bounded.

Proof. Let (M, d) be a metric space.

(1) Let $K \subseteq M$ be compact. For given $r > 0$, $K \subseteq \bigcup_{x \in K} B(x, r)$. Then $\left\{B(x, r)\right\}_{x \in K}$ is an open cover of K. There exists a finite subcover, say $\left\{B(x_i, r) \,\middle|\, \text{for some } x_1, \cdots, x_N \in K\right\}$. Hence, K is totally bounded.

(2) Let $K \subseteq M$ be sequentially compact. Assume that K is not totally bounded. Then there exists $r > 0$ such that for any set consisting of finitely many point, say $y_1, y_2, \cdots, y_N$,

$$K \nsubseteq \bigcup_{i=1}^{N} B(y_i, r).$$

Choose (arbitrarily) a point $x_1 \in K$. Since $K \nsubseteq B(x_1, r)$, there exists $x_2 \in K \backslash B(x_1, r)$. Also, $K \nsubseteq \bigcup_{i=1}^{2} B(x_i, r)$. There exists $x_3 \in K \backslash \bigcup_{i=1}^{2} B(x_i, r)$.

Continue this process, $K \nsubseteq \bigcup_{i=1}^{n} B(x_i, r)$. There exists $x_{n+1} \in K \backslash \bigcup_{i=1}^{n} B(x_i, r)$ and hence $d(x_{n_1}, x_n) > r$ for $i = 1, 2, \cdots, n$. Then $\{x_n\}_{n=1}^{\infty} \subseteq K$ is a sequence with $d(x_i, x_j) > r$ for $i \neq j$. Hence $\{x_n\}_{n=1}^{\infty}$ cannot contain a convergent subsequence. It contradicts that K is sequentially compact.

□

Theorem 2.4.31. *Let (M, d) be a metric space and $K \subseteq M$. The following statements are equivalent.*

(1) K is compact.

(2) K is sequentially compact.

(3) K is totally bounded and complete.

In addition, every one of (1), (2) and (3) implies

(4) K is closed and bounded.

Moreover, if $K \subseteq (\mathbb{R}^n, \|\cdot\|)$, then (1)-(4) are equivalent.

Proof. "(1) $\Longrightarrow$ (2)"
Let $\{x_n\}_{n=1}^{\infty} \subseteq K$ be a sequence. Suppose that $\{x_n\}_{n=1}^{\infty}$ contains at most finitely many different elements. There exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_1} = x_{n_2} = \cdots = x_{n_k} = \cdots$ for every $k \in \mathbb{N}$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ converges.

We may assume that $\{x_n\}_{n=1}^{\infty}$ contains infinitely many different elements. By Theorem 2.4.22, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $x_0 \in K$. Therefore, K is sequentially compact.

"(2) $\Longrightarrow$ (3)"
Let K be sequentially compact. By Theorem 2.4.7 and Lemma 2.4.30, K is complete and totally bounded.

"(3) $\Longrightarrow$ (1)"
Let K be totally bounded and complete. Assume that K is not compact. Then there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of K which does not contain a finite subcover.

Since K is totally bounded, for $r = 1$, there exists $y_1^{(1)}, \cdots, y_{N_1}^{(1)} \in K$ such that $K \subseteq \bigcup_{i=1}^{N_1} B(y_i^{(1)}, 1)$. Then there exists $1 \le \ell_1 \le N_1$ such that $K \cap B(y_{\ell_1}^{(1)}, 1)$ cannot be covered by finitely many elements of $\{U_\alpha\}_{\alpha \in I}$.

Since $K \cap B(y_{\ell_1}^{(1)}, 1)$ is totally bounded, for $r = \frac{1}{2}$, there exists $y_1^{(2)}, \cdots, y_{N_2}^{(2)} \in K \cap B(y_{\ell_1}^{(1)}, 1)$ such that $K \cap B(y_{\ell_1}^{(1)}, 1) \subseteq \bigcup_{i=1}^{N_2} B(y_i^{(2)}, \frac{1}{2})$. Hence, there exists $1 \le \ell_2 \le N_2$ such that $K \cap B(y_{\ell_1}^{(1)}, 1) \cap B(y_{\ell_2}^{(2)}, \frac{1}{2})$ cannot be coverd by finitely many elements of $\{U_\alpha\}_{\alpha \in I}$.

Continue this process, we can choose $z_1, z_2, \cdots \in K$ such that $z_n \in K \cap \Big(\bigcap_{i=1}^{n-1} B(z_i, \frac{1}{i})\Big)$ and $K \cap \Big(\bigcap_{i=1}^{n-1} B(z_i, \frac{1}{i})\Big)$ cannot be covered by finitely many elements of $\{U_\alpha\}_{\alpha \in I}$.

Claim: $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
Proof of claim: Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. For $m, n \geq N$,

$$z_m, z_n \in K \cap \Big(\bigcap_{i=1}^{N} B(z_i, \frac{1}{i})\Big) \subseteq K \cap B(z_N, \frac{1}{N}).$$

Then

$$d(z_m, z_n) \leq d(z_m, z_N) + d(z_N, z_m) \leq \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Hence, $\{z_n\}_{n=1}^{\infty}$ is Cauchy and the claim is proved.

Since K is complete, there exists $z \in K$ such that $\lim_{n\to\infty} z_n = z$. Also, since $\bigcup_{\alpha\in I} U_\alpha \supseteq K$, there exists $\alpha_0 \in I$ such that $z \in U_{\alpha_0}$. Moreover, there exists $\delta > 0$ such that $B(z, \delta) \subseteq U_{\alpha_0}$ since U_{α_0} is open. Choose $L \in \mathbb{N}$ such that $\dfrac{1}{L} < \dfrac{\delta}{2}$ and for $n \geq L$, $d(z_n, z) < \frac{\delta}{2}$.

For $x \in B(z_L, \frac{1}{L})$,

$$d(x, z) \leq d(x, z_L) + d(z_L, z) \leq \frac{1}{L} + \frac{\delta}{2} < \delta.$$

Then $B(z_L, \frac{1}{L}) \subseteq B(x, \delta) \subseteq U_{\alpha_0}$. We have

$$K \cap \Big(\bigcap_{i=1}^{L} B(z_i, \frac{1}{i})\Big) \subseteq K \cap B(z_L, \frac{1}{L}) \subseteq U_{\alpha_0}.$$

It contradicts that $K \cap \Big(\bigcap_{i=1}^{L} B(z_i, \dfrac{1}{i})\Big)$ cannot be covered by finitely many elements of $\{U_\alpha\}_{\alpha\in I}$. Therefore, K is compact.

In $(\mathbb{R}^n, \|\cdot\|)$, (1) $\Longleftrightarrow$ (4) is proved by Heine-Borel Theorem. □

Remark. Let $\{x_n\}_{n=1}^{\infty} \subseteq (M, d)$ converge to x. Then $A = \{x_1, x_2, \cdots\} \cup \{x\}$ is sequentially compact. Hence, A is compact.

Definition 2.4.32. Let (M, d) be a metric space, $U \subseteq M$ be open and $A \subseteq M$.

(1) A is called "*precompact*" if $\overline{A}$ is compact.

(2) Suppose $A \subseteq M$. We say that A is compactly contained in U if A is precompact and $\overline{A} \subseteq U$. Denoted by $A \subset\subset U$.

Proposition 2.4.33. *Every bounded set A in $\mathbb{R}^n$ is precompact.*

Proof. (Exercise) □

Example 2.4.34. Define

$$\ell^\infty(\mathbb{R}) := \Big\{(a_1, a_2, a_3, \cdots) \;\Big|\; a_i \in \mathbb{R} \text{ for every } i \in \mathbb{N} \text{ and } \sup_{i\in\mathbb{N}} |a_i| < \infty\Big\}.$$

For $\mathbf{a} = (a_1, a_2, a_3, \cdots)$, $\mathbf{b} = (b_1, b_2, b_3, \cdots) \in \ell^\infty(\mathbb{R})$, define

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots) \quad \text{and} \quad \alpha\mathbf{a} = (\alpha a_1, \alpha a_2, \cdots) \quad \text{for } \alpha \in \mathbb{R}.$$

Then $\ell^\infty(\mathbb{R})$ is a vector space. Define $\|\cdot\| : \ell^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$\|\mathbf{a}\| = \sup_{i \in \mathbb{N}} |a_i|.$$

Then $\big(\ell^\infty(\mathbb{R}), \|\cdot\|\big)$ is a normed space.

(1) $\big(\ell^\infty(\mathbb{R}), \|\cdot\|\big)$ is complete.

(2) Let

$$\begin{aligned}
A &= \Big\{(a_1, a_2, a_3, \cdots) \in \ell^\infty \ \Big|\ |a_k| \le \frac{1}{k}\Big\} \quad (\text{ex: } (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots) \in A) \\
B &= \Big\{(a_1, a_2, a_3, \cdots) \in \ell^\infty \ \Big|\ \lim_{k\to\infty} a_k = 0\Big\} \quad (\text{ex: } (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots) \in B) \\
C &= \Big\{(a_1, a_2, a_3, \cdots) \in \ell^\infty \ \Big|\ \lim_{k\to\infty} a_k \text{ converges}\Big\} \quad (\text{ex: } (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots) \in C) \\
D &= \Big\{(a_1, a_2, a_3, \cdots) \in \ell^\infty \ \Big|\ \sup_k |a_k| = 1\Big\} \quad (\text{ex: } (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots) \in D)
\end{aligned}$$

A is closed in a complete space. Then A is complete and which implies that A is compact and totally bounded.

For $r > 0$ choose $N \in \mathbb{N}$ such that $\frac{1}{N} < r$. Let

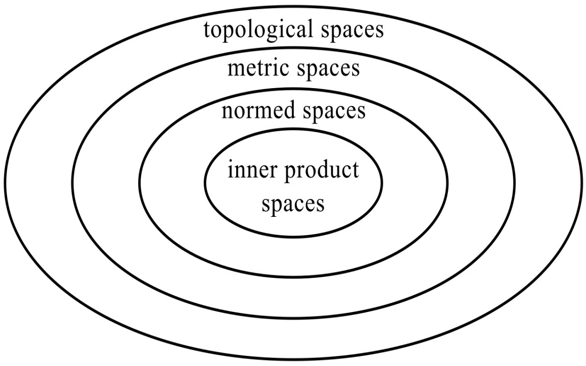
$$\begin{aligned}
T &= \Big\{-\frac{N}{N+1}, -\frac{(N-1)}{N+1}, \cdots, \frac{-1}{N+1}, 0, \frac{1}{N+1}, \cdots, \frac{N}{N+1}\Big\} \text{and} \\
S &= \Big\{(s_1, s_2, \cdots, s_N, 0, 0, 0, \cdots) \in \ell^\infty \ \Big|\ s_i \in T\Big\}.
\end{aligned}$$

Then the size of S is equal to $(2N+1)^N < \infty$ and $A \subseteq \bigcup_{\mathbf{a}\in S} B\big(\mathbf{a}, \frac{1}{N}\big)$.

B and C are not bounded. Hence, they are not compact. D is not totally bounded and hence it is not compact (sequentially compact).

❑ Conclusion

Topological Space	Metric Space	$\mathbb{R}^n$
compact $\neq$ *sequentially compact* $\neq$ *totally bounded + complete* $\neq$ *closed and bounded*	*compact* $=$ *sequentially compact* $=$ *totallybounded + complete* $\Rightarrow$ *closed and bounded* $\not\Leftarrow$	*compact* $=$ *sequentially compact* $=$ *totallybounded + complete* $=$ *closed and bounded*



2.5 Connected Sets

❑ Connected Sets

Definition 2.5.1. Let (M, d) be a metric space and $A \subseteq M$

(1) Let $\mathcal{U}, \mathcal{V} \subseteq M$ be two nonempty open sets in M. We say that $\mathcal{U}$ and $\mathcal{V}$ separate A if

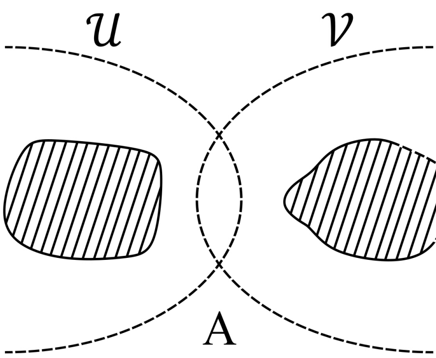
(i) $A \cap \mathcal{U} \cap \mathcal{V} = \emptyset$

(ii) $A \cap \mathcal{U} \neq \emptyset$

(iii) $A \cap \mathcal{V} \neq \emptyset$

(iv) $A \subseteq \mathcal{U} \cup \mathcal{V}$

(2) We say that a set $A \subseteq M$ is disconnected or separated if there exists two open sets $\mathcal{U}$ and $\mathcal{V}$ in M such that $\mathcal{U}$ and $\mathcal{V}$ separate A. If there exists no such pair of open sets, we say that A is connected.

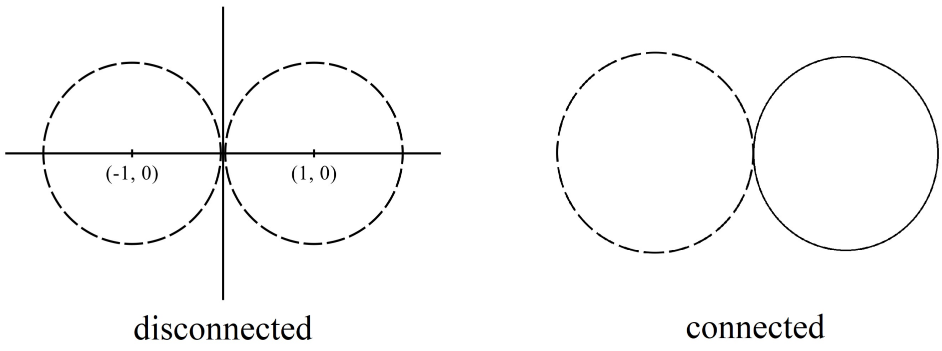


(3) A maximal connected subset of A is called a "*connected component*" of A.

Example 2.5.2. (1) $A = (-1, 0) \cup (0, 1)$ is disconnected.

(2) $B = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x+1)^2 + y^2 < 1\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x-1)^2 + y^2 < 1\right\}$ is disconnected.

(3) $C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x+1)^2 + y^2 \leq 1\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x-1)^2 + y^2 < 1\right\}$ is connected.



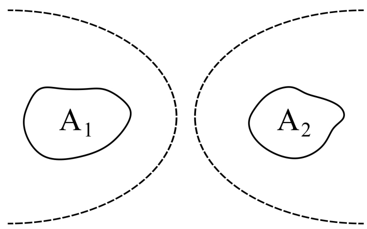
Proposition 2.5.3. *Let (M, d) be a metric space and $A \subseteq M$. Then A is disconnected if and only if there exists two nonempty sets A_1 and A_2 such that*

(i) $A = A_1 \cup A_2$

(ii) $A_1 \cap \overline{A_2} = \emptyset$

(iii) $\overline{A_1} \cap A_2 = \emptyset$.

Proof. ($\Longrightarrow$) If A is disconnected, then there exists open sets $\mathcal{U}$, $\mathcal{V} \subseteq M$ such that (i) $A \cap \mathcal{U} \cap \mathcal{V} = \emptyset$, (ii) $A \cap \mathcal{U} \neq \emptyset$, (iii) $A \cap \mathcal{V} \neq \emptyset$, (iv) $A \subseteq \mathcal{U} \cup \mathcal{V}$.



Let $A_1 = A \cap \mathcal{U} \neq \emptyset$ and $A_2 = A \cap \mathcal{V} \neq \emptyset$. Then $A = A_1 \cup A_2$.
Since $A_1 \cap \mathcal{V} = \emptyset$ and $\mathcal{V}$ is open, $A_1 \subset \mathcal{V}^c$ and $\mathcal{V}^c$ is closed. We have $\overline{A_1} \subseteq \overline{\mathcal{V}^c} = \mathcal{V}^c$. Then $\overline{A_1} \cap \mathcal{V} = \emptyset$ and hence, $\overline{A_1} \cap A_2 = \emptyset$. Similarly, $A_1 \cap \overline{A_2} = \emptyset$.

($\Longleftarrow$) If there exists A_1 $A_2 \neq \emptyset$ satisfying (i), (ii) and (iii). Let $\mathcal{U} = \left(\overline{A_2}\right)^c$ and $\mathcal{V} = \left(\overline{A_1}\right)^c$. Then $\mathcal{U}$ and $\mathcal{V}$ are nonempty. Then

$$A_1 = A \cap \mathcal{U} \neq \emptyset, \quad A_2 = A \cap \mathcal{V} \neq \emptyset \quad \text{and} \quad A = A_1 \cup A_2 \subseteq \mathcal{U} \cup \mathcal{V}.$$

Since $A_1 \cap \left(\overline{A_1}\right)^c = \emptyset$ and $A_2 \cap \left(\overline{A_2}\right)^c = \emptyset$, we have $A_1 \cap \mathcal{V} = \emptyset$ and $A_2 \cap \mathcal{U} = \emptyset$. Then

$$A \cap \mathcal{U} \cap \mathcal{V} = \left(A_1 \cap \mathcal{U} \cap \mathcal{V}\right) \cup \left(A_2 \cup \mathcal{U} \cup \mathcal{V}\right) = \emptyset \cup \emptyset = \emptyset.$$

□

Corollary 2.5.4. *Let (M, d) be a metric space and $A \subseteq M$ be connected. If there exists $A_1, A_2 \subseteq M$ such that (i) $A = A_1 \cup A_2$ and (ii) $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$, then either $A_1 = \emptyset$ or $A_2 = \emptyset$. In other words, $A \subseteq A_1$ or $A \subseteq A_2$.*

Theorem 2.5.5. *Let $A \subseteq \mathbb{R}$ be connected if and only if for $x, y \in A$ and $x < z < y$ then $z \in A$. That is, A is an interval*

Proof. ($\Longrightarrow$) If false, there exists $x < z < y$ for some $x, y \in A$ and $z \notin A$. Let $A_1 = (-\infty, z) \cap A$ and $A_2 = (z, \infty) \cap A$. Since $x \in (-\infty, z)$ and $y \in (z, \infty)$, $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Also, $A = A_1 \cup A_2$ (since $z \notin A$). We have

$$\overline{A_1} \subseteq (-\infty, z] \quad \text{and} \quad \overline{A_2} \subseteq [z, \infty).$$

Then

$$\overline{A_1} \cap A_2 = \emptyset \quad \text{and} \quad A_1 \cap \overline{A_2} = \emptyset.$$

Hence, A is disconnected. It contradicts the hypothesis that A is connected.

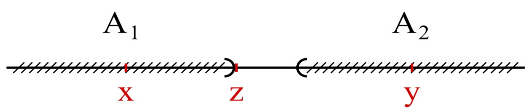
($\Longleftarrow$) Assume that A is disconnected. There exists $A_1, A_2 \subseteq \mathbb{R}$ such that (1) $A_1, A_2 \neq \emptyset$ (2) $A = A_1 \cup A_2$ (3) $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$.

Since $A_1, A_2 \neq \emptyset$, there exists $x \in A_1$ and $y \in A_2$. By (3), $x \neq y$ and we may assume that $x < y$. Let $z = \sup\big([x, y] \cap A_1\big)$. Then $z \in \overline{A_1}$ and thus $z \notin A_2$. There are only two possibilities:

(a) If $z \notin A_1$, then$z \notin A = A_1 \cup A_2$. It contradicts the hypothesis that $z \in A$ since $x < z < y$.

(b) If $z \in A_1$ then $z \notin \overline{A_2}$. There exists $r > 0$ such that $(z, z + r) \cap A_2 = \emptyset$. Thus, $x < z + \frac{r}{2} < y$, but $z + r \notin A_1 \cup A_2 = A$. It contradicts the hypothesis that $z + \frac{r}{2} \in A$.

Therefore, A is connected.

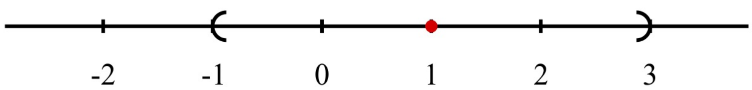


□

2.6 Subspace Topology

Observe that $(\mathbb{N}, |\cdot|) \subseteq (\mathbb{Z}, |\cdot|) \subseteq (\mathbb{Q}, |\cdot|) \subseteq (\mathbb{R}, |\cdot|)$

- $B(1, 2)$ in $(\mathbb{N}, |\cdot|)$ is $\{1, 2\}$.
- $B(1, 2)$ in $(\mathbb{Z}, |\cdot|)$ is $\{0, 1, 2\}$.
- $B(1, 2)$ in $(\mathbb{Q}, |\cdot|)$ is $(-1, 3) \cap \mathbb{Q}$.
- $B(1, 2)$ in $(\mathbb{R}, |\cdot|)$ is $(-1, 3)$.



Recall that $B(x, r)$ in M is defined by $\{y \in M \mid d(x, y) < r\}$. Hence, the set $\{1, 2\}$ is open in $(\mathbb{N}, |\cdot|)$ and in $(\mathbb{Z}, |\cdot|)$ but not open in $(\mathbb{Q}, |\cdot|)$ and $(\mathbb{R}, |\cdot|)$.

If $N \subseteq M$, the metric space $(N, d) \subseteq (M, d)$. A set A could be open in (N, d) but not open in (M, d). For example, $\mathbb{Q}$ is open in $(\mathbb{Q}, |\cdot|)$ but not open in $(\mathbb{R}, |\cdot|)$.

On the other hand, if $(N, d) \subseteq (M, d)$, for $x \in N \subseteq M$, the open ball $B_N(x, r)$ in N is

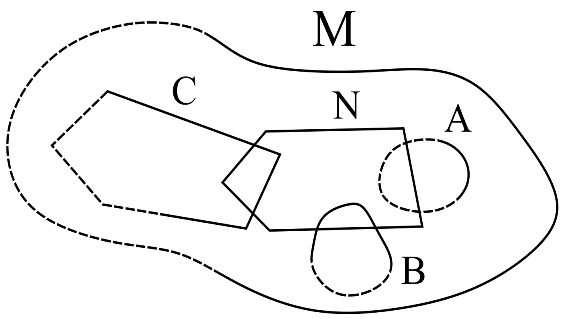
$$\{y \in N \mid d(x, y) < r\} = \{y \in M \mid d(x, y) < r\} \cap N = B_M(x, r) \cap N.$$

Hence, we may define the topology of (N, d) induced by the topology of (M, d) with the intersection of N.

❑ Subspace Topology

Definition 2.6.1. Let (M, d) be a metric space and $N \subseteq M$. Then (N, d) is a metric space and we call the topology of (N, d) "*the subspace topology of* (N, d)".

Example 2.6.2.



A is open relative to N
B is closed relative to N
C is compact relative to N

Remark.

$$E \text{ is open (closed, compact) in } (M, d) \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad E \cap N \text{ is open (closed, compact) in } (N, d).$$

Proposition 2.6.3. *Let (M, d) be a metric space and $N \subseteq M$. A subset $\mathcal{V} \subseteq N$ is open in (N, d) if and only if there exists a set $\mathcal{U} \subseteq M$ which is open in M such that $\mathcal{V} = \mathcal{U} \cap N$.*

Proof. Define a r-ball in (N, d) by $B_N(x, r) = \{y \in N \mid d(x, y) < r\}$ and a r-ball in (M, d) by $B_M(x, r) = \{y \in M \mid d(x, y) < r\}$. Then

$$B_N(x, r) = B_M(x, r) \cap N.$$

$(\Longrightarrow)$ Since $\mathcal{V}$ is open in (N, d), for $x \in \mathcal{V}$, there exists $r_x > 0$ such that $B_N(x, r_x) \subseteq N$. Then $\mathcal{V} \subseteq \bigcup_{x \in \mathcal{V}} B_N(x, r_x) \subseteq \mathcal{V}$ and hence $\mathcal{V} = \bigcup_{x \in \mathcal{V}} B_N(x, r_x)$.

Define $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_M(x, r_x)$. Then $\mathcal{U}$ is open in (M, d) since it is a union of open balls in (M, d). Then

$$\mathcal{U} \cap N = \bigcup_{x \in \mathcal{V}} B_M(x, r_x) \cap N = \bigcup_{x \in \mathcal{V}} \Big(B_M(x, r_x) \cap N \Big) = \bigcup_{x \in \mathcal{V}} B_N(x, r_x) = \mathcal{V}.$$

$(\Longleftarrow)$ For $x \in \mathcal{V} \subseteq \mathcal{U}$, since $\mathcal{U}$ is open in (M, d), there exists $\delta_x > 0$ such that $B_M(x, \delta_x) \subseteq \mathcal{U}$. Then

$$B_N(x, \delta_x) = B_M(x, \delta_x) \cap N \subseteq \mathcal{U} \cap N = \mathcal{V}.$$

Hence, x is an interior point of $\mathcal{V}$ in (N, d) and hence $\mathcal{V}$ is open in (N, d). □

Corollary 2.6.4. *Let (M, d) be a metric space and $N \subseteq M$. A set $E \subseteq N$ is closed in (N, d) if and only if there exists a set $F \subseteq M$ which is closed in (M, d) such that $E = F \cap N$.*

Definition 2.6.5. Let (M, d) be a metric space and $N \subseteq M$. A subset $A \subseteq M$ is said to be "*open (closed, compact) relative to N*" if $A \cap N$ is open (closed, compact) in (N, d).*

Remark. If E is open (closed, compact) in (M, d), then $E \cap N$ is open (closed, compact) in (N, d).

Theorem 2.6.6. *Let (M, d) be a metric space and $K \subseteq N \subseteq M$. Then K is compact in (M, d) if and only if K is compact in (N, d).*

Proof. ($\Longrightarrow$) Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of K in (N, d). Then for each V_α, there exists an open set U_α in (M, d) such that $V_\alpha = U_\alpha \cap N$. Then $K \subseteq \bigcup_{\alpha \in I} V_\alpha \subseteq \bigcup_{\alpha \in I} U_\alpha$. Hence, $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K in (M, d).

Since K is compact in (M, d), $\{U_\alpha\}_{\alpha \in I}$ contains a finite subsover of K, say

$$K \subseteq \bigcup_{i=1}^{L} U_{\alpha_i}.$$

Since $K \subseteq N$, we have

$$K \subseteq \Big(\bigcup_{i=1}^{L} U_{\alpha_i} \Big) \cap N = \bigcup_{i=1}^{L} \big(U_{\alpha_i} \cap N \big) = \bigcup_{i=1}^{L} V_{\alpha_i}$$

Then $\{V_\alpha\}_{\alpha \in I}$ contains a finite subcover of K in (N, d) and K is compact in (N, d).

($\Longleftarrow$) Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of K in (M, d). Since U_α is open in M and $N \subseteq M$, the set $V_\alpha := U_\alpha \cap N$ is open in N. Also, since $K \subseteq \bigcup_{\alpha \in I} U_\alpha$ and $K \subseteq N$,

$$K \subseteq \Big(\bigcup_{\alpha \in I} U_\alpha \Big) \cap N = \bigcup_{\alpha \in I} \big(U_\alpha \cap N \big) = \bigcup_{\alpha \in I} V_\alpha.$$

Then $\{V_\alpha\}_{\alpha \in I}$ is an open cover in (N, d).

Since K is compact in (N, d), $\{V_\alpha\}_{\alpha \in I}$ contains a finite subcover of K, say

$$K \subseteq \bigcup_{i=1}^{L} V_{\alpha_i} = \bigcup_{i=1}^{L} \big(U_{\alpha_i} \cap N \big) = \Big(\bigcup_{i=1}^{L} U_{\alpha_i} \Big) \cap N.$$

Then $K \subseteq \bigcup_{i=1}^{L} U_{\alpha_i}$ and $\{U_\alpha\}_{\alpha \in I}$ contains a finite subcover. Hence, K is compact in (M, d).

□

*We usually say that A is relatively open in N.

Remark. Let (M, d) be a metric space and $K \subseteq N \subseteq M$. Then K is sequentially compact in (M, d) if and only if K is sequentially compact in (N, d).

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K.
($\Longleftarrow$) Clear.
($\Longrightarrow$)

K is sequentially compact in (M, d)	if and only if	there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to a point $x_0 \in K$
	if and only if	since the metric d on M is the same as the metric d on N and $K \subseteq N$, we have $\{x_{n_k}\}_{k=1}^{\infty} \subseteq N$ and $\lim\limits_{k\to\infty} x_{n_k} = x_0$ in (N, d).
	if and only if	K is sequentially compact in (N, d).

□

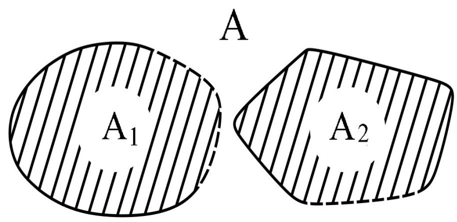
Example 2.6.7. Let $K = [0, 1] \cap \mathbb{Q}$, $N = \mathbb{Q}$ and $M = \mathbb{R}$. Let d be the usual metric induced by $|\cdot|$. Then $[0, 1] \cap \mathbb{Q} \subseteq \mathbb{R}$.

K is closed in $(\mathbb{Q}, |\cdot|)$ but not compact in $(\mathbb{Q}, |\cdot|)$ since it is not sequentially compact.

On the other hand, since K is not compact in $(\mathbb{R}, |\cdot|)$, it is not compact in $(\mathbb{Q}, |\cdot|)$.

Recall: $A \subseteq M$ is disconnected if there are $A_1, A_2 \subseteq M$ such that (1) $A_1, A_2 \neq \emptyset$ (2) $A_1 \cup A_2 = A$ (3) $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$.

By (2) and (3), $A_1 = A \backslash \overline{A_2} = A \cap \underbrace{\left(\overline{A_2}\right)^c}_{open}$. Hence, A_1 is open relative to A. Similarly, A_2 is open relative to A.



Also, since $\overline{A_1} \cap A_2 = \emptyset$ and $A = A_1 \cup A_2$, we obtain $A_1 = A \cap \overline{A_1}$. Therefore, A_1 is closed relative to A. Similarly, A_2 is closed relative to A.

Remark. (1) If $A \subseteq M$ is disconnected, then there exists nonempty sets A_1 and A_2 which are both open and closed relative to A.

(2) If $A \subseteq M$ is connected, then the set which is both open and closed relative to A is either $\emptyset$ or A itself.

Remark. Let $A \subseteq (M, d)$

(1) A is connected if and only if there exists no nonempty sets A_1 and A_2 such that (i) $A = A_1 \cup A_2$, (ii) $A_1 \cap A_2 = \emptyset$ and (iii) A_1 and A_2 are open relative to A.

(2) A is connected if and only if there exists no nonempty sets B_1 and B_2 such that (i) $A = B_1 \cup B_2$, (ii) $B_1 \cap B_2 = \emptyset$ and (iii) B_1 and B_2 are closed relative to A.

(3) A is connected if and only if the only subsets of A which are both closed and open relative to A are A itself or $\emptyset$.

2.7 Normed Spaces and Inner Product Spaces

❑ Normed Spaces

Definition 2.7.1. A "*normed vector space*" $(V, \|\cdot\|)$ is a real vector space associated with a function $\|\cdot\| : V \to \mathbb{R}$ such that

(i) $\|x\| \geq 0$ for every $x \in V$.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(iii) $\|\lambda \cdot x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$ and $x \in V$.

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in V$.

We call the function $\|\cdot\|$ satisfying (i)-(iv) a "*norm*" on V.

Example 2.7.2. (1) Let $V = \mathbb{R}^n$ and $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ where $x = (x_1, \cdots, x_n)$. Then $(V, \|\cdot\|_2)$ is a normed space and $\|\cdot\|_2$ is called 2-norm.
The statements (i), (ii) and (iii) in the definition are trivial. Let's check (iv) here.

$$\begin{aligned} \|x+y\|_2^2 &= \sum_{i=1}^n (x_i + y_i)^2 = \sum_{i=1}^n x_i^2 + 2x_iy_i + y_i^2 \\ &\leq \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n x_iy_i + \sum_{i=1}^n y_i^2 \\ &\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 = (\|x\|_2 + \|y\|_2)^2 \end{aligned}$$

(2) Let $V = \mathbb{R}^n$ and $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ where $x = (x_1, \cdots, x_n)$, $1 \leq p < \infty$. Then $(V, \|\cdot\|_p)$ is a normed space and $\|\cdot\|_p$ is called a p-norm.

(3) Let $V = \mathbb{R}^n$ and $\|x\|_\infty = \max(|x_1|, \cdots, |x_n|)$. Then $(V, \|\cdot\|_\infty)$ is a normed vector space and $\|\cdot\|_\infty$ is called an ∞-norm.

Example 2.7.3. Let $C\big([0,1]\big)$ be the collection of all continuous real-valued function on $[0,1]$. That is,

$$C\big([0,1]\big) := \left\{f : [0,1] \to \mathbb{R} \;\middle|\; f \text{ is continuous.}\right\}$$

Define

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} & 1 \leq p < \infty \\ \max_{x \in [0,1]} |f(x)| & p = \infty. \end{cases}$$

Then $\Big(C\big([0,1]\big), \|\cdot\|_p\Big)$ is a normed space. Check

(1) $C\big([0,1]\big)$ is a vector space.

(2) $\|\cdot\|_p$ satisfies (i)-(iv).

❏ Series on a Normed Space

Definition 2.7.4. Let $(V, \|\cdot\|)$ be a normed space and $\{x_k\}_{k=1}^{\infty}$ be a sequence in V.

(1) We say that $\{x_k\}_{k=1}^{\infty}$ is "*bounded*" if there exists $B > 0$ such that

$$\|x_k\| \le B \quad \text{for every } k \in \mathbb{N}.$$

(2) We say that $\{x_k\}_{k=1}^{\infty}$ converges to x_0 if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_k - x_0\| < \varepsilon \quad \text{whenever } k \ge N.$$

(3) $\{x_k\}_{k=1}^{\infty}$ is said to a "*Cauchy sequence*" if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \quad \text{whenever } m, n \ge N.$$

(4) We say that $(V, \|\cdot\|)$ is complete if every Cauchy sequence in V converges. A complete normed space is called "*Banach space*".

Definition 2.7.5. Let $(V, \|\cdot\|)$ be a normed space and $\{x_k\}_{k=1}^{\infty}$ be a sequence in V.

(1) We define the partial sum of the sequence by

$$s_n = x_1 + \cdots + x_n = \sum_{k=1}^{n} x_k$$

and call $\displaystyle\sum_{k=1}^{\infty} x_k$ a series of $\{x_k\}_{k=1}^{\infty}$.

(2) A series $\displaystyle\sum_{k=1}^{\infty} x_k$ is said to converge to s if the partial sum $\{s_n\}_{n=1}^{\infty}$ converges to s. Denote $s = \displaystyle\sum_{k=1}^{\infty} x_k$.

Theorem 2.7.6. *Let $(V, \|\cdot\|)$ be a normed space and $\{x_k\}_{k=1}^{\infty}$ be a sequence in V.*

(1) If $\displaystyle\sum_{k=1}^{\infty} x_k$ converges, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \ge m \ge N$,

$$\|x_m + x_{m+1} + \cdots + x_n\| < \varepsilon.$$

(2) In addition, if $(V, \|\cdot\|)$ is a Banach space, then $\displaystyle\sum_{k=1}^{\infty} x_k$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \ge m \ge N$,

$$\|x_m + x_{m+1} + \cdots + x_n\| < \varepsilon.$$

Proof. (Exercise) □

Corollary 2.7.7. *If* $\sum_{k=1}^{\infty} x_k$ *converges, then* $\lim_{k\to\infty} \|x_k\| = 0$.

Proof. Since $\sum_{k=1}^{\infty} x_k$ converges, $\{s_n\}_{n=1}^{\infty}$ converges and hence it is Cauchy. We have

$$\lim_{k\to\infty} \|x_k\| = \lim_{k\to\infty} \|s_k - s_{k-1}\| = 0.$$

□

Definition 2.7.8. We say that

(1) A series $\sum_{k=1}^{\infty} x_k$ "*converges absolutely*" if $\sum_{k=1}^{\infty} \|x_k\|$ converges.

(2) A series $\sum_{k=1}^{\infty} x_k$ "*converges conditionally*" if $\sum_{k=1}^{\infty} x_k$ converges but not converges absolutely

Example 2.7.9. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in V with $\|x_k\| = 1$ for $k = 1, 2, \cdots$. Then $\sum_{k=1}^{\infty} \frac{x_k}{2^k}$ converges absolutely.

Theorem 2.7.10. *In a Banach space* $(V, \|\cdot\|)$*, if* $\sum_{k=1}^{\infty} x_k$ *aboslutely converges then* $\sum_{k=1}^{\infty} x_k$ *converges.*

Proof. Since $\sum_{k=1}^{\infty} x_k$ converges aboslutely, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \|x_k\| < \varepsilon$.

For $m, n \geq N$,

$$\|x_m + x_{m+1} + \cdots + x_n\| \leq \|x_m\| + \|x_{m+1}\| + \cdots + \|x_n\| < \varepsilon.$$

Hence, the partial sum $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence and $\sum_{k=1}^{\infty} x_k$ converges. □

Remark. In general, the result of Theorem 2.7.10 is false if $(V, \|\cdot\|)$ is not complete. For example,

- $V = C\big([0,1]\big)$ and $\|f\| = \int_0^1 |f(t)|\, dt$.
- $V = \big\{(a_1, a_2, a_3 \cdots, 0, 0, 0, \cdots)\big\}$ with $\|x\| = \sup |a_i|$.

Example 2.7.11. Let $V = C\big([0,1]\big)$ with $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $f_n(x) = \dfrac{x^n}{n!}$ for $n = 0, 1, 2, \cdots$. Then $f_n \in C\big([0,1]\big)$. The series $\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ converges in $C\big([0,1]\big)$.

The partial sum is $s_n = \sum_{k=0}^{n} f_k(x) = \sum_{k=0}^{n} \dfrac{k!}{x^k}$. Then $s_n - s_m = \sum_{k=m+1}^{n} \dfrac{x^k}{k!}$.

$$\begin{aligned}
\big\|s_n - s_m\big\|_{C\left([0,1]\right)} &= \Big\| \sum_{k=m+1}^{n} \frac{x^k}{k!} \Big\|_{C\left([0,1]\right)} \\
&\le \sum_{k=m+1}^{n} \Big\|\frac{x^k}{k!}\Big\|_{C\left([0,1]\right)} = \sum_{k=m+1}^{n} \frac{1}{k!}\big\|x^k\big\|_{C\left([0,1]\right)} \\
&\le \sum_{k=m+1}^{n} \frac{1}{k!} \max_{x\in[0,1]} |x^k| \\
&\le \sum_{k=m+1}^{n} \frac{1}{k!} < \varepsilon \quad \text{(as } m, n \text{ sufficiently large.)}
\end{aligned}$$

Hence $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C\big([0,1]\big)$. As we know $\Big(C\big([0,1]\big), \|\cdot\|\Big)$ is a Banach space, $\sum_{k=0}^{\infty} f_k(x)$ converges to a continuous function on $[0,1]$.

Definition 2.7.12. Let V be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are "*equivalent*" if there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$$

for every $x \in V$.

Remark. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on V. Then the norm spaces $\big(V, \|\cdot\|_1\big)$ and $\big(V, \|\cdot\|_2\big)$ will have the same topological properties.

❑ Inner Product Spaces

Definition 2.7.13. An "*inner product space*" $(V, <\cdot,\cdot>)$ is a real vector space V associated with a binary function $<\cdot,\cdot>: V \times V \to \mathbb{R}$ such that

(a) $< x, x > \ge 0$ for every $x \in V$.

(b) $< x, x > = 0$ if and only if $x = 0$.

(c) $< x, y + z > = < x, y > + < x, z >$ for every $x, y, z \in V$.

(d) $< \lambda x, y > = \lambda < x, y >$ for every $\lambda \in \mathbb{R}$ and $x, y \in V$.

(e) $< x, y > = < y, x >$ for every $x, y \in V$.

A symmetric bilinear form $< \cdot, \cdot >$ satisfies (a)-(e) is called an "*inner product* on V

Example 2.7.14. Let $V = \mathbb{R}^n$ and $< x, y >= \sum_{i=1}^{n} x_i y_i$ where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then $\big(\mathbb{R}^n, < \cdot, \cdot >\big)$ is an inner product space.

Example 2.7.15. Let $V = C\big([0,1]\big)$ and $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Then $\Big(C\big([0,1]\big), \langle\cdot,\cdot\rangle\Big)$ is an inner vector space.

Remark. (1) A normed vector space $(V, \|\cdot\|)$ is a metric space by defining

$$d(x, y) := \|x - y\| \quad \text{for every } x, y \in V.$$

Check that $d(\cdot, \cdot)$ is a metric on V.

(2) An inner product space $\big(V, < \cdot, \cdot >\big)$ is a normed vector space by defining

$$\|x\| = \sqrt{< x, x >} \quad \text{for every } x \in V.$$

(3) $\|x\| = \|x - 0\| = d(x, 0)$.

Example 2.7.16. (1) $\big(\mathbb{R}^n, < \cdot, \cdot >\big)$ is an inner product vector space by defining

$$< x, y >= \sum_{i=1}^{n} x_i y_i \quad \text{where } x = (x_1, \cdots, x_n) \text{ and } \ y = (y_1, \cdots, y_n).$$

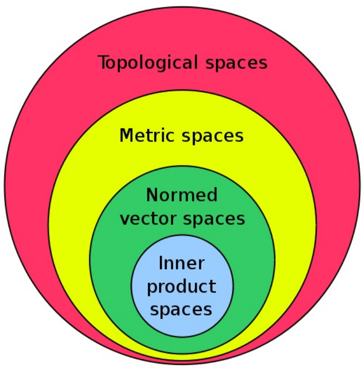
(2) $\big(\mathbb{R}^n, \|\cdot\|\big)$ is a normed space by defining

$$\|x\| = \sqrt{< x, x >} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

(3) $\big(\mathbb{R}^n, d\big)$ is a metric space by defining the "*induced metric*"

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Remark. Since a normed space is also a metric space, we can consider all topological properties on a normed space.



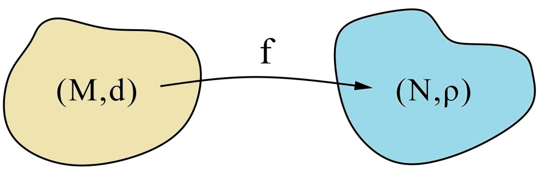
Chapter 3

Continuous Maps

3.1 Continuity

❏ Mappings and Limits

We consider the mappings from one metric space to another one.



Definition 3.1.1. Let (M, d) and (N, ρ) be two metric spaces and $A \subseteq M$.

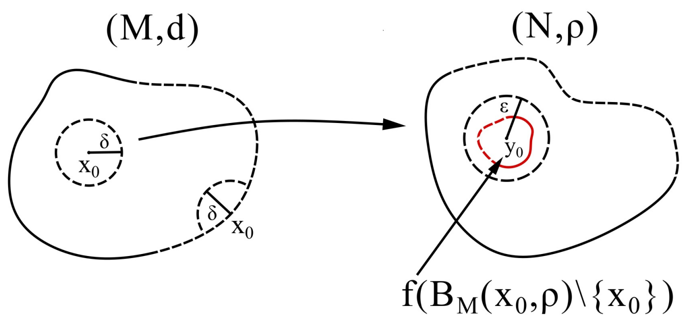
(1) A function $f : A \to N$ between two metric spaces is usually called a "*mapping*".

(2) For $x_0 \in A'$, we say that $y_0 \in N$ is the "*limit*" of f at x_0 if for every $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that every $x \in A$ with $d(x, x_0) < \delta$, then

$$\rho\big(f(x), y_0\big) < \varepsilon.$$

Denoted by

$$\lim_{\substack{x \to x_0 \\ x \in A}} f(x) = y_0 \quad \text{or} \quad f(x) \to y_0 \text{ as } x \to x_0.$$



Proposition 3.1.2. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. For $x_0 \in A'$, $\lim\limits_{x\to x_0} f(x) = y_0$ if and only if for every sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ converging to x_0 in (M,d), the sequence $\{f(x_k)\}_{k=1}^{\infty}$ converges to y_0 in (N,ρ).*

Proof. ($\Longrightarrow$) Given $\varepsilon > 0$, since $\lim\limits_{x\to x_0} f(x) = y_0$, there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that if $x \in A$ with $d(x,x_0) < \delta$, then

$$\rho\big(f(x), y_0\big) < \varepsilon.$$

Let $\{x_k\}_{k=1}^{\infty} \subseteq A$ be a sequence which converges to x_0. Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, $d(x_k,x_0) < \delta$. Therefore,

$$\rho\big(f(x_k), y_0\big) < \varepsilon \quad \text{whenever } k \geq N.$$

We have $\{f(x_k)\}_{k=1}^{\infty}$ converges to y_0.

($\Longleftarrow$) Assume the contrary, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists $x_\delta \in A$ such that $d(x_\delta,x_0) < \delta$ but $\rho\big(f(x_\delta), y_0\big) \geq \varepsilon$.

Let $\delta = \dfrac{1}{k}$, then there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ such that $d(x_k,x_0) < \dfrac{1}{k}$ but $\rho\big(f(x_k), y_0\big) \geq \varepsilon$. Hence, $\{x_k\}_{k=1}^{\infty}$ converges to x_0 but $\{f(x_k)\}_{k=1}^{\infty}$ does not converge to y_0. It contradicts the hypothesis and thus this direction is proved. □

❏ Continuity

Definition 3.1.3. Let (M,d) and (N,ρ) be two metric spaces, $x_0 \in A \subseteq M$ and $f : A \to N$.

(1) f is said to be continuous at x_0 if either $x_0 \in A\backslash A'$ or $\lim\limits_{\substack{x\to x_0\\ x\in A}} f(x) = f(x_0)$.

(2) If f is continuous at every point of A, then f is said to be continuous on A.

Remark. If x_0 is an isolated point of A (that is, $x_0 \in A\backslash A'$) then f is automatically continuous at x_0.

Proposition 3.1.4. *Let (M,d) and (N,ρ) be two metric spaces, $x_0 \in A \subseteq M$, and $f : A \to N$ be a map. Then f is continuous at x_0 if and only if for every $\varepsilon > 0$, there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that*

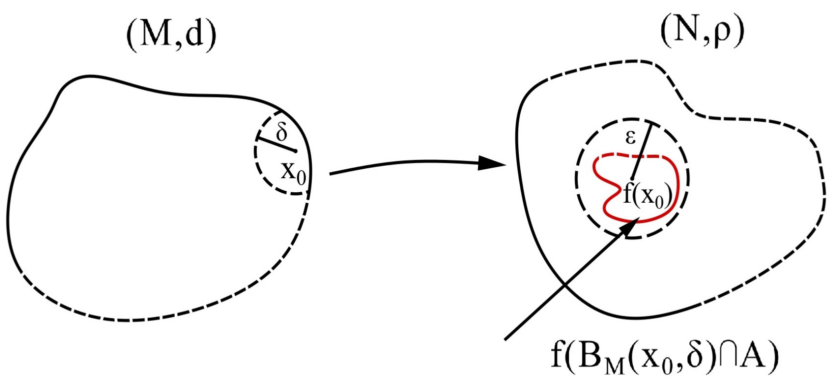
$$\rho\big(f(x), f(x_0)\big) < \varepsilon$$

for all point $x \in A$ with $d(x,x_0) < \delta$.

Proof. (Exercise) □

Remark. f is continuous at x_0 if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f\big(B_M(x_0,\delta)\cap A\big) \subseteq B_N\big(f(x_0),\varepsilon\big).$$



Example 3.1.5. (1) $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_1, x_2, \cdots, x_n) = x_k \quad \text{for some } k = 1, 2, \cdots, n$$

is a continuous function.

(2) A norm $\|\cdot\| : V \to \mathbb{R}$ is a continuous function on V.

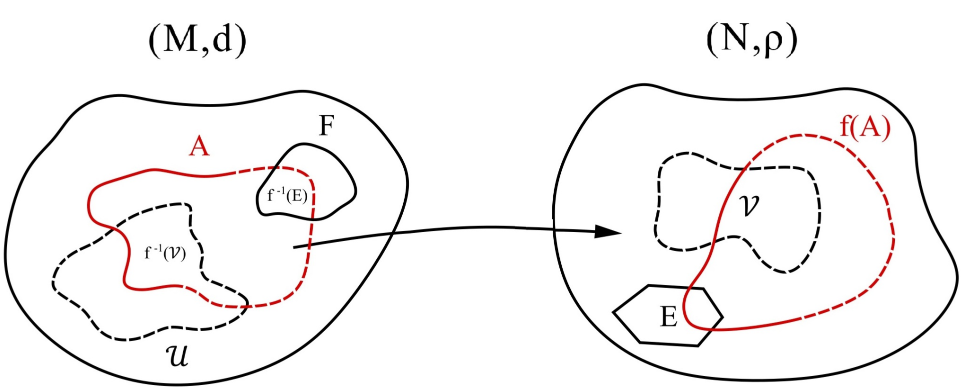
(3) Let (M,d) be a metric space, $A \subseteq M$. The distance function $f(x) = d(x,A)$ is continuous on M.

Theorem 3.1.6. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. Then the following statements are equivalent.*

(1) f is continuous on A.

(2) For every open set $\mathcal{V} \subseteq N$, the preimage $f^{-1}(\mathcal{V}) = \mathcal{U} \subseteq A$ is open relative to A; that is, $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$ for some open set $\mathcal{U}$ in M.

(3) For every closed set $E \subseteq N$, the preimage $f^{-1}(E) \subseteq A$ is closed relative to A; that is, $f^{-1}(E) = F \cap A$ for some closed set F in M.



Proof. **"(1) ⇒ (2)"**
Assume that $f^{-1}(\mathcal{V}) \neq \emptyset$. Let $x_0 \in f^{-1}(\mathcal{V})$, then $f(x_0) \in \mathcal{V}$. Since $\mathcal{V}$ is open, there exists $\varepsilon > 0$ such that $B_N\big(f(x_0), \varepsilon\big) \subseteq \mathcal{V}$. Moreover, since f is continuous at x_0, there exists $\delta_{x_0} > 0$ such that $f\big(B_M(x_0, \delta_{x_0}) \cap A\big) \subseteq B_N\big(f(x_0), \varepsilon\big) \subseteq \mathcal{V}$. Then $B_M(x_0, \delta_{x_0}) \cap A \subseteq f^{-1}(\mathcal{V})$.

Similarly, since f is continuous on A, for every $x \in A$, there exists $\delta_x > 0$ such that $f\big(B_M(x, \delta_x) \cap A\big) \subseteq \mathcal{V}$. Hence, $B_M(x, \delta_x) \cap A \subseteq f^{-1}(\mathcal{V})$.

Define $\mathcal{U} = \bigcup_{x \in f^{-1}(\mathcal{V})} B_M(x, \delta_x)$. Then $\mathcal{U}$ is open in M and $f^{-1}(\mathcal{V}) \subseteq \mathcal{U} \cap A$. On the other hand, since

$$f(\mathcal{U} \cap A) = f\Big(\bigcup_{x \in f^{-1}(\mathcal{V})} B_M(x, \delta_x) \cap A\Big) = \bigcup_{x \in f^{-1}(\mathcal{V})} f\big(B_M(x, \delta_x) \cap A\big) \subseteq \mathcal{V},$$

we have $\mathcal{U} \cap A \subseteq f^{-1}(\mathcal{V})$ and hence $\mathcal{U} \cap A = f^{-1}(\mathcal{V})$.

"(2) ⇒ (1)"
Let $x \in A$ and then $f(x) \in N$. For given $\varepsilon > 0$, $B_N\big(f(x), \varepsilon\big)$ is open in N. By (2), there exists an open set $\mathcal{U} \subseteq M$ such that $\mathcal{U} \cap A = f^{-1}\Big(B_N\big(f(x), \varepsilon\big)\Big)$. Hence, for $x \in \mathcal{U}$, there exists $\delta_x > 0$ such that $B_M(x, \delta_x) \subseteq \mathcal{U}$. Then

$$f\big(B_M(x, \delta_x) \cap A\big) \subseteq f(\mathcal{U} \cap A) = B_N\big(f(x), \varepsilon\big).$$

Thus, for $y \in A$ and $d(x, y) < \delta$, $\rho\big(f(x), f(y)\big) < \varepsilon$. Hence f is continuous at x. Furthermore, since x is an arbitrary point in A, f is continuous on A.

"(2) ⇒ (3)"
Since E is closed in N, the complement E^c is open in N. By (2), $f^{-1}(E^c)$ is open relative to A and there exists $\mathcal{U}$ open in M such that $f^{-1}(E^c) = \mathcal{U} \cap A$.

Let $F = \mathcal{U}^c$. Then F is closed and

$$F \cap A = \mathcal{U}^c \cap A = A \backslash (A \cap \mathcal{U}) = A \backslash f^{-1}(E^c) = f^{-1}(N) \backslash f^{-1}(E^c) = f^{-1}(E)$$

is closed relative to A.

"(3) ⇒ (2)" (Exercise)

□

Remark. Let $f : (M, d) \to (N, \rho)$ be continuous. Then

f^{-1}(open set in N) is open set in M
f^{-1}(closed set in N) is closed set in M.

The pullback of an open (closed) set through a continuous function is open (closed). But the pushforward of an open (closed) set in M by a continuous function may not be open (closed). For example, $f(x) = x^2$ and $f\big((-1, 1)\big) = [0, 1)$.

Example 3.1.7. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. The set

$$\left\{x \in \mathbb{R}^n \;\middle|\; \|f(x)\|_{\mathbb{R}^m} < 1\right\} = f^{-1}\big(B_m(0,1)\big)$$

is open.

Example 3.1.8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Then $f(0,0) = 1$. Along $x = y$, $f(x,x) = 0$ if $x \neq 0$. Hence, f is not continuous at $(0,0)$. The set $\{0\}$ is closed in $\mathbb{R}$. But $f^{-1}\big(\{0\}\big)$ is not closed in $\mathbb{R}^2$.

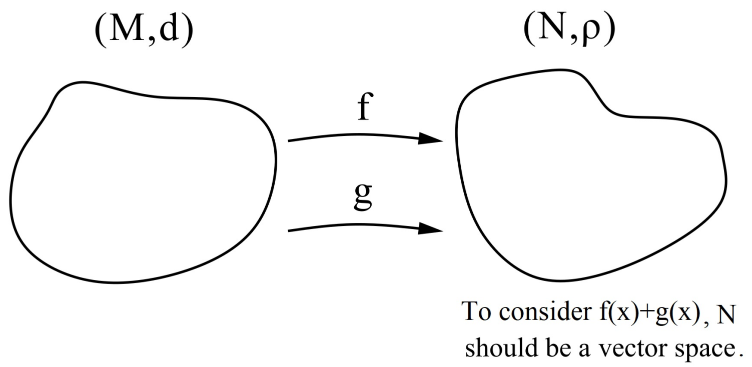
Remark. Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : M \to N$ be a map. We define a map $g : A \to N$ by $g(x) = f(x)$ for $x \in A$. We usually denote g by $f\big|_A$.

Question: If $f : M \to N$ is continuous on M, is $f\big|_A$ continuous on A?
Answer: Yes!
Question: If $f\big|_A$ is continuous on A, is f continuous on M?
Answer: No! For example, $f(x) = 1$ on $\mathbb{Q}$ and $f(x) = 0$ otherwise.

Remark. Let V be a vector space and N be a metric space. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Let $f : V \to N$ be a map. Then f is continuous on $(V, \|\cdot\|_1))$ if and only if f is continuous on $(V, \|\cdot\|_2)$.

Proof. (Exercise) □

3.2 Operations on Continuous Maps



Definition 3.2.1. Let (M,d) be a metric space and $(V, \|\cdot\|)$ be a normed vector space and $A \subseteq M$. Let $f, g : A \to V$ be maps and $h : A \to \mathbb{R}$ be a function. Define

$$\begin{aligned} (f \pm g)(x) &= f(x) \pm g(x) && x \in A \\ (\alpha f)(x) &= \alpha f(x) && x \in A,\ \alpha \in \mathbb{R} \\ (hf)(x) &= h(x)g(x) && x \in A \\ \Big(\frac{f}{h}\Big)(x) &= \frac{f(x)}{h(x)} && x \in A,\ h(x) \neq 0 \end{aligned}$$

Proposition 3.2.2. *Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed space, $A \subseteq M$, $f, g : A \to V$ be maps, $h : A \to \mathbb{R}$ be a function. Suppose that $x_0 \in A'$ and $\lim_{x \to x_0} f(x) = v$, $\lim_{x \to x_0} g(x) = w$, $\lim_{x \to x_0} h(x) = c$. Then*

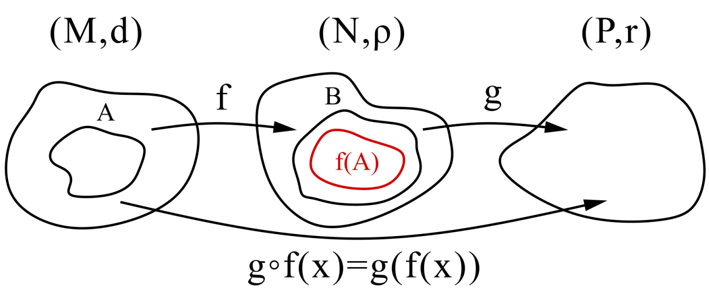
$$\lim_{x \to x_0} (f \pm g)(x) = v \pm w$$
$$\lim_{x \to x_0} (hf)(x) = cv$$
$$\lim_{x \to x_0} \left(\frac{f}{h}\right) = \frac{1}{c} v \quad \text{if } c \neq 0.$$

Proof. (Exercise) □

Corollary 3.2.3. *Under the hypothesis of Proposition 3.2.2, suppoe that f, g and h are continuous at $x_0 \in A$. Then $f \pm g$, hf are continuous at x_0 and $\frac{f}{h}$ is continuous at x_0 if $h(x_0) \neq 0$.*

Corollary 3.2.4. *Under the hypothesis of Proposition 3.2.2, suppose that f, g and h are continuous on A. Then $f \pm g$ and hf are continuous on A and $\frac{f}{h}$ is continuous on $\{x \in \mid h(x) \neq 0\}$.*

Definition 3.2.5. Let (M, d), (N, ρ) and (P, r) be metric spaces, $A \subseteq M$, $B \subseteq N$ and $f : A \to N$, $g : N \to P$ be maps such that $f(A) \subseteq B$. The composite function $g \circ f : A \to P$ is the map defined by $g \circ f(x) = g\big(f(x)\big)$.



Theorem 3.2.6. *Let (M, d), (N, ρ) and (P, r), f and g satisfy the hypothesis of Definition 3.2.5.*

(1) Suppose that f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

(2) Suppose that f is continouos on A and g is continuous on $f(A)$. Then $g \circ f$ is continuous on A.

Proof. (1) (Exercise)

(2) Let $\mathcal{W}$ be an open set in P. Since g is continuous on $f(A) \subseteq N$, $g^{-1}(\mathcal{W})$ is open relative to $f(A)$. Thus, there exists $\mathcal{V}$ which is open in N such that $g^{-1}(\mathcal{W}) = \mathcal{V} \cap f(A)$.

Similarly, since f is continuous on A, $f^{-1}(\mathcal{V})$ is open relative in A. There exists $\mathcal{U}$ which is open in M such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Then

$$\left(g \circ f\right)^{-1}(\mathcal{W}) = f^{-1}\left(g^{-1}(\mathcal{W})\right) = f^{-1}\left(\mathcal{V} \cap f(A)\right) = f^{-1}(\mathcal{V}) \cap A = \mathcal{U} \cap A$$

is open relative to A.

Since $\mathcal{W}$ is an arbitrary open set in P, $g \circ f$ is continuous on A.

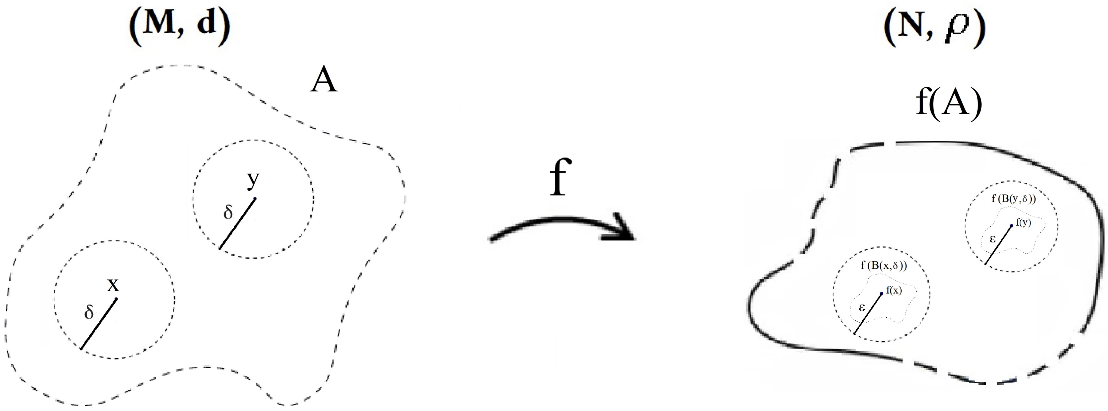
□

3.3 Uniform Continuity

Definition 3.3.1. Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. We say that f is "*uniformly continuous*" on A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\rho\left(f(x), f(y)\right) < \varepsilon$$

for all $x, y \in A$ for which $d(x, y) < \delta$.



Uniformly Continuous function

Remark. (1) Continuity is a property of a function at a single point. Uniform continuity is a property of a function on a set.

(2) For a uniformly continuous function, δ only depneds on ε but independent of x.

Proposition 3.3.2. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. If f is uniformly continuous on A, then f is continuous on A.*

Proof. (Exercise) □

Example 3.3.3. $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$ since $\left|f(x) - f(y)\right| = \left||x| - |y|\right| \leq |x - y|$.

Example 3.3.4. $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a, \infty)$ for all $a > 0$ but is not uniformly continuous on $(0, \infty)$.
Fix $a > 0$, for $a < x < y < \infty$, by mean value theorem,

$$f(y) - f(x) = f'(c)(y - x) = -\frac{1}{c^2}(y - x) \quad \text{for some } c \in (x, y).$$

Hence,

$$\big|f(y) - f(x)\big| = \frac{1}{c^2}|y - x| < \frac{1}{a^2}|y - x|.$$

Example 3.3.5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| < M$ for every $x \in \mathbb{R}$. Then f is uniformly continuous.

Definition 3.3.6. (1) Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that f is "*Lipschitz function*" is there exists $K > 0$ such that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K$$

for every $x, y \in A$ and $x \neq y$, or

$$|f(x) - f(y)| \leq K|x - y|.$$

for every $x, y \in A$.

(2) Let (M, d) and (N, ρ) be two metric spaces and $f : M \to N$. We say that f is "*Lipschitz function*" if there exists $K > 0$ such that

$$\frac{\rho\big(f(x), f(y)\big)}{d(x, y)} \leq K \quad \text{for every } x, y \in M \text{ and } x \neq y.$$

Note. A Lipschitz function is uniformly continous.

Definition 3.3.7. We say that a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is "*Hölder continuous with exponent* α" if there exists $K > 0$ and $0 < \alpha \leq 1$ such that

$$|f(x) - f(y)| \leq K|x - y|^{\alpha} \quad \text{for every } x, y \in A.$$

Note. A function f which is Hölder continuous with exponent α is uniformly continuous.

Remark.

$$\begin{aligned} \text{Bounded first derivative fuctnions} &\Rightarrow \text{Lipschitz functions} \\ &\Rightarrow \text{Hölder continuous functions} \\ &\Rightarrow \text{Uniformly continous functions} \end{aligned}$$

Theorem 3.3.8. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. Then f is uniformly continuous on A if and only if for any two sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subseteq A$, if $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$ then $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) = 0$.*

Proof. ($\Longrightarrow$) Suppose the contrary. There exists two seuqnece $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subseteq A$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$ but $\lim_{n\to\infty} d\big(f(x_n), f(y_n)\big) \neq 0$. That is, there exists $\varepsilon > 0$ such that for every $k \in \mathbb{N}$, there exists $n_k \geq k$ such that $\rho\big(f(x_{n_k}), f(y_{n_k})\big) > \varepsilon$.

Since f is uniformly continuous, there exists $\delta > 0$ such that for every $x, y \in A$ with $d(x, y) < \delta$, $\rho\big(f(x), f(y)\big) < \varepsilon$. Since $\lim_{n\to\infty} d(x_n, y_n) = 0$, there exists $M \in \mathbb{N}$ such that for every $n \geq M$, $d(x_n, y_n) < \delta$. Then we can choose $n_M \geq M$ and we have $d(x_{n_M}, y_{n_M}) < \delta$ but $\rho\big(f(x_{n_M}), f(y_{n_M})\big) > \varepsilon$.

($\Longleftarrow$) Suppose the contrary. There exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $x_n, y_n \in A$ with $d(x_n, y_n) < \dfrac{1}{n}$, but $\rho\big(f(x_n), f(y_n)\big) > \varepsilon$. Then for the sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subseteq A$ with $\lim_{n\to\infty} d(x_n, y_n) = 0$, we have $\lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) \neq 0$ and obtain a contradiction. □

Remark. Let (M, d) and (N, ρ) be metric space, $A \subseteq M$ and $f : A \to N$ be a map. Then the following statements are equivalent

(1) f is NOT uniformly continuous on A.

(2) There exists two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} d(x_n, y_n) = 0$ but $\limsup_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(3) There exists two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} d(x_n, y_n) = 0$ but $\lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(4) There exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists two point $x_n, y_n \in A$ such that $d(x_n, y_n) < \dfrac{1}{n}$ but $\rho\big(f(x_n), f(y_n) > \varepsilon$.

Remark. (1) Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function with $|f'(x)| < M$ for all $x \in I$. Then f is uniformly continuous on I by using mean value theorem.

(2) The converse of above statement is false. A differentiable and uniformly continuous function may not have bounded derivatives. For example, $f(x) = \sqrt{x}$ on $[0, 1]$.

Example 3.3.9. (1) $f(x) = x^2$ is uniformly continuous on $[0, M]$ for any $M > 0$ but it is not uniformly continuous on $\mathbb{R}$. Let $x_n = n$, $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| \to 0$ but

$$\big|f(n) - f(n + \frac{1}{n})\big| = |2 - \frac{1}{n^2}| > 1 \quad \text{for every } n.$$

(2) $f(x) = \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$.

(3) $f(x) = \sin \dfrac{1}{x}$ is not uniformly continuous on $(0, 1)$.

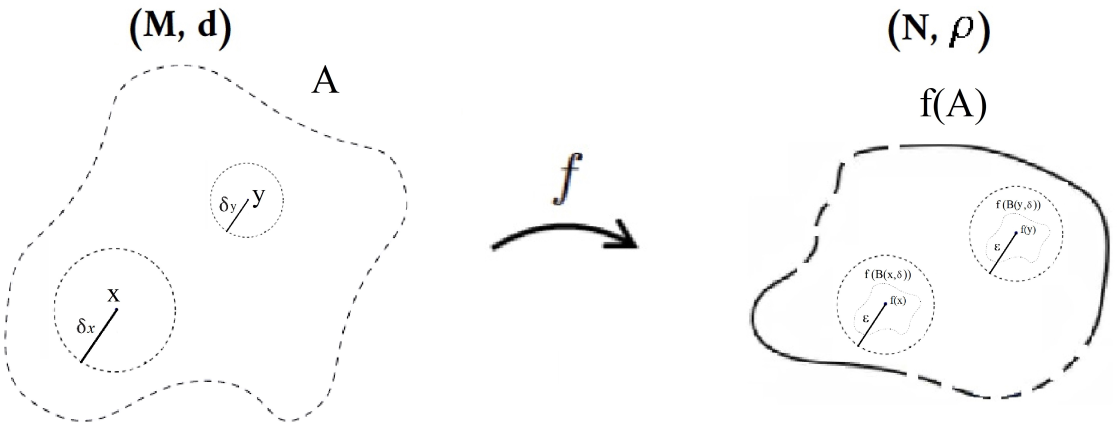
❑ Continuity v.s. Uniform Continuity

Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$.

• Suppose f is **continuous on** A. For given $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that if $d(x, y) < \delta$ then $\rho\big(f(x), f(y)\big) < \varepsilon$. Therefore,

$$f\big(B_M(x, \delta) \cap A\big) \subseteq B_N\big(f(x), \varepsilon\big).$$

Note that δ depends on ε and x.

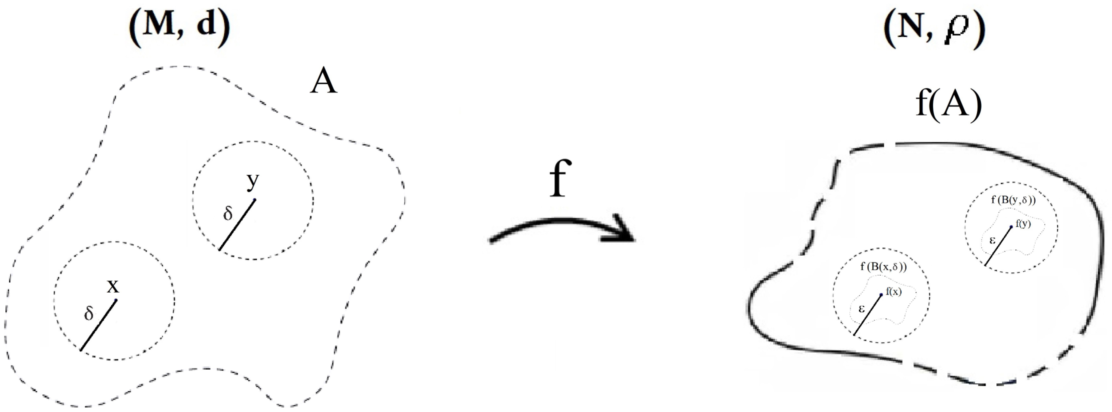


Continuous function

• Suppose f is **uniformly continuous on** A. For given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $d(x, y) < \delta$ then $\rho\big(f(x), f(y)\big) < \varepsilon$. Therefore,

$$f\big(B_M(x, \delta) \cap A\big) \subseteq B_N\big(f(x), \varepsilon\big).$$

Note that δ depends only on ε but indenpendent of x.

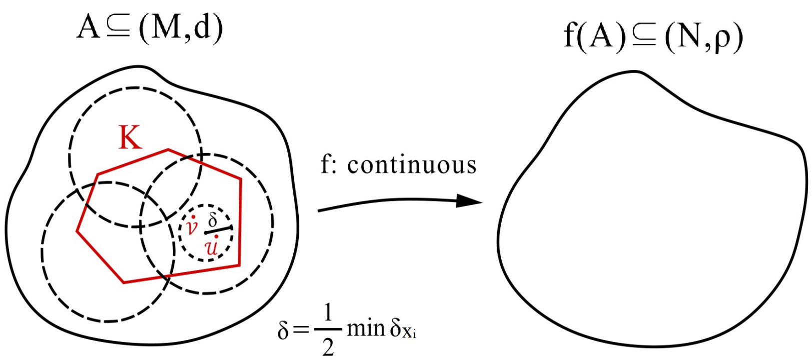


Uniformly Continuous function

Remark. Let $f : A \to f(A)$ be continuous. For every $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that $\rho\big(f(x), f(y)\big) < \varepsilon$ whenere $x, y \in A$ with $d(x, y) < \delta(x, \varepsilon)$. Define $\delta_f(\varepsilon) := \inf_{x \in A} \delta(x, \varepsilon) > 0$. If $\delta_f(\varepsilon) > 0$ for every $\varepsilon > 0$, then f is uniformly continuous on A.

Note. This is the idea that continuity on a compact set gives rise to the uniform continuity.

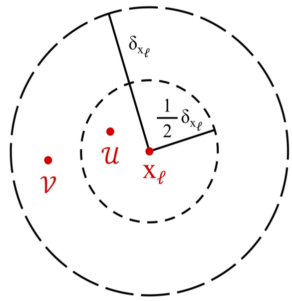
Theorem 3.3.10. *Let (M,d) and (N,ρ) be metric space, $A \subseteq M$ and $f : A \to N$ be a map. If $K \subseteq A$ is compact and f is continuous on K, then f is uniformly continuously on K.*



Proof. Since f is continuous on K, given $\varepsilon > 0$, for every $x \in K$, there exists $\delta_x > 0$ such that for $\underbrace{y \in K \text{ with } d(x,y) < \delta_x}_{y \in B_M(x,\delta_x)\cap K}$ then

$$\rho\big(f(x), f(y)\big) < \frac{\varepsilon}{2}.$$

Since K is compact and $K \subseteq \bigcup_{x\in K} B_M(x, \frac{\delta_x}{2})$, there exists $x_1, \cdots, x_L \in K$ such that $K \subseteq \bigcup_{i=1}^{L} B_M(x_i, \frac{\delta_{x_i}}{2})$.



Define $\delta = \dfrac{1}{2} \min\limits_{1\le i\le L} \delta_{x_i}$. Let $u, v \in K$ with $d(u,v) < \delta$. Since $K \subseteq \bigcup_{i=1}^{L} B_M(x_i, \frac{\delta_{x_i}}{2})$, there exists $1 \le \ell \le L$ such that $u \in B_M(x_\ell, \frac{\delta_{x_\ell}}{2}) \subseteq B_M(x_\ell, \delta_{x_\ell})$. Thus,

$$d(v, x_\ell) \le d(v,u) + d(u, x_\ell) < \delta + \frac{\delta_{x_\ell}}{2} \le \delta_{x_\ell}.$$

Then $v \in B(x_\ell, \delta_{x_\ell})$ and we have

$$\rho\big(f(u), f(v)\big) \le \rho\big(f(u), f(x_\ell)\big) + \rho\big(f(x_\ell), f(v)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, f is uniformly continuous on K. □

Lemma 3.3.11. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$ and $f : A \to N$ be uniformly continuous. If $\{x_n\}_{n=1}^{\infty} \subseteq A$ is a Cauchy sequence in (M,d), then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence in (N,ρ).*

Proof. Since f is uniformly continuous on A, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in A$ with $d(x,y) < \delta$, then

$$\rho\big(f(x), f(y)\big) < \varepsilon.$$

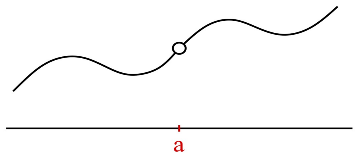
Since $\{x_n\}_{n=1}^{\infty} \subseteq A$ is Cauchy in (M,d), there exists $L \in \mathbb{N}$ such that if $m, n \ge L$ then $d(x_n, x_m) < \delta$. Thus, for $m, n \ge L$, we have

$$\rho\big(f(x_m), f(x_n)\big) < \varepsilon.$$

Hence, $\big\{f(x_n)\big\}_{n=1}^{\infty}$ is Cauchy in (N,ρ). □

❏ Extension of a Function

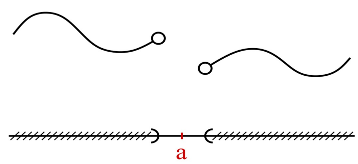
Let $f : \mathbb{R}\backslash\{a\} \to \mathbb{R}$ be continuous and $\lim_{x\to a} f(x) = L$.



Question: Is there a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous on $\mathbb{R}$ and $g(x) = f(x)$ on $\mathbb{R}\backslash\{a\}$?

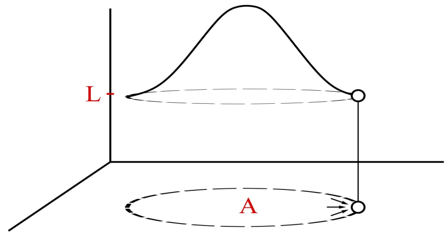
Answer: Yes! Define $g(x) = \begin{cases} f(x) & x \neq a \\ L & x = a. \end{cases}$

Suppose that $f : A \to \mathbb{R}$ be a continuous function.



It doesn't make sense to define value of $\overline{f}$ at a.

Question: Is there a continuous function $g : \overline{A} \to \mathbb{R}$ such that $g(x) = f(x)$ on A?



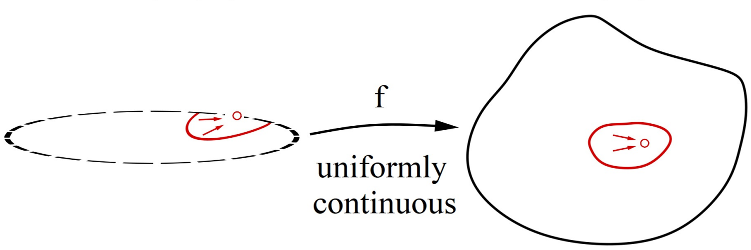
Answer: No! For example, $f(x) = \frac{1}{x}$ on $(0, 1)$. There exists no continuous function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ on $(0, 1)$.

Corollary 3.3.12. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be uniformly continuous. If N is complete, then f has a unique extension to a continuous function on $\overline{A}$, that is $g : \overline{A} \to N$ such that*

(1) g is uniformly continuous on $\overline{A}$.

(2) $g(x) = f(x)$ on A.

(3) (uniqueness) If there is $h : \overline{A} \to N$ satisfying (1) and (2), then $g(x) = h(x)$ on $\overline{A}$.



Proof. We only need to define the value of g on $\overline{A}\backslash A$. Let $x \in \overline{A}\backslash A$. There exists $\{x_n\}_{n=1}^{\infty} \subseteq A$ converging to x. Hence, $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in A. Since f is uniformly continuous on A, by Lemma 3.3.11, $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence in (N, ρ).

Since N is complete, there exists $y = y_x \in N$ such that $\lim_{n\to\infty} f(x_n) = y$. Define

$$g(x) = \begin{cases} f(x) & x \in A \\ y_x & x \in \overline{A}\backslash A \end{cases}$$

(I) To check that $g(x)$ is well-defined.
For $x \in \overline{A}\backslash A$, let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be two sequences in A which both converge to x. Then $d(x_n, z_n) \to 0$ as $n \to \infty$. Since f is uniformly continuous on A, $\rho\big(f(x_n), f(z_n)\big) \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} f(x_n) = y_x = \lim_{n\to\infty} f(z_n)$. We have $g(x)$ is well-defined on $\overline{A}$ and the statement (2) holds.

(II) Check that g is uniformly continuous on $\overline{A}$
Since f is uniformly continuous on A, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in A$ with $d(x, y) < \delta$, then

$$\rho\big(f(x), f(y)\big) < \frac{\varepsilon}{3}.$$

Let $r = \dfrac{\delta}{3}$. For $u, v \in \overline{A}$ with $d(u, v) < r$, by the definition of g, there are $v', u' \in A$ with $d(u, u') < r$ and $d(v, v') < r$ such that

$$\rho\big(f(u), f(u')\big) < \frac{\varepsilon}{3} \quad \text{and} \quad \rho\big(f(v), f(v')\big) < \frac{\varepsilon}{3}.$$

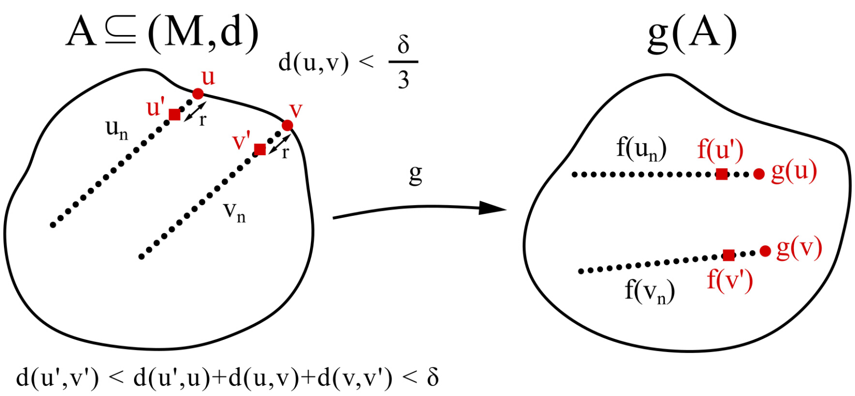
Then

$$d(u', v') \le d(u', u) + d(u, v) + d(v, v') < 3r = \delta.$$

We have $\rho\big(f(u'), f(v')\big) < \dfrac{\varepsilon}{3}$ and hence

$$\rho\big(g(u), g(v)\big) \le \rho\big(g(u), g(u')\big) + \rho\big(g(u'), g(v')\big) + \rho\big(g(v'), g(v)\big) < \varepsilon.$$

The statement (1) is proved.



(III) To check the extension is unique.
If there exists $h : \overline{A} \to N$ satisfying statements (1), (2) and (3), then $h(x) = f(x) = g(x)$ for every $x \in A$. Let $x \in \overline{A}\backslash A$. Given $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that if $d(x, y) < \delta_1$

then $\rho\big(g(x), g(y)\big) < \frac{\varepsilon}{2}$ and if $d(x,y) < \delta_2$, then $\rho\big(h(x), h(y)\big) < \frac{\varepsilon}{2}$.
Since $x \in \overline{A}\backslash A$, there exists $y \in A$ such that $d(x,y) < \min(\delta_1, \delta_2)$. Then

$$\begin{aligned} \rho\big(g(x), h(x)\big) &\le \rho\big(g(x), g(y)\big) + \rho\big(g(y), h(y)\big) + \rho\big(h(y), h(x)\big) \\ &< \frac{\varepsilon}{2} + \underbrace{\rho\big(g(y), h(y)\big)}_{=0 \text{ since } y\in A} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is an arbitrary positive number, we obtain $g(x) = h(x)$ and hence $g(x) = h(x)$ for every $x \in \overline{A}$. The statement (3) is proved.

□

3.4 Continuous Maps on Compact Sets

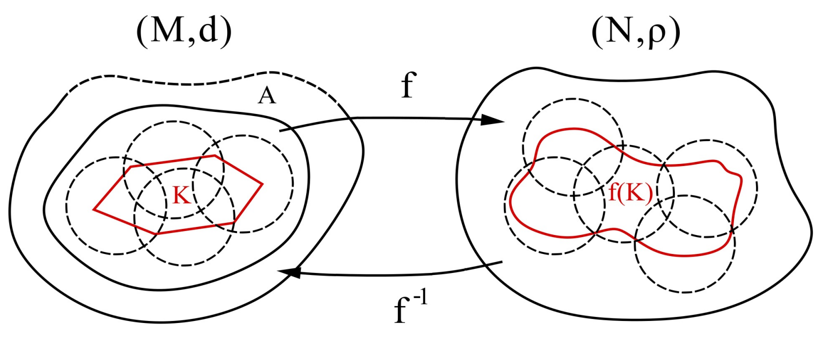
Theorem 3.4.1. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$ and $f : A \to N$ be a continuouos map. If $K \subseteq A$ is compact, then $f(K)$ is compact in (N,ρ)*

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $f(K)$. Since $f : A \to N$ is continuous and U_α is open for every $\alpha \in I$, $f^{-1}(U_\alpha)$ is open in A. Therefore, there exists V_α which is open in M such that $f^{-1}(U_\alpha) = V_\alpha \cap A$ for every $\alpha \in I$.

Since $f(K) \subseteq \bigcup_{\alpha\in I} U_\alpha$ and $K \subseteq A$, we have $K \subseteq \bigcup_{\alpha\in I} V_\alpha$. That is, $\{V_\alpha\}_{\alpha\in I}$ is an open cover of K. The compactness of K implies that there exists $\alpha_1, \cdots, \alpha_n \in I$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \cap A = \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}).$$

Then $f(K) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ and hence $f(K)$ is compact.



□

Corollary 3.4.2. *Let (M,d) be a metric space and $K \subseteq M$ be compact. If $f : M \to \mathbb{R}$ is continuous, then f attains its maximum and minimum in K. That is, there exists $x_0, x_1 \in K$ such that*

$$f(x_0) = \max_{x\in K} f(x) \quad \textit{and} \quad f(x_1) = \min_{x\in K} f(x).$$

Proof. Since f is continuous and K is compact, $f(K)$ is compact in $\mathbb{R}$. Hence, $f(K)$ is sequentially compact in $\mathbb{R}$ and then f attains its extreme values in K. □

Corollary 3.4.3. *(Extreme Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum.*

Proof. (Exercise) □

Remark. Let (M, d) and (N, ρ) be two metric spaces and $f : M \to N$ be continuous.

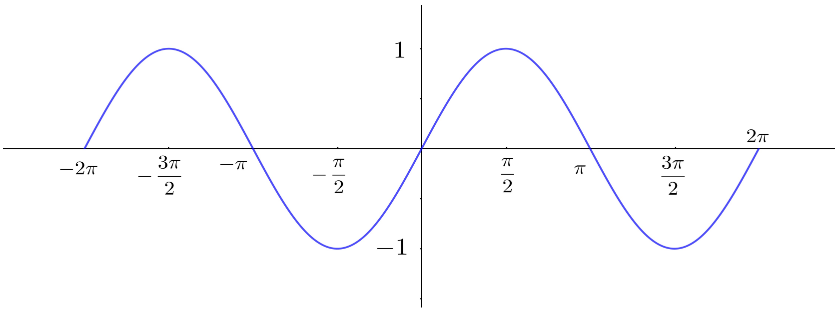
(1) Continuous maps send compact sets to compact sets. But the converse is false. For example, f is a constant map on $\mathbb{R}$.

$$\text{Compact Set} \ \overset{f}{\longrightarrow} \ \text{Compact Set}$$
$$\underset{f^{-1}}{\not\longleftarrow}$$

(2) Continuous maps send connected sets to connected sets. But the converse is false. For example, $f(x) = x^2$ on $\{1, -1\}$.

$$\text{Connected Set} \ \overset{f}{\longrightarrow} \ \text{Connected Set}$$
$$\underset{f^{-1}}{\not\longleftarrow}$$

Remark. Let $f : M \to \mathbb{R}$ be continuous and $K \subseteq M$ be compact. Then f attains its extreme in K. The extreme points are not unique. For example, $f(x) = \sin x$ on $[-2\pi, 2\pi]$.



Corollary 3.4.4. *Let (M, d) be a metric space, $K \subseteq M$ be compact and $f : K \to \mathbb{R}$ be a continuous map. Then the set $\left\{x \in K \,\middle|\, f(x) \text{ is the maximum of } f \text{ on } K\right\}$ is a nonempty compact set.*

Proof. Let $L = \sup_{x \in K} f(x)$. Then $f^{-1}(L) \neq \emptyset$. Since $\{L\} \subseteq \mathbb{R}$ is closed and f is continuous, $f^{-1}(\{L\})$ is closed. Hence, $f^{-1}(\{L\}) \cap K$ is closed set in K and it is compact. □

Theorem 3.4.5. *Let E be a noncompact set in $\mathbb{R}$. Then*

(1) there exists a continuous function on E which is unbounded.

(2) there exists a bounded and continuous function on E which has no maximum.

Proof. (1) Since E is noncompact in $\mathbb{R}$, either E is not bounded or E not closed (or both). If E is unbounded, then the function $f(x) = x$ on E is an unbounded function.

If E is bounded but not closed, then there exists $x_0 \in \overline{E}\backslash E$. Hence, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ such that $\lim\limits_{x\to x_0} x_n = x_0$. The function $f(x) = \dfrac{1}{x - x_0}$ is defined on E but not bounded.

(2) If E is unbounded, we define $f(x) = \dfrac{x^2}{1+x^2}$. Then $\sup\limits_{x\in E} f(x) = 1$ but $f(x) < 1$ for every $x \in E$.

If E is unbounded but not closed, let x_0 and $\{x_n\}_{n=1}^{\infty}$ be defined as above. We define $f(x) = \dfrac{1}{1+(x-x_0)^2}$. Then $\sup\limits_{x\in E} f(x) = 1$ but $f(x) < 1$ for every $x \in E$.

□

Theorem 3.4.6. *Let (M, d) and (N, ρ) be metric space, $A \subseteq M$ and $f : A \to N$ be a map. If $K \subseteq A$ is compact and f is continuous on K, then f is uniformly continuously on K.*

Proof. Since f is continuous on K, given $\varepsilon > 0$, for every $x \in K$, there exists $\delta_x > 0$ such that for $\underbrace{y \in K \text{ with } d(x, y) < \delta_x}_{y\in B_M(x,\delta_x)\cap K}$ then

$$\rho\big(f(x), f(y)\big) < \frac{\varepsilon}{2}.$$

Since K is compact and $K \subseteq \bigcup\limits_{x\in K} B_M(x, \frac{\delta_x}{2})$, there exists $x_1, \cdots, x_L \in K$ such that $K \subseteq \bigcup\limits_{i=1}^{L} B_M(x_i, \frac{\delta_{x_i}}{2})$.

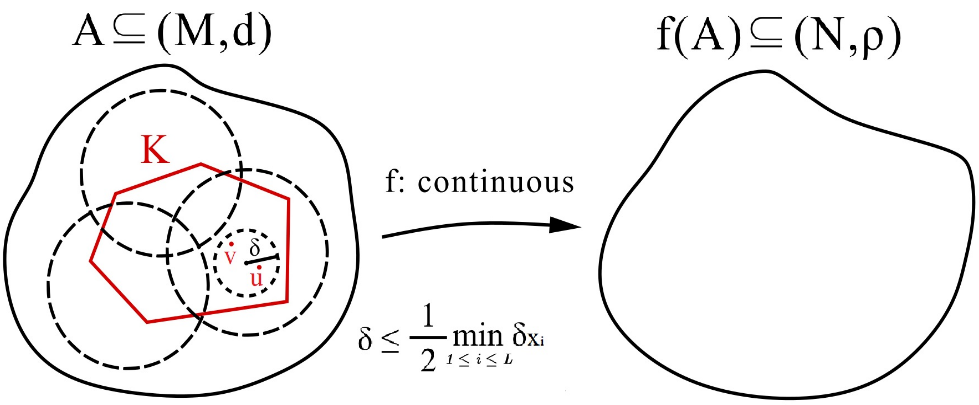
Define $\delta = \dfrac{1}{2}\min\limits_{1\le i\le L} \delta_{x_i}$. Let $u, v \in K$ with $d(u, v) < \delta$. Since $K \subseteq \bigcup\limits_{i=1}^{L} B_M(x_i, \frac{\delta_{x_i}}{2})$, there exists $1 \le \ell \le L$ such that $u \in B_M(x_\ell, \frac{\delta_{x_\ell}}{2}) \subseteq B_M(x_\ell, \delta_{x_\ell})$. Thus,

$$d(v, x_\ell) \le d(v, u) + d(u, x_\ell) < \delta + \frac{\delta_{x_\ell}}{2} \le \delta_{x_\ell}.$$

Then $v \in B(x_\ell, \delta_{x_\ell})$ and we have

$$\rho\big(f(u), f(v)\big) \le \rho\big(f(u), f(x_\ell)\big) + \rho\big(f(x_\ell), f(v)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, f is uniformly continuous on K.



□

❏ Appendix

Definition 3.4.7. Let V be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are quivalent if there exists $\alpha,\beta > 0$ such that

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_2 \quad \text{for every } x \in V.$$

Remark. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on V. Then U is open in $(V, \|\cdot\|_1)$ if and only if U is open in $(V, \|\cdot\|_2)$.

Example 3.4.8. Let $V = \mathbb{R}^n$ and $\|\cdot\|_2$ be the usual norm on $\mathbb{R}^n$. That is, $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ where $x = (x_1, x_2, \cdots, x_n)$. Then every norm $\|\cdot\|$ on V is equivalent to $\|\cdot\|_2$. This implies that every two norms on $\mathbb{R}^n$ are equivalent.

Let $\|\cdot\|$ be a norm of $\mathbb{R}^n$. To prove that $\|\cdot\| \sim \|\cdot\|_2$. Let $e_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$. Then $x = (x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i e_i$. Then

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \ge \max(|x_1|, \cdots, |x_n|).$$

Let $\beta = \sqrt{\sum_{i=1}^{n} \|e_i\|^2}$. Then $\beta \ge \max(\|e_1\|, \cdots, \|e_n\|)$. We have

$$\|x\| = \sum_{i=1}^{n} x_i e_i\| \le \sum_{i=1}^{n} |x_i|\|e_i\| \le \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}\Big(\sum_{i=1}^{n} \|e_i\|^2\Big)^2 \le \beta\|x\|_2.$$

On the other hand, since $\|\cdot\|$ is a norm, $f(x) := \|x\|$ is continuous on $(\mathbb{R}^n, \|\cdot\|_2)$. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. Then $\mathbb{S}^{n-1}$ is compact in $(\mathbb{R}^n, \|\cdot\|_2)$. Therefore, there exists $a \in \mathbb{S}^{n-1}$ such that $0 < f(a) = \min_{x \in \mathbb{S}^{n-1}} f(x)$. For $x \in \mathbb{S}^{n-1}$, $f(x) \ge f(a)$.

Consider $0 \ne y \in \mathbb{R}^n$, $\frac{y}{\|y\|_2} \in \mathbb{S}^{n-1}$ and

$$f(a) \le f\Big(\frac{y}{\|y\|_2}\Big) = \Big\|\frac{y}{\|y\|_2}\Big\| = \frac{1}{\|y\|_2}\|y\|.$$

Hence, $\underbrace{f(a)}_{=\alpha}\|y\|_2 \le \|y\|$.

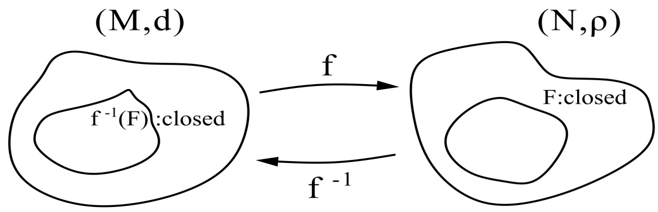
Remark. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V, N be a metric space and $f : V \to N$ be a map. Suppose that $\|\cdot\|_1 \sim \|\cdot\|_2$. Then f is continuous on $(V, \|\cdot\|_1)$ if and only if f is continuous on $(V, \|\cdot\|_2)$.

❑ Review of Continuous Maps

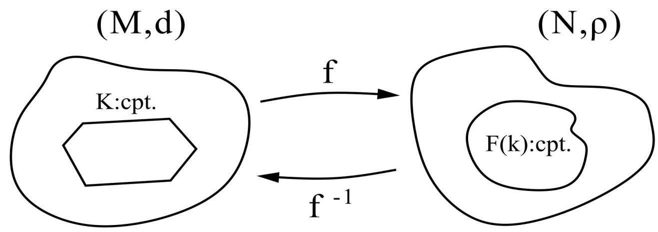
Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f : M \to N$ be a map.

(1)

$$\begin{aligned} & f \text{ is continuous on } M. \\ \iff & f^{-1}(F) \text{ is closed for every closed subset } F \subseteq N. \\ \iff & f^{-1}(U) \text{ is open for every open subset } U \subseteq N. \end{aligned}$$



(2) Suppose that f is continuous on M. Then $f(K)$ is compact for every compact subset $K \subseteq N$.



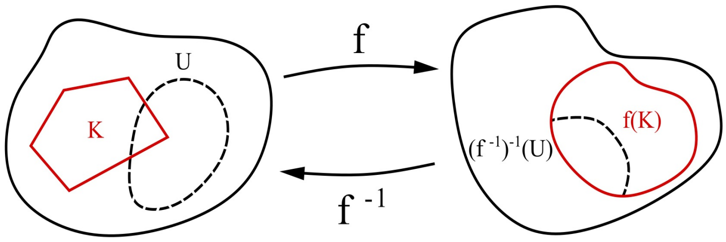
Observation: Let $f : A \to f(A) \subseteq N$.

- The inverse function of f may not exist. If f is 1-1, then the inverse fucntion of f exists and denoted by $f^{-1} : f(A) \to A$.

- If $f : A \to f(A)$ is 1-1 and continuous, is the inverse function $f^{-1} : f(A) \to A$ continuous?
  Idea: let M be compact and $E \subseteq M$ be closed. Then E is compact in M. The set $f(E)$ is compact in N since f is continuous. This implies that $f(E)$ is closed in N. We have f send every closed set E in M to a closed set $f(E)$ in N. Therefore, f^{-1} is continuous.

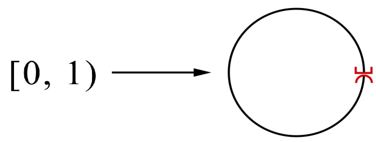
Theorem 3.4.9. *Let (M, d) and (N, ρ) be two metric spaces, $K \subseteq M$ be compact and $f : K \to N$ be a 1-1 and continuous function. Then the inverse function $f^{-1} : f(K) \to K$ is continuous.*

Proof. It suffices to prove that for every closed set E in M, the preimage, $\left(f^{-1}\right)^{-1}(E)$ of E under f^{-1} is relatively closed in $f(K)$.



Since E is closed in M and K is compact in M, the intersection $E \cap K$ is compact in M. Also, since f is continuouos, $f(E \cap K)$ is compact in N and hence it is closed in N. Moreover, since f is 1-1, we have $f(E \cap K) = \left(f^{-1}\right)^{-1}(E)$ is closed in $f(K)$. Therefore, f^{-1} is continuous on $f(K)$. □

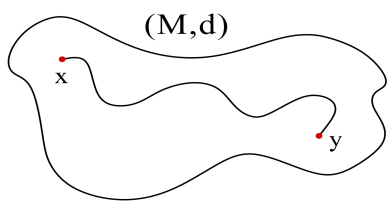
Remark. Theorem 3.4.9 is false if K is not compact. For example, $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (\cos t, \sin t)$ on $K = [0, 2\pi)$. Then f is 1-1 and continuous on $[0, 2\pi)$ and $f\big([0, 2\pi)\big) = \mathbb{S}^1$. But f^{-1} is not continuous at $(1, 0) = f(0)$.



3.5 Continuous Maps on Connected Sets and Path Connected Sets

❏ Path Connected Sets

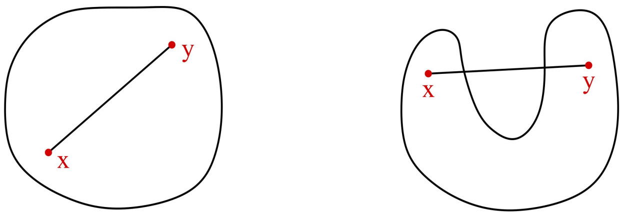
Definition 3.5.1. Let (M, d) be a metric space, $x, y \in M$. We say that a path in M from x to y is a continuous map $\phi : [0, 1] \to M$ such that $\phi(0) = x$ and $\phi(1) = y$.



Remark. We can replace $[0, 1]$ by $[a, b]$. If $\phi : [a, b] \to M$ such that $\phi(a) = x$ and $\phi(b) = y$, define $\phi(t) = \phi\big(a + (b - a)t\big)$. Then $\phi : [0, 1] \to M$ such that $\phi(0) = a$ and $\phi(1) = b$.

Definition 3.5.2. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be "*path connected*" if every pair of points $x, y \in A$ can be joined by a path in M. That is, there is a continuous map $\phi : [0, 1] \to A$ such that $\phi(0) = x$ and $\phi(1) = y$.

Definition 3.5.3. A set A in a vector space V is called "*convex*" if for all $x, y \in A$, the line segment joining x and y, denoted by $\overline{xy}$, lies in A.

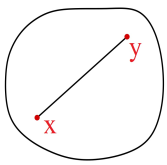


convex not convex

Example 3.5.4. An open (closed) ball in a vector space is convex. If $M = \mathbb{N}$ the open ball $B(3, 2) = \{2, 3, 4\}$ is not convex.

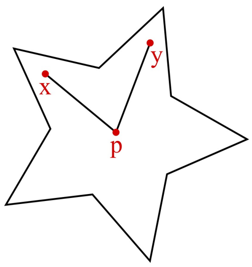
Remark.

A convex set in a normed space is path connected by taking $\phi(t) = (1 - t)x + ty$.

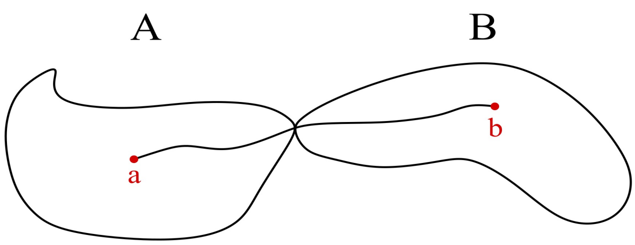


φ(t)=(1-t)x+ty

Example 3.5.5. A set S in a vector space V is called "*star-shaped*", if there exists $p \in S$ such that for every $q \in S$, the line segment joining p and q lies in S. Note that A star-shaped set is path connected by taking $\phi(t) = \begin{cases} (1-2t)x + 2tp & t \in [0, \frac{1}{2}] \\ (2-2t)p + (2t-1)y & t \in [\frac{1}{2}, 1]. \end{cases}$



Remark. Let $A, B \subseteq M$ be path-connected. If there exists $a \in A$ and $b \in B$ and a path in $A \cup B$ joining a and b, then $A \cup B$ is path connected.



Theorem 3.5.6. *Let (M, d) be a metric space and $A \subseteq M$. If A is path-connected, then A is connected.*

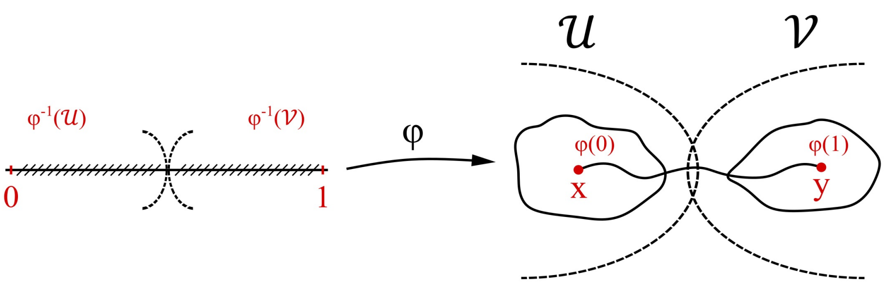
Proof. Assume that A is disconnected. Then there exist open sets $\mathcal{U}$ and $\mathcal{V}$ in M such that

$$\text{(i) } A \subseteq \mathcal{U} \cup \mathcal{V} \quad \text{(ii) } A \cap \mathcal{U} \neq \emptyset \quad \text{(iii) } A \cap \mathcal{V} \neq \emptyset \quad \text{(iv) } A \cap \mathcal{U} \cap \mathcal{V} = \emptyset$$

By (ii) and (iii), choose $x \in A \cap \mathcal{U}$ and $y \in A \cap \mathcal{V}$. Since A is path connected, there exists a continuous map $\phi : [0,1] \to A$ such that $\phi(0) = x$ and $\phi(1) = y$. then $\phi^{-1}(\mathcal{U})$ and $\phi^{-1}(\mathcal{V})$ are open relative to $[0,1]$. Hence, there exist $\mathcal{W}_1$ and $\mathcal{W}_2$ open in $[0,1]$ such that $\phi^{-1}(\mathcal{U}) = \mathcal{W}_1$ and $\phi^{-1}(\mathcal{V}) = \mathcal{W}_2$.

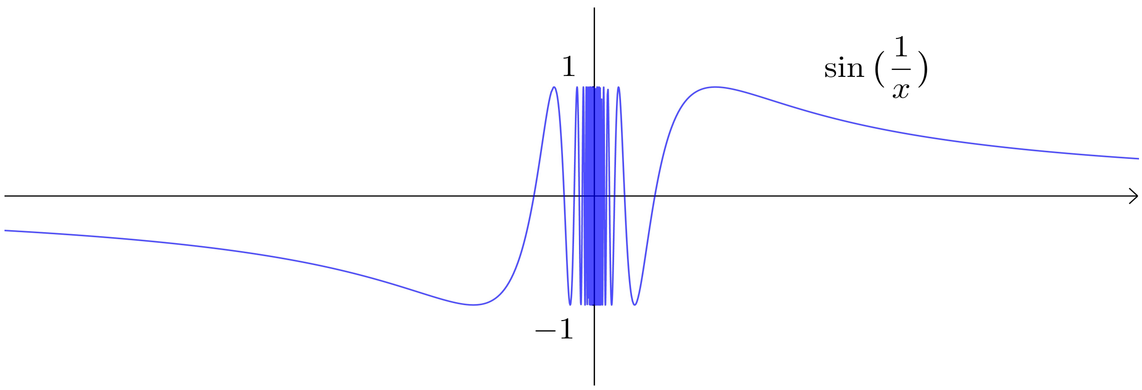
By (i), $[0,1] \subseteq \mathcal{W}_1 \cup \mathcal{W}_2$. Also,by (ii) and (iii), $0 \in \mathcal{W}_1$ and $1 \in \mathcal{W}_1$. We have $[0,1] \cap \mathcal{W}_1 \neq \emptyset$ and $[0,1] \cap \mathcal{W}_2 \neq \emptyset$.

By (iv), $[0,1] \cap \mathcal{W}_1 \cap \mathcal{W}_2 = \phi^{-1}(\mathcal{U}) \cap \phi^{-1}(\mathcal{V}) = \emptyset$. (Otherwise, there exists $t_0 \in [0,1] \cap \mathcal{W}_1 \cap \mathcal{W}_2$. Then $\phi(t_0) \in A \cap \mathcal{U} \cap \mathcal{V}$). It contradicts the fact that $[0,1]$ is connected.



□

Remark. From the above theorem, a path-connected set is connected. But the converse is false. For example, $A = \left\{(x, \sin\frac{1}{x}) \mid x \in (0,1)\right\} \cup \Big(\{0\} \times [-1,1]\Big)$ is connected but not path connected. Let $x = (1, \sin 1)$ and $y = (0,1)$. There exists no path in A joining x and y.



Theorem 3.5.7. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a continuous map.*

(1) If A is connected, then $f(A)$ is connected.

(2) If A is path connected, then $f(A)$ is path connected.

Proof. (1) Assume that $f(A)$ is disconnected. Then there exists $\mathcal{U}$ and $\mathcal{V}$ open in N such that

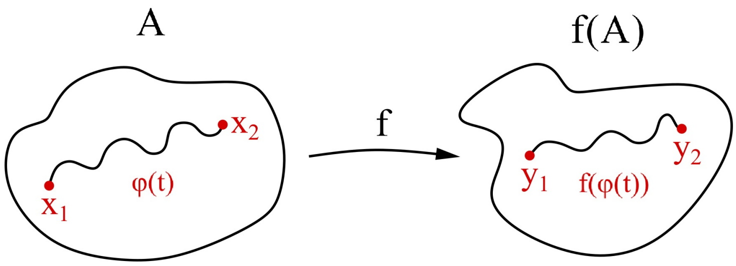
$$\text{(i) } f(A) \subseteq \mathcal{U} \cup \mathcal{V} \quad \text{(ii) } f(A) \cap \mathcal{U} \neq \emptyset \quad \text{(iii) } f(A) \cap \mathcal{V} \neq \emptyset \quad \text{(iv) } f(A) \cap \mathcal{U} \cap \mathcal{V} = \emptyset$$

Since f is continuous and $\mathcal{U}$ and $\mathcal{V}$ are open in N, $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V})$ are open relative to A. Therefore, there exists $\mathcal{W}_1$ and $\mathcal{W}_2$ open in M such that $f^{-1}(\mathcal{U}) = A \cap \mathcal{W}_1$ and $f^{-1}(\mathcal{V}) = A \cap \mathcal{W}_2$.

By (i), $A \subseteq \mathcal{W}_1 \cup \mathcal{W}_2$. From (ii) and (iii), $A \cap \mathcal{W}_1 \neq \emptyset$ and $A \cap \mathcal{W}_2 \neq \emptyset$. By (iv), $A \cap \mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. Hence, A is disconnected and we obtain a contradiction.

(2) Let $y_1, y_2 \in f(A)$. Then there exists $x_1, x_2 \in A$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since A is path connected, there exists a continuous map $\phi : [0,1] \to A$ such that $\phi(0) = x_1$ and $\phi(1) = x_2$.

Define $\psi(t) := f(\phi(t))$. Clearly, $\psi(t)$ maps from $[0,1]$ to $f(A)$. Since f and ϕ are continuous, ψ is continuous on $[0,1]$. $\psi(0) = f\big(\phi(0)\big) = f(x_1) = y_1$ and $\psi(1) = f\big(\phi(1)\big) = f(x_1) = y_1$. Hence, ψ is a path in $f(A)$ joining y_1 and y_2. Since y_1 and y_2 are arbitrary pair of points in $f(A)$, $f(A)$ is path connected.



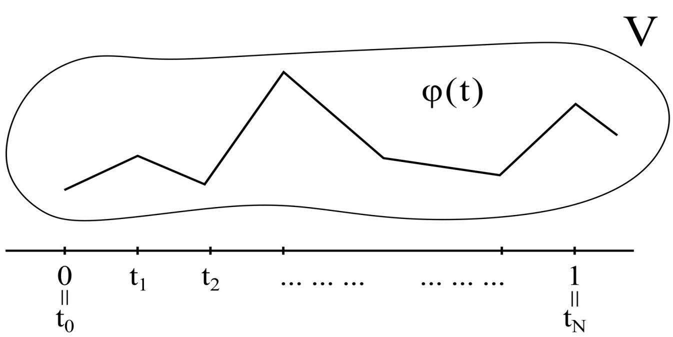
□

Corollary 3.5.8. *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) \neq f(b)$, then for any value L between $f(a)$ and $f(b)$, there exists $c \in (a,b)$ such that $f(c) = L$.*

Proof. Since $[a,b]$ is connected and f is continuous, $f\big([a,b]\big)$ is connected in $\mathbb{R}$. Hence, for $f(a), f(b) \in f\big([a,b]\big)$ and L between $f(a)$ and $f(b)$ then $L \in f\big([a,b]\big)$. Therefore, there exists $c \in [a,b]$ such that $f(c) = L$. Since $L \neq f(a)$ and $L \neq f(b)$, we obtain $c \in (a,b)$.

□

Definition 3.5.9. Let V be a vector space and $\phi : [0,1] \to V$ be a continuous map. We say that ϕ is "*piecewise linear*" if there exists $t_0, t_1, \cdots, t_n \in [0,1]$ with $a = t_0 < t_1 < \cdots < t_n = 1$ such that ϕ is a linear map on each $[t_{i-1}, t_i]$ for $i = 1, 2, \cdots, n$.



Remark. Let A be a convex or star-shaped subset in a vector space V. Then for any pair of points $x, y \in A$, there exists a piecewise linear mapping in A joining x and y.

Lemma 3.5.10. *Let $x, y, z \in V$. If there are piecewise linear mappings $\phi_1, \phi_2 : [0,1] \to V$ such that ϕ_1 joins x and y and ϕ_2 joins y and z, then there exists a piecewise linear mapping $\phi : [0,1] \to V$ such that ϕ joins x and z.*

Theorem 3.5.11. *Let G be a connected and open set in a vector space V. Then for any $x, y \in G$, there exists a piecewise linear mapping $\phi : [0,1] \to G$ such that $\phi(0) = x$ and $\phi(1) = y$.*

Proof. Let $x \in G$. Define

$$G_1 = \left\{ z \in G \;\middle|\; \text{there exists a piecewise linear mapping } \phi_z(t) : [0,1] \to G \text{ such that } \phi_z(0) = x \text{ and } \phi_z(1) = z. \right\}$$

Clearly, $x \in G_1$. It sufficies to show that $G_1 = G$.

Claim 1: G_1 is open.

Proof of Claim 1: Let $z \in G$. Since G is open, there exists $\delta > 0$ such that $B(z, \delta) \subseteq G$. Since $B(z, \delta)$ is convex, for any point $z \in B(z, \delta)$, there exists a piecewise linear mapping joining z and z_1. Hence, by Lemma3.5.10 there is a piecewise linear mapping joining x and z_1. Then $z_1 \in G_1$ and hence $B(z, \delta) \subseteq G_1$. Thus, G_1 is open.

Claim2: $G \backslash G_1$ is open.

Proof of Claim 2: If $w \in G \backslash G_1$, then there exists no piecewise linear mapping joining x and w. Since G is open, there exists $r > 0$ such that $B(w, r) \subseteq G$. For any point $w_1 \in B(w, r)$, there is a piecewise linear mapping joining w and w_1.

Assume that $w_1 \in G_1$. Then, by Lemma3.5.10, there exists a piecewise linear mapping joining x and w. Thus, $w \in G_1$ and we obtain a contradiction. Hence, $B(z, r) \subseteq G \backslash G_1$. Then $G \backslash G_1$ is open.

By Claim 2, G_1 is closed in G. Then G_1 is both open and closed relative to G. Since G is connected, either $G_1 = \emptyset$ or $G_1 = G$. But $G_1 \neq \emptyset$ and hence $G_1 = G$.

□

Chapter 4

Uniform Convergence and the Space of Continuous Functions

4.1 Pointwise and Uniform Convergence

L^∞ convergence
uniform convergence

pointwise convergence

L^2 convergence

Definition 4.1.1. Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f_k, f : A \to N$ be maps for $k = 1, 2, \cdots$. The sequence $\{f_k\}_{k=1}^{\infty}$ is said to converge (pointwise) to f on A if

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big)$$

for every $a \in A$. We denote $f_k \to f$ pointwise (p.w.)

Remark. (precise definition) Suppose $f_k \to f$ pointwise if for every $\varepsilon > 0$ and every $a \in A$, there exists $N = N(\varepsilon, a) \in \mathbb{N}$ such that if $k \geq N$,

$$\rho\big(f_k(a), f(a)\big) < \varepsilon.$$

Note that N depends on ε and a.

Definition 4.1.2. Let (M, d) and (N, ρ) be two metrics, $B \subseteq A \subseteq M$ and $f_k, f : A \to N$ be maps for $k = 1, 2, \cdots$. We say that the sequence $\{f_k\}_{k=1}^{\infty}$ "*uniformly converges*" to f on B if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that for every $x \in B$, if $k \geq N$

$$\rho\big(f_k(x), f(x)\big) < \varepsilon.$$

We write $f_k \to f$ uniformly on B.

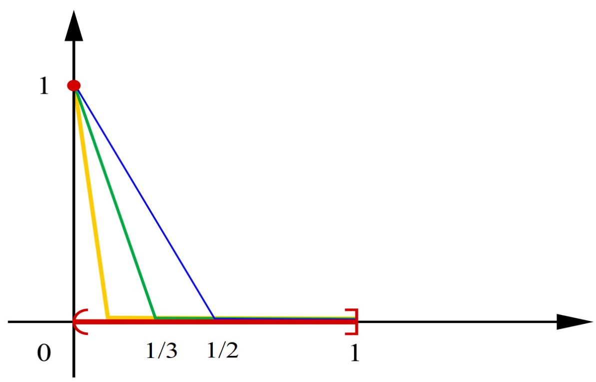
Note that this "N" only depends on ε but is indenpendent of x.

Remark. Let (M, d) and (N, ρ) be two metrics, $B \subseteq A \subseteq M$ and $f_k, f : A \to N$ be maps for $k = 1, 2, \cdots$. We say that the sequence $\{f_k\}_{k=1}^{\infty}$ "*uniformly converges*" to f on B if

$$\lim_{k\to\infty} \Big(\sup_{x\in B} \rho\big(f_k(x), f(x)\big) \Big) = 0.$$

Example 4.1.3. Let $f_k, f : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 0, & x \in [\frac{1}{k}, 1] \\ -kx + 1, & x \in [0, \frac{1}{k}) \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0, & x \in (0, 1] \\ 1, & x = 0 \end{cases}$$



Then f_k converges to f pointwise but does not converges to f uniformly.

Example 4.1.4. Let $f_k : [0, 1] \to \mathbb{R}$ by $f_k(x) = x^k$ and $f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$

(1) Fix $x \in [0,1)$,

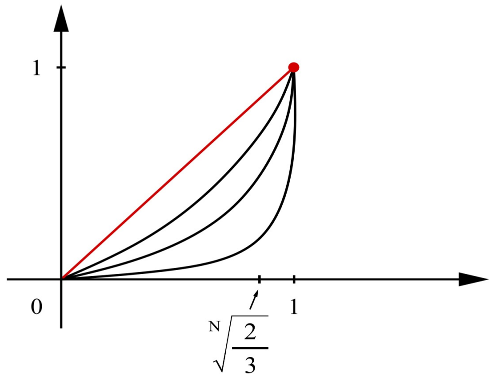
$$x^k \to 0 \quad \text{as } k \to \infty.$$

Clearly, $1^k = 1$ for every k. Hence, $f_k(x) \to f(x)$ pointwise on $[0,1]$.

(2) Let $\varepsilon = \frac{1}{2}$. For every $N \in \mathbb{N}$, choose $x_N = \sqrt[N]{\frac{2}{3}} \in [0,1)$. Then

$$\left|f_N(x_N) - f(x_N)\right| = \left|\frac{2}{3} - 0\right| > \varepsilon.$$

Hence, $\{f_k\}_{k=1}^{\infty}$ does not converges to f on $[0,1]$ uniformly.



(3) Fix $0 < a < 1$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $N > \dfrac{\ln \varepsilon}{\ln a}$. For every $x \in [0,a]$ and $k \geq N$,

$$\left|f_k(x) - f(x)\right| = |x^k - 0| \leq a^k < \varepsilon.$$

Hence, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0,a]$ for every $0 \leq a < 1$.

Example 4.1.5. Let $f_k : \mathbb{R} \to \mathbb{R}$ by $f_(x) = \dfrac{\sin x}{k}$ and $f(x) = 0$. Then $f_k \to f$ uniformly on $\mathbb{R}$.

Proposition 4.1.6. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f_k, f : A \to N$ be maps for $k = 1,2,\cdots$. If $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly, then $\{f_k\}_{k=1}^{\infty}$ converges to f pointwise.*

Proof. (Exercise) □

Remark.

$$\text{Uniform Convergence} \quad \Rightarrow \quad \text{Pointwise Convergence}$$
$$\not\Leftarrow$$

Proposition 4.1.7. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f_k : A \to N$ be a sequence of maps. Suppose that (N,d) is complete. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on A if and only if for every $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that for every $x \in A$ and $m,n \geq L$,*

$$\rho\big(f_m(x), f_n(x)\big) < \varepsilon.$$

Proof. ($\Longrightarrow$) Let $f : A \to N$ be a map where $f_k \to f$ uniformly on A. Given $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that for every $x \in A$ and $k \geq L$,

$$\rho\big(f_n,(x), f(x)\big) < \frac{\varepsilon}{2}.$$

For every $m,n \geq L$,

$$\rho\big(f_m(x), f_n(x)\big) \leq \rho\big(f_(x), f(x)\big) + \rho\big(f(x), f_m(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Longleftarrow$) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of maps on A with the Cauchy criterion. Fix $a \in A$, the sequence $\{f_k(a)\}_{k=1}^{\infty}$ is a Cauchy sequence in N. Since (N, ρ) is complete, there exists $y_a \in N$ such that $f_k(a) \to y_a$ in (N, ρ).
By the same argument, for every $x \in A$, there exists $y_x \in A$ such that $f_k(x) \to y_x$. Define a map $f : A \to N$ by $f(x) = y_x$. Then $f_k \to f$ pointwise on A.

Given $\varepsilon > 0$, by the Cauchy creiterion, there exists $L \in \mathbb{N}$ such that for every $x \in A$ and $m, n \geq L$,

$$\rho\big(f_m(x), f_n(x)\big) < \frac{\varepsilon}{2}.$$

Since $f_k \to f$ pointwise on A, for every $x \in A$, there exists $L_x \geq L$ such that if $m > L_x$,

$$\rho\big(f_m(x), f(x)\big) < \frac{\varepsilon}{2}.$$

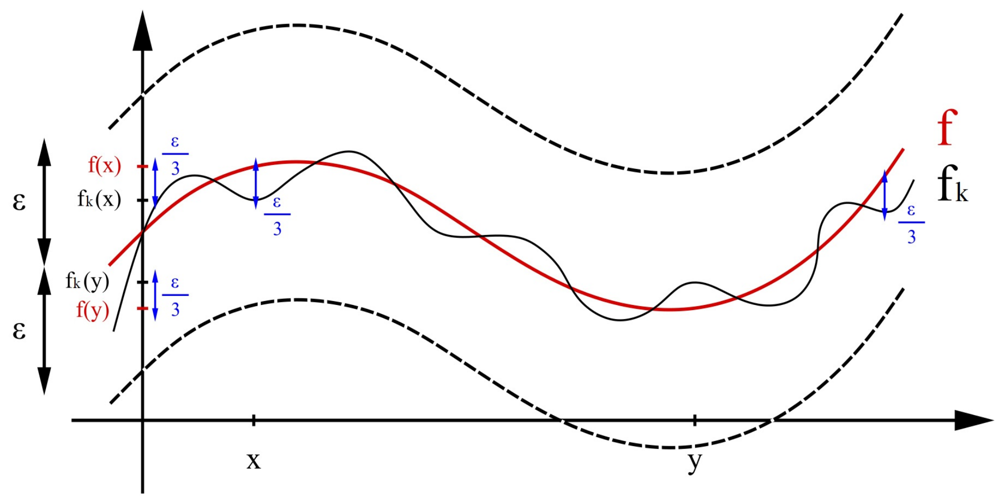
Hence, for every $x \in A$ and $k \geq L$, we choose $m_x \geq L_x \geq L$. Then

$$\rho\big(f_k(x), f(x)\big) \leq \underbrace{\rho\big(f_k(x), f_{m_x}(x)\big)}_{\textit{Cauchy criterion}} + \underbrace{\rho\big(f_{m_x}(x), f(x)\big)}_{\textit{pointwise convergence}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $f_k \to f$ uniformly on A. □

Remark. The completeness of (N, ρ) is NOT necessay in the direction ($\Longrightarrow$).

Theorem 4.1.8. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f_k : A \to N$ be a sequence of continuous maps converging to $f : A \to N$ uniformly on A. Then f is continuouos on A.*



Proof. Since $f_k \to f$ uniformly on A, for given $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that for every $x \in A$ and $k \geq L$,

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3}.$$

Since f_L is continuous on A, for $a \in A$, there exists $\delta_a > 0$ such that if $x \in A$ with $d(x, a) < \delta_a$,

$$\rho\big(f_L(a), f_L(x)\big) < \frac{\varepsilon}{3}.$$

Hence,

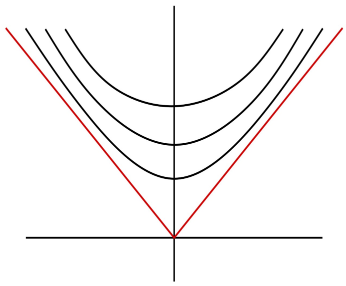
$$\begin{aligned}\rho\big(f(x), f(a)\big) &\le \rho\big(f(x), f_L(x)\big) + \rho\big(f_L(x), f_L(a)\big) + \rho\big(f_L(a), f(a)\big)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for every $x \in A$ with $d(x, a) < \delta_a$. Thus, f is continuous at a. Since a is an arbitrary point in A, f is continuous on A. □

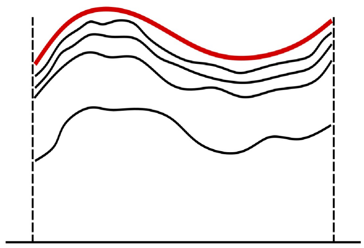
Remark. (1) The uniform convergence of $\{f_k\}_{k=1}^{\infty}$ suggests a switch of the limit of points and the limit of sequence. That is, $f_k \to f$ uniformly on A and $a \in \overline{A}$, then

$$\lim_{x\to a}\Big(\lim_{k\to\infty} f_k(x)\Big) = \lim_{k\to\infty}\Big(\lim_{x\to a} f_k(x)\Big).$$

(2) The uniform limit of a sequence of continuous functions might not be uniformly continuous.
Question: How about the uniform limit of a sequence of uniformly continuous functions? Is it uniformly continuous?
Answer: Yes.



Suppose $f_k : I \to \mathbb{R}$ uniformly converges to f.
Question: If each f_k is differentiable, is f differentiable? If yes, does $f_k' \to f$?
Answer: No.



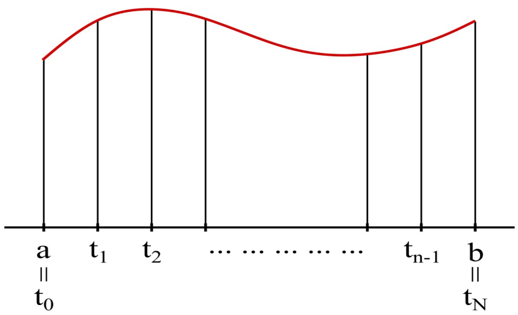
Question: If each f_k is integrable, is f integrable? If yes, does $\displaystyle\int_I f_k\,dx \to \int_I f\,dx$?
Answer: Yes.

Recall: Let $f : [a, b] \to \mathbb{R}$ and $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$.

The upper and lower sums of P for f are

$$U(P, f) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

where $M_i = \sup_{t\in[t_{i-1},t_i]} f(t)$ and $m_i = \inf_{t\in[t_{i-1},t_i]} f(t)$. We have



(i) $L(P, f) \le U(P, f)$.

(ii) If P_1 is a refinement of P (that is, $P \subseteq P_1$), then

$$L(P, f) \le L(P_1, f) \le U(P_1, f) \le U(P, f).$$

(iii) For any two partitions P_1 and P_2 of $[a, b]$, we have

$$L(P_1, f) \le U(P_2, f).$$

(iv) $\underline{\int_a^b} f(x)\, dx := \sup_P L(P, f)$ and $\overline{\int_a^b} f(x)\, dx := \inf_P U(P, f)$. Clearly,

$$\underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx.$$

If $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$, we say f is (Riemannian) integrable on $[a, b]$ and denoted by $\int_a^b f(x)\, dx$.

(v) A function f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Theorem 4.1.9. *(Uniform convergence and integration) Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of integrable functions which converge uniformly to f on $[a, b]$. Then f is integrable and*

$$\lim_{k\to\infty} \int_a^b f_k(x)\, dx = \int_a^b f(x)\, dx$$

Proof. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[a, b]$, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $k \ge N$ and $x \in [a, b]$,

$$|f_k(x) - f(x)| < \varepsilon.$$

Since f_N is integrable on $[a, b]$, there exists a partition P of $[a, b]$ such that $U(P, f_N) - L(P, f_N) < \varepsilon$. Let

$$M_i = \sup_{t\in[t_{i-1},t_i]} f(t),\ m_i = \inf_{t\in[t_{i-1},t_i]} f(t),\ M_i^{(N)} = \sup_{t\in[t_{i-1},t_i]} f_N(t),\ m_i^{(N)} = \inf_{t\in[t_{i-1},t_i]} f_N(t).$$

Then

$$\left|M_i - M_i^{(N)}\right| \le \sup_{t\in[t_{i-1},t_i]} |f(t) - f_N(t)| < \varepsilon \quad \text{and} \quad \left|m_i - m_i^{(N)}\right| \le \sup_{t\in[t_{i-1},t_i]} |f(t) - f_N(t)| < \varepsilon.$$

We have

$$\begin{aligned} U(P, f) - L(P, f) &\le \left|U(P, f) - U(P, f_N)\right| + \left|U(P, f_N) - L(P, f_N)\right| + \left|L(P, f_N) - L(P, f)\right| \\ &< \sum_{i=1}^{n} \left|M_i - M_i^{(N)}\right|(t_i - t_{i-1}) + \varepsilon + \sum_{i=1}^{n} \left|m_i - m_i^{(N)}\right|(t_i - t_{i-1}) \\ &< 2\varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) + \varepsilon \\ &= [2(b - a) + 1]\varepsilon. \end{aligned}$$

Hence, f is integrable on $[a, b]$. Moreover, for $k \geq N$,

$$\Big|\int_a^b f(x)\,dx - \int_a^b f_k(x)\,dx\Big| = \Big|\int_a^b f(x) - f_k(x)\,dx\Big| \leq \int_a^b |f(x) - f_k(x)|\,dx < \varepsilon(b-a).$$

Therefore,

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} \int_a^b f_k(x)\,dx.$$

□

Example 4.1.10. (Using the integrabtion to determine the convergence is not uniform)
The set $\mathbb{Q} \cap [0, 1]$ is countable. Write $\mathbb{Q} \cap [0, 1] = \{q_k \mid k \in \mathbb{N}\}$ Define $f_k(x) : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 1 & x \in \{q_1, q_2, \cdots, q_k\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_k \to f$ pointwise on $[0, 1]$. On the other hand, every $f_k(x)$ is integrable on $[0, 1]$, but f is not integrable on $[0, 1]$. Hence, $\{f_k(x)\}_{k=1}^{\infty}$ does not converge to f uniformly.

Note that we can check this result directly.

Remark. Suppose that $f_k \to f$ pointwise and $\int f_k\,dx \to \int f\,dx$. It cannot imply that $f_k \to f$ uniformly.

Theorem 4.1.11. *(Uniform convergence and differentiation) Let $I \subseteq \mathbb{R}$ be a finite interval. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions which are differentiable on I and such that $\{f_k(a)\}_{k=1}^{\infty}$ converges for some $a \in I$. If $\{f_k'\}_{k=1}^{\infty}$ converges uniformly to g on I, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on I to a function f, and*

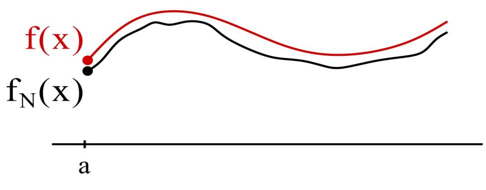
$$f'(x) = \frac{d}{dx}\Big(\lim_{k\to\infty} f_k(x)\Big) = \lim_{k\to\infty}\Big(\frac{d}{dx} f_k(x)\Big) = \lim_{k\to\infty} f_k'(x).$$

Proof. Since $\{f_k(a)\}_{k=1}^{\infty}$ converges, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$,

$$\big|f_n(a) - f_m(a)\big| < \frac{\varepsilon}{2}.$$

Since $\{f_k'\}_{k=1}^{\infty}$ converges uniformly on I, there exists $N_1 \in N$ such that for every $x \in I$ and for $m, n \geq N_1$,

$$\big|f_n'(x) - f_m'(x)\big| < \frac{\varepsilon}{2|I|}.$$



For $x \in I$ and $m, n \geq \max(N, N_1)$, by M.V.T, there exists $c_x \in (x_0, x)$ [or $c_x \in (x, x_0)$], such that

$$\Big|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\Big| = \big|f_n'(c_x) - f_m'(c_x)\big||x - x_0| < \frac{\varepsilon}{2|I|}\cdot|I| = \frac{\varepsilon}{2}. \tag{4.1}$$

Then

$$\big|f_n(x) - f_m(x)\big| \leq \big|f_n(x_0) - f_m(x_0)\big| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ converges uniformly on I. That is, there exists $f : I \to \mathbb{R}$ such that $f_k \to f$ uniformly on I.

(**To prove** $f'(x) = \lim_{k\to\infty} f_k'(x)$.) Fix $x \in I$.
Define

$$\phi_k(t) = \begin{cases} \dfrac{f_k(t) - f_k(x)}{t - x} & t \in I,\ t \neq x \\ f_k'(x) & t = x \end{cases} \quad \text{and} \quad \phi(t) = \begin{cases} \dfrac{f(t) - f(x)}{t - x} & t \in I,\ t \neq x \\ g(x) & t = x \end{cases}$$

Then $\lim_{k\to\infty} \phi_k(t) = \phi(t)$ and $\lim_{t\to x} \phi_k(t) = f_k'(x)$ for $k = 1, 2, 3, \cdots$.

Given $\varepsilon > 0$, for $m, n \geq N_1$ and for every $t \in I\backslash\{x\}$,

$$\begin{aligned} \Big|\phi_n(t) - \phi_m(t)\Big| &= \frac{1}{|t-x|}\Big|\big[f_n(t) - f_n(x)\big] - \big[f_m(t) - f_m(x)\big]\Big| \\ &\leq \frac{1}{|t-x|}\Big|f_n'(c_{t,x}) - f_m'(c_{t,x})\Big||t-x| \quad \text{for some } c_{t,x} \in (t, x) \\ &\leq \frac{\varepsilon}{2|I|}. \end{aligned}$$

Hence $\{\phi_k(x)\}_{k=1}^{\infty}$ satisfies the Cauchy criterion on $I\backslash\{x\}$. Then $\{\phi_k(x)\}_{k=1}^{\infty}$ converges uniformly on $I\backslash\{x\}$. Moreover,

$$\lim_{k\to\infty} f'(x) = \lim_{k\to\infty}\Big(\lim_{t\to x}\phi_k(t)\Big) = \lim_{t\to x}\Big(\lim_{k\to\infty}\phi_k(t)\Big) = \lim_{t\to x}\phi(t) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = f'(x).$$

□

Remark. Under the same hypothesis of the theorem, assume that $\{f_k'\}_{k=1}^{\infty}$ is a sequence of continuous function. We can use the F.T.C to prove it.

Theorem 4.1.12. *(Uniform convergence and differentiation) Let $I \subset \mathbb{R}$ be a finite interval, $f_k : I \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\{f_k(a)\}_{k=1}^{\infty}$ converges for some $a \in I$ and $\{f_k'\}_{k=1}^{\infty}$ converges uniformly to a function g on I. Then*

(1) $\{f_k\}_{k=1}^{\infty}$ converges uniformly to some function f on I.

(2) the limit function f is differentiable on I and $f'(x) = g(x)$ for all $x \in I$. That is,

$$g(x) = \lim_{k\to\infty}\Big(\frac{d}{dx}f_k(x)\Big) = \lim_{k\to\infty} f_k'(x) = f'(x) = \frac{d}{dx}\Big(\lim_{k\to\infty} f_k(x)\Big).$$

Proof. For $x \in I$, by the F.T.C,

$$f_k(x) = f_k(a) + \int_a^x f_k'(t)\,dt.$$

Since $\{f_k(a)\}_{k=1}^{\infty}$ converges and by Theorem 4.1.9, $\{\int_a^x f_k'(t)\,dt\}_{k=1}^{\infty}$ converges for every $x \in I$. Since $\{f_k(x)\}_{k=1}^{\infty}$ converges for every $x \in I$, we can define

$$f(x) = \lim_{k\to\infty}\Big(f_k(a) + \int_a^x f_k'(t)\,dt\Big).$$

Hence,

$$f(a) = \lim_{k\to\infty} f_k(a) \quad \text{and} \quad f(x) - f(a) = \lim_{k\to\infty} \int_a^x f_k'(t)\,dt = \int_a^x g(t)\,dt.$$

We have $f'(x) = g(x)$.

(**To check that** $f_k \to f(x)$ **uniformly on** I.)
Since $\lim_{k\to\infty} f_k(a) = f(a)$, given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $k \ge N_1$, $|f_k(a) - f(a)| < \frac{\varepsilon}{2}$.
Since $f_k'(x) \to g(x)$ uniformly on I, there exists $N_2 \in \mathbb{N}$ such that for every $x \in I$ and $k \ge N_2$,

$$|f_k'(x) - g(x)| < \frac{\varepsilon}{2|I|}.$$

Therefore,

$$\begin{aligned} |f(x) - f_k(x)| &= \left| \left[f(a) + \int_a^x g(t)\,dt\right] - \left[f_k(a) + \int_a^x f_k'(t)\,dt\right] \right| \\ &\le |f(a) - f_k(a)| + \int_a^x |g(t) - f_k'(t)|\,dt \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

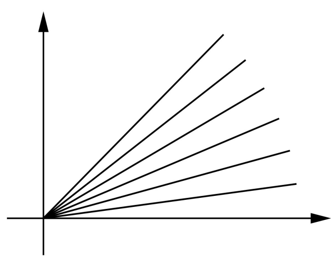
Thus, $f_k \to f$ uniformly on I.

□

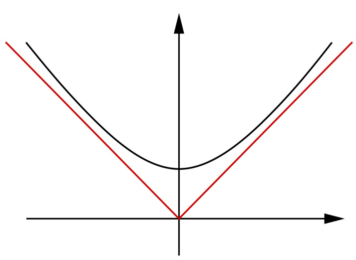
Remark. In the theorem,

(1) the conditioin "$f_k(a) \to f(a)$" is necessary. For example, $f_k(x) = k$, then $f_k' \equiv 0$. But $\{f_k\}_{k=1}^{\infty}$ does not converge.

(2) the finiteness of the interval is necessary. For example $f_k(0) = 0$ and $f_k'(x) = \frac{1}{k}$. Then $f_k'(x) \to g(x) \equiv 0$. But $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly.



Remark. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of differentiable functions and $f_k \to f$ uniformly. It cannot imply that f is differentiable. For example



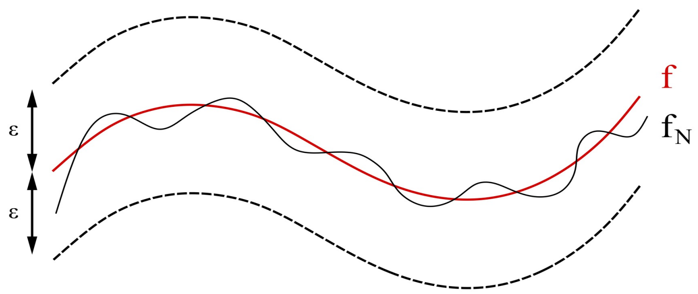
$$f_k(x) = \begin{cases} \frac{k}{2}x^2 & \text{if } |x| \le \frac{1}{k} \\ |x| - \frac{1}{2k} & \text{if } \frac{1}{k} \le |x| \le 1 \end{cases} \quad \text{and } f(x) = |x|.$$

Then $f_k \to f$ uniformly and f_k is differentiable. But f is not differentiable at 0.

❑ Pointwise Convergence v.s. Uniform Convergence

A sequence of functions $\{f_k\}_{k=1}^{\infty}$ which converges uniformly to f automatically converges to f pointwise. But the converse is false. For example, $f_k(x) = k^2x^2(1-x^2)^k$ on $[0,1]$. Then $f_k(x) \to 0$ pointwise but does not converge uniformly.

Question: Is the converse true under certain conditions? By observing the convergence of a



monotonic sequence and monotone sequence property, we know that a monotonic sequence will be closer and closer to its limit. We hope this situation will occur on a sequence of functions. We hope that

$$\left\{x \,\middle|\, \left|f_n(x) - f(x)\right| < \varepsilon\right\} \subseteq \left\{x \,\middle|\, \left|f_{n+1}(x) - f(x)\right| < \varepsilon\right\}.$$

However, there may have some possible troubles.

1. For some $a \in A$, it is possible that $a \in \left\{\left|f_N - f\right| < \varepsilon\right\}$ but $a \notin \left\{\left|f_{N+1} - f\right| < \varepsilon\right\}$. We expect an additional condition that

$$f_1 \le f_2 \le \cdots \le f_n \le f_{n+1} \le \cdots.$$

2. The rate of the convergence of the sequence at some point is too slow. If the domain contains finitely many points, it will not be a trouble. But when the domain contains inifitely many points, this situation will be happend. A compact domain may be overcome this trouble.

Theorem 4.1.13. *(Dini's Theorem) Suppose that K is compact and*

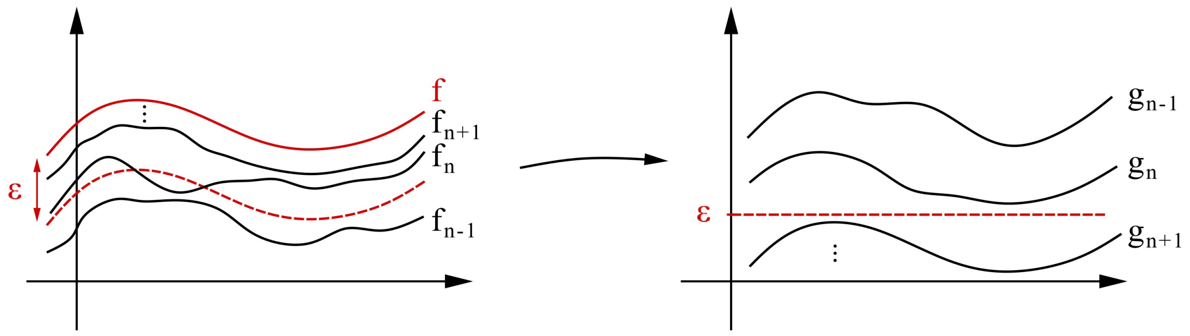
(a) $f_n : K \to \mathbb{R}$ *is continuous on K for* $n = 1, 2, 3, \cdots$;

(b) $\{f_n\}_{n=1}^{\infty}$ *converges pointwise to a continuous function f on K;*

(c) $f_n \le f_{n+1}$ *for all* $n \in \mathbb{N}$.

Then $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to f on K.*

Proof. Define $g_n = f - f_n$ for all $n \in \mathbb{N}$. Since $f_n \to f$ pointwise, $f_n \le f_{n+1}$ and f, f_n are continuous on K for every $n \in \mathbb{N}$, we have $g_n \to 0$ pointwise, $g_n \ge g_{n+1}$, $g_n \ge 0$ for every $n \in \mathbb{N}$ and g_n are continuous on K. It sufficies to show that $g_n \to 0$ uniformly on K.



Given $\varepsilon > 0$, we define $K_n = \{x \in K \mid g_n(x) \geq \varepsilon\}$. Since $g_n(x)$ is continuous on K, $K_n = g_n^{-1}\big([\varepsilon, \infty)\big)$ is closed in K. Then K_n is compact since K is compact. Moreover, since $g_n \geq g_{n+1}$ for every $n \in \mathbb{N}$, we obatin $K_{n+1} \subseteq K_n$ for every $n \in \mathbb{N}$. Fix $x \in K$. Since $g_n(x) \to 0$ as $n \to \infty$, $x \notin K_n$ as n is sufficiently large. We have $x \notin \bigcap_{n=1}^{\infty} K_n$ for every $x \in K$. That is, $\bigcap_{n=1}^{\infty} K_n = \emptyset$. By the finite intersection property, there exists $N \in \mathbb{N}$ such that

$$\bigcap_{n=1}^{N} K_n = \emptyset.$$

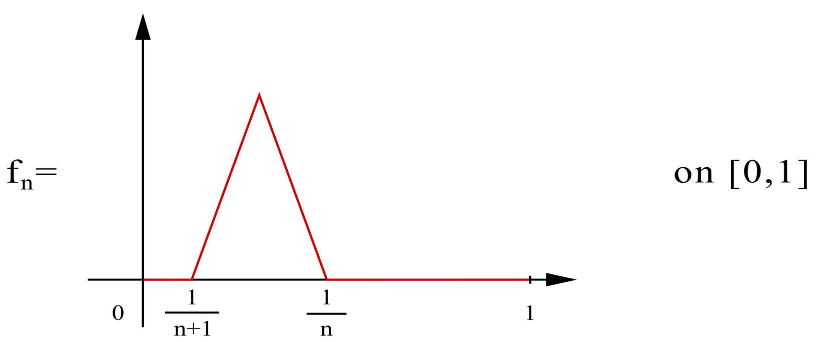
That is, if $n \geq N$, $g_n(x) < \varepsilon$ for every $x \in K$. Hence, $f_n \to f$ uniformly on K.

□

Remark. (1) The result of Theorem 4.1.13 is true if the condition (c) is replaced by $f_n \geq f_{n+1}$.

(2) The compactness is necessary. For example, $f_n(x) = \dfrac{1}{nx+1}$ on $(0, 1)$. Then $f_n \to 0$ pointwise but not uniformly.

(3) The monotonicity is necessary. For exampe, $f_n = \begin{cases} 0 & x \in [0, \dfrac{1}{n+1}] \\ 2n(n+1)(x - \frac{1}{n+1}) & x \in [\frac{1}{n+1}, \frac{2n+1}{2n(n+1)}] \\ -2n(n+1)(x - \frac{1}{n}) & x \in [\frac{2n+1}{2n(n+1)}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$



4.2 Series of Functions

Definition 4.2.1. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed space, $A \subseteq M$ and $g_k, g : A \to V$ be functions.

(1) We say that the series $\sum_{k=1}^{\infty} g_k$ converges pointwise to g if the sequence of partial sum $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n(x) = \sum_{k=1}^{n} g_k(x)$$

converges pointwise to g.

(2) We say that $\sum_{k=1}^{\infty} g_k$ converge to g uniformly on A if $\{s_n\}_{n=1}^{\infty}$ converges to g uniformly on A.

Example 4.2.2. For the geometric series $\sum_{k=0}^{\infty} x^k$, $s_n(x) = \sum_{k=0}^{n} x^k = \begin{cases} \dfrac{1-x^{n+1}}{1-x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1 \end{cases}$

(1) For $x \in (-1, 1)$, $s_n \to \dfrac{1}{1-x}$. Hence, $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$ converges pointwise on $(-1, 1)$.

(2) For $x \in (-\infty, -1] \cup [1, \infty)$, $\{s_n\}_{n=1}^{\infty}$ diverges. Hence $\sum_{k=0}^{\infty} x^k$ diverges on $(-\infty, -1] \cup [1, \infty)$

(3) Let $0 < a < 1$ and $g(x) = \dfrac{1}{1-x}$. For $x \in [-a, a]$,

$$\left|s_n(x) - g(x)\right| = \left|\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}\right| = \left|\frac{x^{n+1}}{1-x}\right| \leq \frac{|a|^{n+1}}{1-a} \to 0.$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that if $n \geq N$, then $\dfrac{|a|^{n+1}}{1-a} < \varepsilon$ and thus $\left|s_n(x) - g(x)\right| < \varepsilon$ for every $x \in [-a, a]$ whenever $n \geq N$. Hence, $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-a, a]$.

(4) $\sum_{k=0}^{\infty} x^k$ does not converge uniformly on $(-1, 1)$ since $\sup_{x \in (-1,1)} \left|s_n(x) - g(x)\right| = \infty$.

Theorem 4.2.3. *(Cauchy Criterion) Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed space, $A \subseteq M$ and $g_k : A \to V$ be functions. If $\sum_{k=1}^{\infty} g_k$ converges uniformly on A, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > m \geq N$,*

$$\left\|\sum_{m+1}^{n} g_k(x)\right\| < \varepsilon$$

for every $x \in A$. In addition, if $(V, \|\cdot\|)$ is a Banach space, then the converse holds.

Proof. ($\Longrightarrow$)
Let $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to $g(x)$ on A. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\left\| s_n(x) - g(x) \right\| < \frac{\varepsilon}{2} \quad \text{for every } x \in A.$$

Hence, for $n > m \geq N$,

$$\Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| = \left\| s_n(x) - s_m(x) \right\| \leq \left\| s_n(x) - g(x) \right\| + \left\| s_m(x) - g(x) \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $x \in A$.

($\Longleftarrow$)
Suppose that $(V, \|\cdot\|)$ is a Banach space. Fix $x \in A$, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > m \geq N$,

$$\left\| s_n(x) - s_m(x) \right\| = \Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| < \varepsilon.$$

Hence, $\left\{ s_n(x) \right\}_{n=1}^{\infty}$ is a Cauchy sequence in V. Since $(V, \|\cdot\|)$ is complete, $\left\{ s_n(x) \right\}_{n=1}^{\infty}$ converges in V, say $\lim_{n\to\infty} s_n(x) = g(x)$. Hence, $\left\{ s_n(x) \right\}_{n=1}^{\infty}$ converges to $g(x)$ pointwise on A.

Now, we check that $s_n(x) \to g(x)$ uniformly on A. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > m \geq N$,

$$\left\| s_n(x) - s_m(x) \right\| < \frac{\varepsilon}{2} \quad \text{for every } x \in A.$$

Since $s_n(x) \to g(x)$ pointwise on A, for every $x \in A$, there exists $m_x \geq N$ such that $\left\| s_{m_x}(x) - g(x) \right\| < \frac{\varepsilon}{2}$. Then, for every $x \in A$ and $n \geq N$,

$$\left\| s_n(x) - g(x) \right\| \leq \left\| s_n(x) - s_{m_x}(x) \right\| + \left\| s_{m_x}(x) - g(x) \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $s_n(x) \to g(x)$ uniformly on A. □

Corollary 4.2.4. *If $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on A, then g_k converges to 0 (0-function) uniformly on A.*

Theorem 4.2.5. *Let (M, d) be a metric space and $(V, \|\cdot\|)$ be a normed space, $A \subseteq M$ and $g_k, g : A \to V$ be functions. If $g_k : A \to V$ are continuous and $\sum_{k=1}^{\infty} g_k(x)$ converges to g uniformly on A, then g is continuous.*

Proof. Since g_k are continuous on A for every $k \in \mathbb{N}$, $s_n(x) = \sum_{k=1}^{n} g_k(x)$ are continuous on A for every $n \in \mathbb{N}$. Since $\{s_n(x)\}_{n=1}^{\infty}$ converges to $g(x)$ uniformly on A, $g(x)$ is continuous on A. □

Corollary 4.2.6. *If $g_k : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and $g(x) = \sum_{k=1}^{\infty} g_k(x)$ converges uniformly on $[a,b]$, then*

$$\int_a^b g(x)\,dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

Proof. (Exercise) □

❑ Weierstrass M-Test

Theorem 4.2.7. *(Weierstrass M-Test) Let (M,d) be a metric space, $(V, \|\cdot\|)$ be a Banach space, $A \subseteq M$ and $g_k : A \to V$ be a sequence of functions. Suppose that there exists $M_k > 0$ such that $\sup_{x\in A} \|g_k(x)\| \le M_k$ for every $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} M_k$ converges. Then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly and absolutely (that is, $\sum_{k=1}^{\infty} \|g_k(x)\|$ converges uniformly) on A.*

Proof. To show that $\{s_n\}_{n=1}^{\infty}$ satisfies the Cauchy criterion. Since $\sum_{k=1}^{\infty} M_k$ converges, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m \ge N$, then

$$\sum_{k=m+1}^{n} M_k < \varepsilon.$$

Thus,

$$\sup_{x\in A} \left\| s_n(x) - s_m(x) \right\| = \sup_{x\in A} \left\| \sum_{k=m+1}^{n} g_k(x) \right\| \le \sum_{k=m+1}^{n} \sup_{x\in A} \|g_k(x)\| \le \sum_{k=m+1}^{n} M_k < \varepsilon.$$

Hence, $\left\{ s_n(x) \right\}_{n=1}^{\infty}$ converges uniformly on A. Similarly, let $t_n(x) = \sum_{k=1}^{n} \|g_k(x)\|$. for $n > m \ge N$,

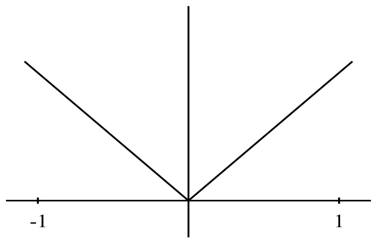
$$\sup_{x\in A} \left| t_n(x) - t_m(x) \right| = \sup_{x\in A} \sum_{k=m+1}^{n} \|g_k(x)\| \le \sum_{k=m+1}^{n} \sup_{x\in A} \|g_k(x)\| \le \sum_{k=m+1}^{n} M_k < \varepsilon.$$

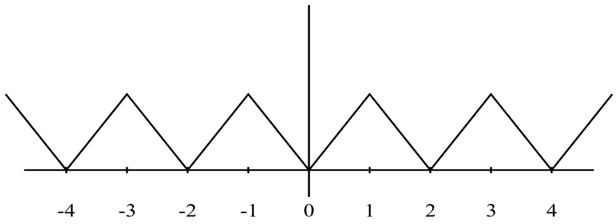
Therefore, $\sum_{k=1}^{\infty} \|g_k(x)\|$ converges uniformly on A. □

■ Application

(I) (Continuous and nowhere differentiable function) (Reference from [Rudin])

Theorem 4.2.8. *There exists a real continuous function on $\mathbb{R}$ which is nowhere differentiable.*





Proof. Define $\phi(x) = |x|$ on $[-1, 1]$ and extend $\phi(x)$ to a 2-period function on $\mathbb{R}$ (still call $\phi(x)$). Then, $\phi(x + 2) = \phi(x)$ for every $x \in \mathbb{R}$.

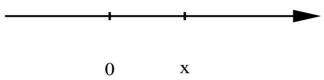
Thus, for $s, t \in \mathbb{R}$,

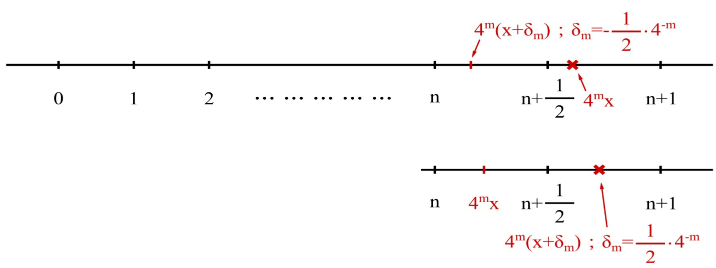
$$\left|\phi(s) - \phi(t)\right| \leq |s - t| \tag{4.2}$$

and ϕ is continuous on $\mathbb{R}$.

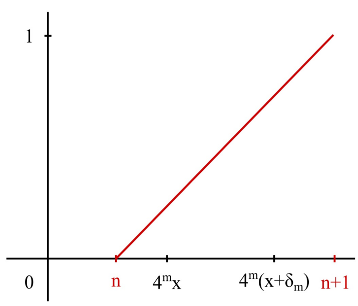
Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \underbrace{\phi(4^n x)}_{\leq 1}$. Since $0 \leq \phi(x) \leq 1$, by M-Test, the series converges uniformly on $\mathbb{R}$. Hence, $f(x)$ is continuous on $\mathbb{R}$.

Now, we want to prove that f is nowhere differentiable. Fix $x \in \mathbb{R}$. (We will show that $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ does not exist). Fix $m \in \mathbb{N}$ and let $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$ where the sign is chosen such that $\mathbb{Z} \cap \left(4^m x, 4^m(x + \delta_m)\right) = \emptyset$ or $\mathbb{Z} \cap \left(4^m(x + \delta_m), 4^m x\right) = \emptyset$.





Define $r_n = \dfrac{\phi\left(4^n(x + \delta_m)\right) - \phi(4^n x)}{\delta_m}$.



- When $n > m$, then $4^n \delta_m = \pm\frac{1}{2} \cdot 4^{n-m}$ is an even integer. Since $\phi(x + 2) = \phi(x)$, $\phi\left(4^n(x + \delta_m)\right) - \phi(4^n x) = 0$ and hence $r_n = 0$.
- When $0 \leq n \leq m$, by (4.2),

$$|r_n| = \frac{|4^n(x+\delta_m) - 4^n x|}{\delta_m} = 4^n \quad \text{and} \quad \phi\big(4^m(x+\delta_m)\big) - \phi(4^m x) = \pm 4^m \delta_m$$

and hence, $|r_m| = 4^m$. We obtain

$$\begin{aligned} \Big|\frac{f(x+\delta_m) - f(x)}{\delta_m}\Big| &= \Big|\sum_{n=0}^{m} \big(\frac{3}{4}\big)^n r_n\Big| \geq \big(\frac{3}{4}\big)^m |r_m| - \Big|\sum_{n=0}^{m-1} \big(\frac{3}{4}\big)^n r_n\Big| \\ &\geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1) \to \infty \quad \text{as } m \to \infty. \end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$, we have $\lim\limits_{m\to\infty} \dfrac{f(x+\delta_m) - f(x)}{\delta_m}$ does not exist and f is not differentiable at x. □

(II) (Approximate a smooth function by polynomials)
Let f have n-th derivatives at a. We want to use a n-the degree polynomial $P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$ to approximate f near a. We have $a_k = \frac{f^{(k)}(a)}{k!}$ for $k = 0, 1, \cdots, n$ and the polynomail is called the Taylor polynomail of n-th degree for f at a. We have known that

$$\frac{f(x) - P_n(x)}{(x-a)^n} \to 0 \quad \text{as } x \to a.$$

Question: If f has infinite derivatives at a, what can we say about $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$?

(i) Does the series converge at x?

(ii) How much is the Taylor series close to $f(x)$? Consider $R_n(x) = f(x) - P_n(x)$ and use the mean value theorem to estimate the errors.

Moreover, by the same ideas, for a continuous function, we want to use a power series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ to approximate it. Hence, we nned to consider the issues of the convergence of the series.

4.3 Taylor Series and Power Series

❑ Power Series

Definition 4.3.1. We call a series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k$$

a "*power series about a*" (or "*centered at a*") for some sequence $\{c_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. In particular, if $a = 0$, we call the series

$$\sum_{k=0}^{\infty} c_k x^k$$

a "*Maclaurin series*".

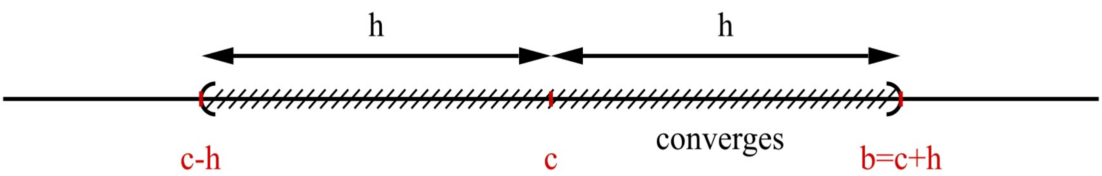
Remark. We can also define a power series in complex number.

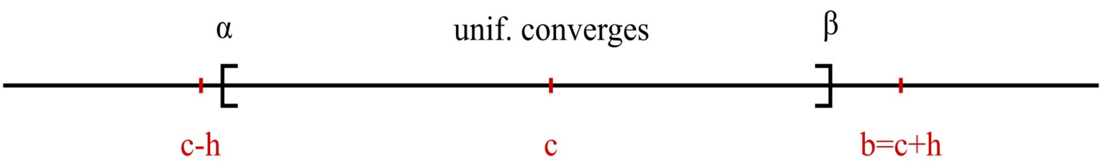
Definition 4.3.2. For $z \in \mathbb{C}$, we call a series of the form

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k$$

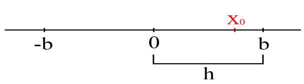
a power series about z_0 (or centered at z_0) for some $\{c_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$.

Theorem 4.3.3. *Let* $\sum_{k=0}^{\infty} c_k (x-a)^k$ *be a power series in* $\mathbb{R}$. *Suppose that the series converges at some point* $b \neq a$ *and define* $h := |b - a|$. *Then the series converges on* $(a-h, a+h)$. *Moreover, the series converges uniformly on* $[\alpha, \beta]$ *if* $[\alpha, \beta] \subseteq (a-h, a+h)$.





Proof. W.L.O.G, we may assume that $a = 0$ and the series $\sum_{k=0}^{\infty} c_k x^k$ converges at some $b \neq 0$. Then $h = |b|$.



Since $\sum_{k=0}^{\infty} c_k b^k$ converges, $|c_k| h^k = |c_k b^k| \to 0$ as $k \to \infty$. Then there exists $N \in \mathbb{N}$ such that for every $k \geq N$, $|c_k| h^k < 1$. Thus

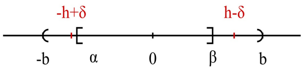
$$|c_k| < \frac{1}{h^k} \quad \text{for every } k \geq N.$$

For $x_0 \in (-h, h)$,

$$\sum_{k=N}^{\infty} |c_k x_0^k| = \sum_{k=N}^{\infty} |c_k||x_0|^k \leq \sum_{k=N}^{\infty} \frac{1}{h^k}|x_0|^k = \sum_{k=N}^{\infty} \left(\frac{|x_0|}{h}\right)^k < \infty.$$

Then $\sum_{k=0}^{\infty} c_k x_0^k$ converges absolutely and hence $\sum_{k=0}^{\infty} c_k x_0^k$ converges. Since x_0 is an arbitrary number in $(-h,h)$, $\sum_{k=0}^{\infty} c_k x^k$ converges on $(-h,h)$.

For $[\alpha,\beta] \subseteq (-h,h)$, choose $0 < \delta < h$ such that $[\alpha,\beta] \subseteq (-h+\delta, h-\delta)$.



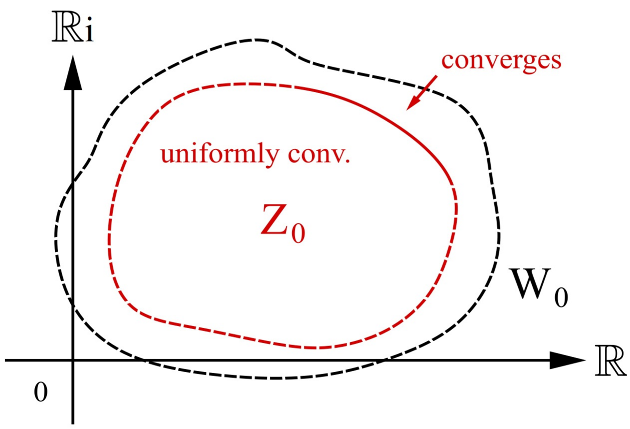
Then for $x_0 \in [\alpha,\beta]$, $\frac{|x_0|}{h} < 1 - \frac{\delta}{h}$. Hence, for $\varepsilon > 0$ and $m > n \geq N$,

$$\Big|\sum_{k=m+1}^{n} c_k x_0^k\Big| \leq \sum_{k=m+1}^{n} |c_k| h^k \cdot \Big(1 - \frac{\delta}{h}\Big)^k < \varepsilon$$

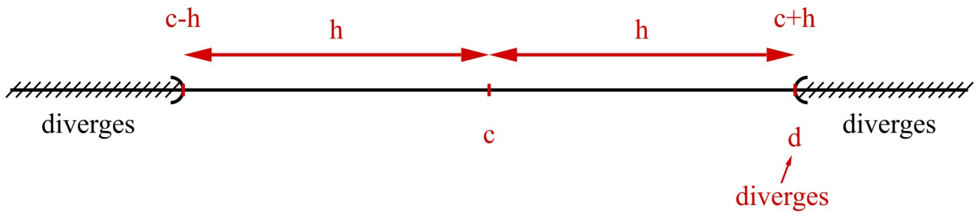
as m, n sufficiently large. Therefore, $\sum_{k=0}^{\infty} c_k x^k$ converges uniformly on $[\alpha,\beta]$. □

Remark. Every series is convergent at the center.

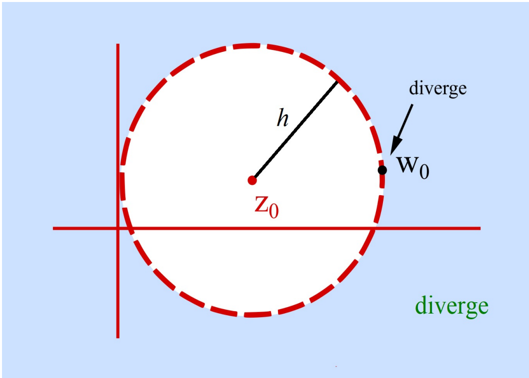
Remark. Let $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ be a complex power series. Suppose that the series converges at some $w_0 \neq z_0$ and define $h := |w_0 - z_0|$. Then the series converges on the set $B(z_0,h)$. The series converges uniformly on any set A where $\overline{A} \subseteq B(z_0,h)$.



Corollary 4.3.4. *(1) Let* $\sum_{k=0}^{\infty} c_k (x-a)^k$ *be a real power series and diverges at some* $d \neq a$. *Define* $h := |d-a|$. *Then the series diverges on* $(-\infty, a-h) \cup (a+h, \infty)$.



(2) Let $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ *be a complex power series and diverge at some* $w_0 \in \mathbb{C}$. *Define* $h := |w_0 - z_0|$. *Then the series diverges outside* $\overline{B(z_0,h)}$.



Definition 4.3.5. (1) A number R is called the radius of convergence of the real power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ if the series converges for all $x \in (a-R, a+R)$ but diverges if $x \in (-\infty, a-R) \cup (a+R, \infty)$.

(2) A number R is called the radius of convergence of the complex power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ if the series converges for all $z \in \{z \in \mathbb{C} \mid |z-z_0| < R\}$ but diverges for all $z \in \{z \in \mathbb{C} \mid |z-z_0| > R\}$.

Remark. (1) $R = \sup\left\{r \geq 0 \,\middle|\, \sum_{k=0}^{\infty} c_k(x-a)^k \text{ converges in } [a-r, a+r]\right\}$.

(2) $R = \sup\left\{r \geq 0 \,\middle|\, \sum_{k=0}^{\infty} c_k(z-z_0)^k \text{ converges in } B(z_0, r)\right\}$.

Question: How to find the radius of convergence of $\sum_{k=0}^{\infty} c_k(x-a)^k$?

By Raito Test (or Root Test), consider the series $\sum_{k=0}^{\infty} b_k$.

$$\text{If} \quad \limsup_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| < 1 \quad \Rightarrow \quad \text{the series converges.}$$

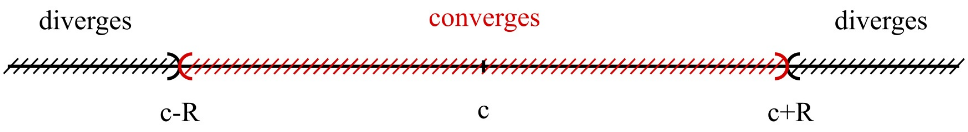
$$\text{If} \quad \liminf_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| > 1 \quad \Rightarrow \quad \text{the series diverges.}$$

For $x \neq a$, let $c_k(x-a)^k = b_k$ then $\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k}\right| = \left|\frac{c_{k+1}}{c_k}\right||x-a|$. Consider

$$\limsup_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right||x-a| < 1 \iff |x-a| < \frac{1}{\limsup_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|} = \liminf_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$$

$$\liminf_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right||x-a| > 1 \iff |x-a| > \frac{1}{\liminf_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|} = \limsup_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$$

If $\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ converges, then $\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = \liminf_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = \limsup_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = R$. Hence, the number R is the radius of converges and the series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges on $(a-R, a+R)$ and diverges on $(-\infty, a-R) \cup (a+R, \infty)$.



Question: How to find the interval of convergence?

(1) Find the radius of convergence

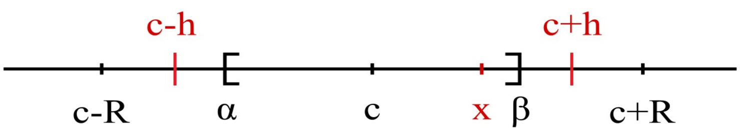
(2) Check whether the series converges at the endpoints $a-R$ and $a+R$.

Theorem 4.3.6. *Let* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *be a power series with the radius of convergence R, and* $[\alpha,\beta] \subseteq (a-R, a+R)$. *Then*

(a) the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ *converges uniformly on* $[\alpha,\beta]$.

(b) the power series $\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-a)^k$ *converges pointwise on* $(a-R, a+R)$ *and converges uniformly on* $[\alpha,\beta]$.

Proof. It suffices to prove (2). For $x \in [\alpha,\beta] \subseteq (a-R, a+R)$, choose $0 < h < R$ such that $[\alpha,\beta] \subseteq (a-h, a+h) \subseteq (a-R, a+R)$.

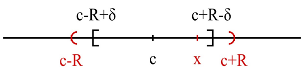


Then $r = \frac{|x-a|}{h} < 1$. Since $a+h \in (a-R, a+R)$, $\sum_{k=0}^{\infty} c_k\big((a+h)-a\big)^k = \sum_{k=0}^{\infty} c_k h^k$ converges. Thus $|c_k h^k| \to 0$ as $k \to \infty$. Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $|c_k h^k| < 1$. Therefore,

$$\sum_{k=N}^{\infty}\left|(k+1)c_k(x-a)^k\right| = \sum_{k=N}^{\infty}(k+1)\underbrace{|c_k h^k|}_{<1}\cdot\underbrace{\left(\frac{|x-a|}{h}\right)^k}_{=\,r^k} < \sum_{k=N}^{\infty}\underbrace{(k+1)r^k}_{M_k} < \infty \quad \text{since } r < 1$$

By Weierstrass M-test, $\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-a)^k$ converges uniformly on $[\alpha,\beta]$.

For $x \in (a-R, a+R)$, choose $0 < \delta < R$ such that $x \in [a-R+\delta, a+R-\delta]$.



Since $\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-a)^k$ converges uniformly on $[a-R+\delta, a+R-\delta]$, the series converges at x and hence the series converges pointwise on $(a-R, a+R)$. □

Corollary 4.3.7. *Let* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *be a power series with the radius of convergence R. Then the power series* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *is differentiable on* $(a-R, a+R)$*. Moreover,*

$$\frac{d}{dx}\Big[\sum_{k=0}^{\infty} c_k(x-a)^k\Big] = \sum_{k=0}^{\infty} \frac{d}{dx}\Big[c_k(x-a)^k\Big] = \sum_{k=1}^{\infty} kc_k(x-a)^{k-1}.$$

Remark. Check that

(1) $\sum_{k=1}^{\infty} kc_k(x-a)^{k-1}$ converges uniformly on $[\alpha,\beta]$ for every $[\alpha,\beta] \subseteq (a-R, a+R)$.

(2) $\sum_{k=1}^{\infty} kc_k(x-a)^{k-1}$ converges pointwise on $(a-R, a+R)$.

Corollary 4.3.8. *Let* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *be a power series with the radius of convergence R and* $[\alpha,\beta] \subseteq (a-R, a+R)$*. Then the power sereis* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *is integrable on* $[\alpha,\beta]$*. Moreover,*

$$\int_{\alpha}^{\beta} \sum_{k=0}^{\infty} c_k(x-a)^k\, dx = \sum_{k=0}^{\infty} \int_{\alpha}^{\beta} c_k(x-a)^k\, dx.$$

Example 4.3.9. The function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges on $\mathbb{R}$.

$$\int_0^t e^x\, dx = \int_0^t \sum_{k=0}^{\infty} \frac{x^k}{k!}\, dx = \sum_{k=0}^{\infty} \int_0^t \frac{x^k}{k!}\, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}\Big|_0^t = \sum_{k=1}^{\infty} \frac{t^k}{k!} = \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!}}_{e^t} - 1 = e^t - 1.$$

Remark. If $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ converges on $(a-R, a+R)$, then $f', f'', \cdots, f^{(k)}$ converge on $(a-R, a+R)$ and we can take the derivatives term by term.

Example 4.3.10. The series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges on $(-1, 1)$. Hence

$$\frac{d}{dx}\Big(\sum_{k=1}^{\infty} \frac{x^k}{k}\Big) = \sum_{k=1}^{\infty} \frac{d}{dx}\frac{x^k}{k} = \sum_{k=0}^{\infty} x^{k-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for every } x \in (-1, 1).$$

On the other hand, for $t \in (-1, 1)$,

$$\sum_{k=1}^{\infty} \frac{t^k}{k} \overset{F.T.C}{=} \int_0^t \frac{d}{dx}\Big(\sum_{k=1}^{\infty} \frac{x^k}{k}\Big)\, dx = \int_0^t \frac{1}{1-x}\, dx = -\ln(1-t).$$

Question: How about when $t = -1$?

Observe that $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by alternating series test.

Question: $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \cdots = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \overset{??}{=} -\ln 2$?

Check whether $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k} = -\ln 2$. Consider

$$\frac{d}{dx}\Big(\sum_{k=1}^{n} \frac{x^k}{k}\Big) = \sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

Then

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} = \int_{-1}^{0} \frac{d}{dx}\Big(\sum_{k=1}^{n} \frac{x^k}{k}\Big)\, dx = \int_{-1}^{0} \frac{1}{1-x}\, dx - \int_{-1}^{0} \frac{x^n}{1-x}\, dx.$$

We have

$$\Big|\sum_{k=1}^{n} \frac{(-1)^k}{k} - (-\ln 2)\Big| = \underbrace{\Big|\int_{-1}^{0} \frac{1}{1-x}\, dx - (-\ln 2)\Big|}_{=0} + \Big|\int_{-1}^{0} \frac{x^n}{1-x}\, dx\Big| < \Big|\int_{-1}^{0} x^n\, dx\Big| = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln 2.$

Example 4.3.11. Find a function $y(x)$ such that

$$y''(x) + y(x) = 0.$$

Suppose that a solution in the form of a power series about $a = 0$,

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

and assume that the series converges on $(-\delta, \delta)$. Then

$$y'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{and} \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}.$$

Plugging into the equation, we obtain

$$0 = y'' + y = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} + \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \big[(k+2)(k+1) c_{k+2} + c_k\big] x^k.$$

The coefficients satisfy $(k+2)(k+1)c_{k+2} + c_k = 0$ for $k = 0, 1, 2\cdots$ and thus the recurrence relation is $c_{k+2} = -\dfrac{c_k}{(k+1)(k+2)}$. We have

$$c_k = \begin{cases} \frac{(-1)^n}{(2n)!}c_0 & \text{if } k = 2n \\ \frac{(-1)^n}{(2n+1)!}c_1 & \text{if } k = 2n+1 \end{cases}$$

Therefore,

$$\begin{aligned} y &= c_0\big[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\big] + c_1\big[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\big] \\ &= c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \\ &= c_0\cos x + c_1\sin x. \end{aligned}$$

❑ Taylor Series

Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$ and $f : I \to \mathbb{R}$ where $f', f'', \cdots, f^{(n)}$ exist at a. We want to use polynomials to approximate f when x is near a. In Elementary Calculus, we have knwon that the n-th Taylor polynomials for f at a would be the best approximation among all n-degree polynomials near a.

Definition 4.3.12. Suppose that f is a function such that $f'(a), f''(a), \cdots, f^{(n)}(a)$ exist. Define

$$P_{n,a,f}(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

where $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 0, 1, \cdots, n$. The polynomial $P_{n,a,f}(x)$ is called the "*Taylor polynomial* of degree n for f at a".

Theorem 4.3.13. *Suppose that f is a function such that $f'(a),\ f''(a), \cdots, f^{(n)}(a)$ exist. Then*

$$\lim_{x\to a}\frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$$

Proof. Consider

$$\frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \frac{f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k}{(x-a)^n} - \frac{f^{(n)}(a)}{n!}.$$

Let $Q(x) = f(x) - \displaystyle\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k$ and $g(x) = (x-a)^n$. Then, for $1 \le i \le n-1$,

$$Q^{(i)}(x) = f^{(i)}(x) - f^{(i)}(a) - f^{(i+1)}(a)(x-a) - \cdots - \frac{f^{(n-1)}(a)(x-a)^{n-i-1}}{1\cdot 2\cdots(n-i-1)}.$$

Hence, $\lim\limits_{x\to a} Q^{(i)}(x) = 0$ for $i = 0, 1, 2, \cdots, n-1$. On the other hand,

$$g^{(i)}(x) = n(n-1)\cdots(n-i+1)(x-a)^{n-i}$$

and hence $\lim_{x\to a} g^{(i)}(x) = 0$ for $i = 0, 1, 2, \cdots, n-1$. By applying L'Hôpital's Rule $n-1$ times,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} &= \lim_{x\to a} \frac{Q(x)}{g(x)} - \frac{f^{(n)}(a)}{n!} \\
&\overset{L.H.}{=} \lim_{x\to a} \frac{Q'(x)}{g'(x)} - \frac{f^{(n)}(a)}{n!} \\
&\overset{L.H.}{=} \quad \vdots \\
&\overset{L.H.}{=} \lim_{x\to a} \frac{Q^{(n-1)}(x)}{g^{(n-1)}(x)} - \frac{f^{(n)}(a)}{n!} \\
&= \lim_{x\to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{n!(x-a)} - \frac{f^{(n)}(a)}{n!} \\
&= 0.
\end{aligned}$$

□

Theorem 4.3.14. *Let P and Q be two polynomials in $(x-a)$, of degree less than or equal to n. Suppose that P and Q are equal up to order n at a. Then $P = Q$.*

Proof. We claim that if $R(x)$ is a polynomial of degree less than or equal to n and $\lim_{x\to a} \frac{R(x)}{(x-a)^n} = 0$, then $R(x) \equiv 0$.

Proof of claim: Expressing $R(x)$ as a polynomial in $(x-a)$

$$R(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n,$$

we want to show that $b_i = 0$ for $i = 0, 1, 2, \cdots, n$ by induction.

Since $\lim_{x\to a} \frac{R(x)}{(x-a)^n} = 0$, we have

$$0 \le \lim_{x\to a} |R(x)| \le \lim_{x\to a} |(x-a)|^n = 0.$$

Then $R(a) = \lim_{x\to a} R(x) = 0$. Thus, for $i = 0$, $b_0 = 0$ and $R(x) = b_1(x-a) + \cdots + b_n(x-a)^n$.

If $b_0 = b_1 = \cdots = b_i = 0$ for $1 \le i < n$, then $R(x) = b_{i+1}(x-a)^{i+1} + \cdots + b_n(x-a)^n$. By using the similar argument as above, since $\lim_{x\to a} \frac{R(x)}{(x-a)^n} = 0$, we have

$$\lim_{x\to a} \left| \frac{R(x)}{(x-a)^{i+1}} \right| \le \lim_{x\to a} |x-a|^{n-(i+1)} = 0.$$

Hence,

$$0 = \lim_{x\to a} \frac{R(x)}{(x-a)^{i+1}} = \lim_{x\to a} b_{i+1} + b_{i+2}(x-a) + \cdots + b_n(x-a)^{n-(i+1)} = b_{i+1}.$$

By the induction, we have $b_0 = b_1 = \cdots = b_n = 0$ and the claim is proved.

Now, define $R(x) = P(x) - Q(x)$. Since P and Q are equal up to order n at a, $R(x)$ is a polynomial of degree less than or equal to n and

$$\lim_{x\to a} \frac{R(x)}{(x-a)^n} = \lim_{x\to a} \frac{P(x) - Q(x)}{(x-a)^n} = 0.$$

By the claim, $R(x) \equiv 0$ and hence $P(x) \equiv Q(x)$. □

Corollary 4.3.15. *Suppose that f has nth derivative at a and P is a polynomial in $(x-a)$ of degree less than or equal to n which equals f up to order n at a. Then $P(x) = P_{n,a,f}(x)$.*

Proof. Since

$$\lim_{x\to a} \frac{P(x) - P_{n,a,f}(x)}{(x-a)^n} = \lim_{x\to a} \frac{P(x) - f(x)}{(x-a)^n} + \lim_{x\to a} \frac{f(x) - P_{n,a,f}(x)}{(x-a)^n} = 0,$$

$P(x)$ and $P_{n,a,f}(x)$ are equal up to order n at a. Also, P and $P_{n,a,f}(x)$ are polynomials of degree less than or equal to n. By Theorem 4.3.14, $P(x) = P_{n,a,f}(x)$. □

Question: Can we estimate the difference between $f(x)$ and $P_{n,a}(x)$ when x is in some interval of a?

Definition 4.3.16. We define the remainder term $R_{n,a}(x)$ by

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$

By the definition of the remainder,

$$f(x) = P_{n,a}(x) + R_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Observe that

$$\begin{aligned}
f(x) &\overset{F.T.C}{=} f(a) + \underbrace{\int_a^x f'(t)\,dt}_{R_{0,a}(x)} \\
&\overset{I.B.P}{=} f(a) + f'(t)t\Big|_a^x - \int_a^x f''(t)t\,dt \\
&= f(a) + f'(x)x - f'(a)a - \int_a^x f''(t)t\,dt \\
&= f(a) + f'(a)(x-a) - f'(a)x + f'(x)x - \int_a^x f''(t)t\,dt \\
&= f(a) + f'(a)(x-a) + \big(f'(x) - f'(a)\big)x - \int_a^x f''(t)t\,dt \\
&\overset{I.B.P}{=} f(a) + f'(a)(x-a) + \Big(\int_a^x f''(t)\,dt\Big)x - \int_a^x f''(t)t\,dt \\
&= f(a) + f'(a)(x-a) + \underbrace{\int_a^x f''(t)(x-t)\,dt}_{R_{1,a}(x)} \\
&\overset{I.B.P}{=} f(a) + f'(a)(x-a) - f''(t)\cdot\frac{(x-t)^2}{2}\Big|_a^x + \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \underbrace{\int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt}_{R_{2,a}(x)}
\end{aligned}$$

By induction, if $f^{(n+1)}$ is continuous on $[a, x]$, then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt \qquad \text{(integral form)}$$

❑ Taylor Theorem

Theorem 4.3.17. *(Taylor Theorem) Let $f(t)$ be a $n+1$ times differentiable function on $[a, x]$ and $R_{n,a}(x)$ be defined by*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Then

(a) (Cauchy form)

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a) \qquad \textit{for some } \xi \in (a, x).$$

(b) (Lagrange form)

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad \textit{for some } \xi \in (a, x).$$

(c) (Integral form)

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$$

Proof.

Recall the Cauchy Mean Value Theorem: If F and G are continous on $[a, x]$ and differentiable on (a, x), there exists $\xi \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(\xi)}{G'(\xi)}.$$

Define F on $[a, x]$ by

$$F(t) = f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n.$$

Let G be a differentiable function on $[a, x]$ such that $G'(t) \neq 0$ on (a, x). By the Cauchy Mean Value Theorem, there exists a number $\xi \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(\xi)}{G'(\xi)}. \tag{4.3}$$

Also,

$$F(x) - F(a) = f(x) - \left[f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\right] = R_{n,a}(x)$$

and

$$F'(\xi) = \cancel{f'(\xi)} - \cancel{f'(\xi)} + \cancel{f''(\xi)(x-\xi)} - \cancel{f''(\xi)(x-\xi)} + \cancel{\cdots} + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n.$$

By (4.3),

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n \cdot \frac{G(x)-G(a)}{G'(\xi)}.$$

(a) Let $G(t) = t - a$. Then $G(x) - G(a) = x - a$ and $G'(\xi) = 1$. Hence,

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a).$$

(b) Let $G(t) = (x-t)^{n+1}$. Then $G(x) - G(a) = -(x-a)^{n+1}$ and $G'(\xi) = -(n+1)(x-\xi)^n$. Hence,

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

The part(c) is proved by using integration by parts. □

Remark. In Theorem 4.3.17,

(i) the ξ in part(a) and part(b) are usually different.

(ii) the ξ in part(a) and part(b) depend on a and x.

(iii) by part(b), if $\left|f^{(n+1)}(t)\right| < M$ for all $t \in [a, x]$, then

$$\left|R_{n,a}(x)\right| < M \cdot \frac{|x-a|^{n+1}}{(n+1)!}.$$

(iv) by part(c), if $\left|f^{(n+1)}(t)\right| < M$, then

$$\left|R_{n,a}(x)\right| \le \frac{M}{n!}\left|\int_a^x (x-t)^n\, dt\right| = \frac{M}{(n+1)!}\left| -(x-t)^{n+1}\Big|_a^x \right| = \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Theorem 4.3.18. *Let $f : (\alpha, \beta) \to \mathbb{R}$ be an infinitely differentiable function and $a \in (\alpha, \beta)$.*

(1) For every $n \in \mathbb{N}$ and $x \in (\alpha, \beta)$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt.$$

(2) Moreover, for some $0 < h < \infty$ *such that* $(a-h, a+h) \subseteq (\alpha, \beta)$*, suppose that there exists* $M > 0$ *such that* $\left|f^{(k)}(x)\right| \leq M$ *for all* $x \in (a-h, a+h)$ *and* $k \in \mathbb{N}$*. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \textit{for all } x \in (a-h, a+h).$$

Proof. It suffices to prove (2). Let $s_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. For $x \in (a-h, a+h)$,

$$\begin{aligned} |s_n(x) - f(x)| &\leq \left| \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt \right| \\ &\leq \frac{M}{n!} h^n \cdot |x-a| \\ &\leq M \cdot \frac{h^{n+1}}{n!} \quad \text{(independent of } x\text{)} \end{aligned}$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$|s_n(x) - f(x)| \leq M \frac{h^{n+1}}{n!} < \varepsilon.$$

Hence, $\{s_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ uniformly on $(a-h, a+h)$. We obtain $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ uniformly on $(a-h, a+h)$.

□

4.4 The Space of Continuous Functions

■ **Some common spaces of functions**

Let X, Y be two sets (metric spaces, normed spaces). We introduce some specific spaces of functions which are often used.

- $\mathcal{C}(X;Y)$, $\mathcal{C}(X) = \mathcal{C}(X;X)$, $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{R})$. For example, $\mathcal{C}([a,b])$.
- $\mathcal{C}_b(X;Y)$
- $\mathcal{L}(X;Y)$, $\mathcal{L}(X;X)$.
- $\mathcal{B}(X;Y)$.

Remark. The above notations are usually used for the field of mathematical analysis but not for all fields in mathematic. Also, those notations are not universal for all authors. Different books may have different definitions for every notation.

Definition 4.4.1. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed vector space and $A \subseteq M$. We define

$$\mathcal{C}(A;V) := \{f : A \to V \mid f \text{ is continuous on } A.\}.$$

and

$$\mathcal{C}_b(A;V) := \{f : A \to V \mid f \text{ is continuous and bounded on } A.\}.$$

Remark.

(1) $C_b\big(A;V\big) \subseteq C\big(A;V\big)$.

(2) Both $C\big(A;V\big)$ and $C_b\big(A;V\big)$ are vector spaces.

(3) If $K \subseteq M$ is compact, then $C\big(K;V\big) = C_b\big(K;V\big)$.

Definition 4.4.2. We can define a norm on $C_b\big(A;V\big)$ by

$$\|f\|_\infty := \sup_{x\in A} \|f(x)\| \quad \text{for every } f \in C_b\big(A;V\big)$$

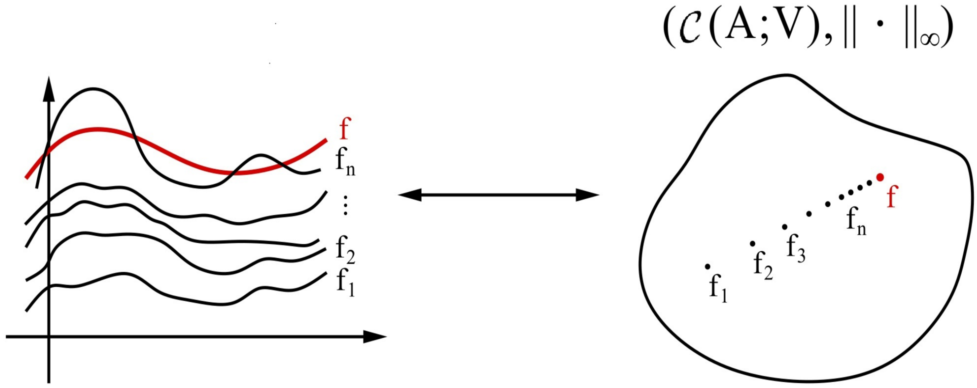
We call "$\|\cdot\|_\infty$" the "*sup-norm*" of f.

Note: We should be careful that $\|\cdot\|$ is the norm on V and $\|\cdot\|_\infty$ is the norm on $C_b\big(A;V\big)$.

Remark. $\|\cdot\|_\infty$ is not a norm on $C\big(A;V\big)$ since it is possible that there esists a function $f \in C\big(A;V\big)$ such that $\|f\|_\infty = \infty$.

Proposition 4.4.3. *Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a normed space and $A \subset M$. Then $\Big(C_b\big(A:V\big), \|\cdot\|_\infty\Big)$ is a nomed vector space.*

Proposition 4.4.4. *Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a normed spaces, $A \subseteq M$ and $f_k, f \in C_b\big(A;V\big)$ for every $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A if and only if $\{f_k\}_{k=1}^\infty$ converges to f in $\Big(C_b\big(A;V\big), \|\cdot\|_\infty\Big)$.*



Proof. (Exercise) □

Theorem 4.4.5. *Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a normed vector space and $A \subseteq M$. If $(V,\|\cdot\|)$ is complete, then $\Big(C_b\big(A;V\big), \|\cdot\|_\infty\Big)$ is complete.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy seuqnece in $\Big(C_b\big(A;V\big), \|\cdot\|_\infty\Big)$.

To prove that there is $f \in C_b\big(A;V\big)$ such that $\{f_k\}_{k=1}^\infty$ converges to f in $\Big(C_b\big(A;V\big), \|\cdot\|_\infty\Big)$. That is, $\lim_{k\to\infty} \|f_k - f\|_\infty = 0$.

For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then

$$\|f_m - f_n\|_\infty = \sup_{x \in A} \|f_m(x) - f_n(x)\| < \varepsilon.$$

Hence, for every $x \in A$ and $m, n \geq N$,

$$\|f_m(x) - f_n(x)\| < \varepsilon.$$

Since $(V, \|\cdot\|)$ is complete, by the Cauchy criterion, there exists a function $f : A \to V$ such that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A.

Since $\{f_k\}_{k=1}^{\infty}$ is a sequence of continuouos functions, f is continuous on A. That is, $f \in \big(C(A; V), \|\cdot\|_\infty\big)$.

Now, we need to show that f is bounded on A. Since $f_k \to f$ uniformly on A, there exists $N_1 \in \mathbb{N}$ such that if $k \geq N_1$,

$$\|f_k(x) - f(x)\| < 1$$

for every $x \in A$.

Since $f_{N_1} \in C_b(A; V)$, there exists $M > 0$ such that $\|f_{N_1}(x)\| < M$ for every $x \in A$. Then

$$\|f(x)\| \leq \|f(x) - f_{N_1}(x)\| + \|f_{N_1}(x)\| \leq 1 + M \quad \text{for every } x \in A.$$

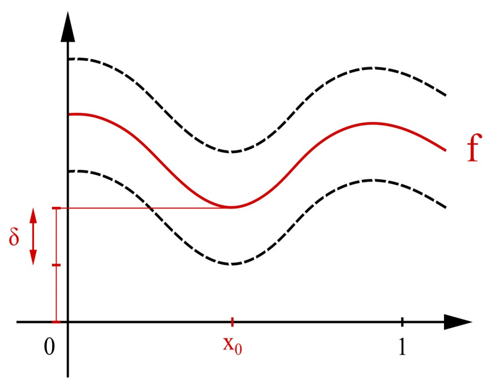
Hence, $f \in C_b(A; V)$ and this implies that $\{f_k\}_{k=1}^{\infty}$ converges to f in $\big(C_b(A; V), \|\cdot\|_\infty\big)$. □

Example 4.4.6. The set $U = \big\{f \in C([0,1]; \mathbb{R}) \mid f(x) > 0 \text{ for every } x \in [0,1]\big\}$ is open in $\big(C_b([0,1]; \mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. Let $f \in U$. to prove that there exists $\delta > 0$ such that the ball

$$B(f, \delta) = \big\{g \in C([0,1]; \mathbb{R}) \mid \|f - g\|_\infty < \delta\big\} \subseteq U.$$

Since f is continuous on $[0,1]$, there exists $x_0 \in [0,1]$ such that $f(x_0) = \min_{x \in [0,1]} f(x) > 0$. Choose $\delta = \frac{1}{2} f(x_0)$. For $g \in B(f, \delta)$ and for every $x \in [0,1]$,



$$\begin{aligned} g(x) &= f(x) - \big(f(x) - g(x)\big) \\ &\geq f(x) - |f(x) - g(x)| \\ &\geq f(x_0) - \sup_{x \in [0,1]} |f(x) - g(x)| \\ &= f(x_0) - \|f - g\|_\infty \\ &\geq \frac{1}{2} f(x_0) \\ &> 0. \end{aligned}$$

Hence $g \in U$ and this implies $B(f, \delta) \subseteq U$. Since f is an arbitrary element in U, U is open in $\big(C_b(A; V), \|\cdot\|_\infty\big)$.

□

4.5 Arzelà-Ascoli Theorem

Review: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of bounded real numbers. By the Bolzano-Weierstrass theorem, there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ and $a \in \mathbb{R}$ such that $a_{n_k} \to a$ as $k \to \infty$.

Let (M, d) be a metric space, $A \subseteq M$ and $f_k : A \to \mathbb{R}$ be a sequence of functions such that for every $x \in A$, $\{f_n(x)\}_{n=1}^{\infty}$ is a bounded sequence of real numbers. [That is, for every $x \in A$, there exists $M_x > 0$ such that $|f_n(x)| < M_x$ for every $n \in \mathbb{N}$. But it may not have a universal number $M > 0$ such that $|f_n(x)| < M$ for every $n \in \mathbb{N}$.]
Question: Is there a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and $f : A \to \mathbb{R}$ such that $f_{n_k} \to f$ on A (pointwise or uniformly).

Definition 4.5.1. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed vector space, $A \subseteq M$ and $\mathscr{F}$ be a family of function from A into V. (That is, $\mathscr{F} \subseteq \left\{ f \mid f : A \to V \right\}$).

(1) We say that $\mathscr{F}$ is pointwise bounded (precompact, compact) on A if for every $x \in A$, the set $F_x := \left\{ f(x) \mid f \in \mathscr{F} \right\}$ is bounded (precompact, compact) in $(V, \|\cdot\|)$.

(2) We say that $\mathscr{F}$ is uniformly bounded on A if the set $F := \bigcup_{x \in A} F_x = \left\{ f(x) \mid f \in \mathscr{F},\ x \in A \right\}$ is bounded in $(V, \|\cdot\|)$.

(3) In particular, if $\mathscr{F}$ is pointwise bounded, then there exists a function $\phi : A \to \mathbb{R}$ such that for every $x \in A$ and every $f \in \mathscr{F}$

$$\|f(x)\| \leq \phi(x).$$

Moreover, if $\mathscr{F}$ is uniformly bounded, there exists $M > 0$ such that for every $f \in \mathscr{F}$,

$$\|f(x)\| \leq M.$$

Example 4.5.2. (1) $f_n(x) = \dfrac{1}{nx}$ on $(0, 1)$. Then $\{f_n\}_{n=1}^{\infty}$ is a pointwise bounded sequence of functions on $(0, 1)$, but is not uniformly bounded on $(0, 1)$.

(2) $f_n(x) = \sin(nx)$ on $\mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on $\mathbb{R}$.

Rewritten Question: Suppose $\{f_n\}_{n=1}^{\infty}$ is pointwise (uniformly) bounded sequence of real-valued functions on A. Is there a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges on A?

Answer: No! Even if $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded sequence of continuous function on a compact set A, there need not exist a subsequence which converges (pointwise) on A.

Example 4.5.3. Let $f_n(x) = \sin(nx)$ on $[0, 2\pi]$. Assume that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges (pointwise) on $[0, 2\pi]$. Then

$$\lim_{k \to \infty} \left[\sin(n_k x) - \sin(n_{k+1} x) \right]^2 = 0$$

for every $x \in [0, 2\pi]$. Thus,

$$\lim_{k \to \infty} \int_0^{2\pi} \left[\sin(n_k x) - \sin(n_{k+1} x) \right]^2 dx = 0. \quad \text{(Skip the proof)}$$

But

$$\int_0^{2\pi} \left[\sin(n_k x) - \sin(n_{k+1} x) \right]^2 dx = 2\pi$$

for every $k \in \mathbb{N}$.

Remark. If A is countable, it is doable.

Theorem 4.5.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a pointwise bounded sequence of real-valued functions on a countable set A, then $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges for every $x \in A$.*

Proof. Since A is countable, we can write $A = \{x_i \mid i = 1, 2, 3, \cdots\}$. Since $\{f_n(x_1)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, $\{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{1,k}\}_{k=1}^{\infty}$ such that $\{f_{1,k}(x_1)\}_{k=1}^{\infty}$ converges. Denote this subsequence S_1.

Take x_2 into $\{f_{1,k}\}_{k=1}^{\infty}$. Since $\{f_{1,k}(x_2)\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, $\{f_{1,k}\}_{k=1}^{\infty}$ contains a subsequence $\{f_{2,k}\}_{k=1}^{\infty}$ such that $\{f_{2,k}(x_2)\}_{k=1}^{\infty}$ converges. Denote this subsequence S_2.

Continue this procedure, there exists $S_1, S_2, S_3, \cdots$ which we represent by the array

$$\begin{array}{lllllllll}
S_1: & \boxed{f_{1,1}}, & f_{1,2}, & f_{1,3}, & f_{1,4}, & f_{1,5}, & \cdots & & \\
S_2: & f_{2,1}, & \boxed{f_{2,2}}, & f_{2,3}, & f_{2,4}, & f_{2,5}, & \cdots & & \\
S_3: & f_{3,1}, & f_{3,2}, & \boxed{f_{3,3}}, & f_{3,4}, & f_{3,5}, & \cdots & & \\
S_4: & f_{4,1}, & f_{4,2}, & f_{4,3}, & \boxed{f_{4,4}}, & f_{4,5}, & \cdots & & \\
 & & & \vdots & & \ddots & & & \\
S_n: & f_{n,1}, & f_{n,2}, & f_{n,3}, & f_{n,4}, & f_{n,5}, & \cdots, & \boxed{f_{n,n}}, & \cdots \\
 & & & \vdots & & & & &
\end{array}$$

and which have the following properties.

(a) S_{n+1} is a subsequence of S_n for $n = 1, 2, 3, \cdots$

(b) $\{f_{n,k}(x_n)\}_{k=1}^{\infty}$ converges as $k \to \infty$.

(c) The order where the functions appears is the same in each seuqence.

Now, we choose the sequence $S = \{f_{k,k}\}_{k=1}^{\infty}$. By (c), S (except possible its first $n-1$ terms) is a subsequence of S_n for $n = 1, 2, \cdots$. Hence, by (b), $\{f_{k,k}(x_i)\}_{k=1}^{\infty}$ converges for every $x_i \in A$. □

Review: Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ and $f_k, f : A \to N$ be maps. If $f_k \to f$ uniformly on A, then $f_k \to f$ pointwise on A. But the converse is false.

Question: Under what additional conditions, the pointwise convergence implies the uniform convergence?

By Dini's theorem, suppose that

(1) $K \subseteq M$ is compact,

(2) $f_n, f : K \to \mathbb{R}$ are continuous, and

(3) $f_{n+1} \geq f_n$ for every $n \in \mathbb{N}$.

Then if $f_n \to f$ pointwise on K, we have $f_n \to f$ uniformly on K.

Remark. Conditions (1), (2) are reasonable hypotheses, but the monotonic condition (3) is unusual. Is there any substitute condition?

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❑ Equicontinuous Family of Functions

Let (M, d) be a metric space and $(V, \|\cdot\|)$ be a normed vector space and $A \subseteq M$. Recall that if a function $f : A \to V$ is uniformly continuous on A, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in A$ with $d(x, y) < \delta$,

$$\|f(x) - f(y)\| < \varepsilon.$$

Consider $f_1, f_2 : A \to V$ are both uniformly continuous on A. Then for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that for $x, y \in A$ with $d(x, y) < \delta_1$,

$$\|f_1(x) - f_1(y)\| < \varepsilon$$

and for $x, y \in A$ with $d(x, y) < \delta_2$,

$$\|f_2(x) - f_2(y)\| < \varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then if $x, y \in A$ with $d(x, y) < \delta$,

$$\|f_1(x) - f_1(y)\| < \varepsilon \quad \text{and} \quad \|f_2(x) - f_2(y)\| < \varepsilon.$$

Question: How about $\mathscr{F}$ is a family of infinitely many uniformly continuous function on A?

In general, if $\mathscr{F}$ consists of infinitely many uniformly continuous function on A, for given $\varepsilon > 0$, it is impossible to find $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $x, y \in A$ with $d(x, y) < \delta$ for every $f \in \mathscr{F}$.

Example 4.5.5. Let $\{f_n\}_{n=1}^\infty$ be a sequence of function defined by $f_n(x) = \dfrac{x^2}{x^2+(1-nx)^2}$ on $[0,1]$. Then $|f_n(x)| \le 1$ and hence $\{f_n\}_{n=1}^\infty$ is uniformly bounded on $[0,1]$. Also, for every $x \in [0,1]$, $\lim\limits_{n\to\infty} f_n(x) = 0$ but $f_n(\frac{1}{n}) = 1$ for every $n \in \mathbb{N}$. Therefore, there exists no subsequence which can converge uniformly on $[0,1]$.

Definition 4.5.6. Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a normed vector space and $A \subseteq M$. A family $\mathscr{F}$ of continuous function in $C\big(M;V\big)$ is said to be "*equicontinuous*" if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x) - f(y)\| < \varepsilon$$

whenever $x, y \in A$ with $d(x,y) < \delta$ and $f \in \mathscr{F}$.

Remark.

(1) A subfamily of an equicontinuous family of functions is equicontinuous. (That is, if $\mathscr{F}$ is equicontinuous and $\mathscr{G} \subseteq \mathscr{F}$ then $\mathscr{G}$ is equicontinuous.)

(2) Let $\mathscr{F}$ be an equicontinuous family of functions. For every $f \in \mathscr{F}$, f is uniformly continuous.

(3) Every family consists of finitely many uniformly continuous functions is equicontinuous.

Example 4.5.7. (1) Let $f(x) = \dfrac{x^2}{x^2+(1-nx)^2}$ on $[0,1]$. Then $\mathscr{F} = \big\{f_n \bigm| n \in \mathbb{N}\big\}$ is not equicontinuous.

(2) Let $\mathscr{F} = \big\{f : \mathbb{R} \to \mathbb{R} \bigm| |f'(x)| \le M \text{ for every } x \in \mathbb{R}\big\}$. Then $\mathscr{F}$ is equicontinuous.

Lemma 4.5.8. *Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a normed vector space and $K \subseteq M$ be a compact subset. If B is precompact in $\Big(C\big(K;V\big), \|\cdot\|_\infty\Big)$, then B is equicontinuous*

Proof.

Assume that B is not equicontinuous. There exists $\varepsilon > 0$, a sequence of functions $\{f_k\}_{k=1}^\infty$ in B and $x_k, y_k \in K$ with $d(x_k, y_k) < \frac{1}{k}$, but

$$\|f_k(x_k) - f_k(y_k)\| \ge \varepsilon.$$

Compactness implies (1) infinity → finiteness; (2) convergent subsequence
Precompactness implies convergent subsequence

Since B is precompact in $\Big(C\big(K;V\big), \|\cdot\|_\infty\Big)$ and K is compact in M, there exists subsequences $\{f_{k_j}\}_{j=1}^\infty$ and $\{x_{k_j}\}_{j=1}^\infty$ such that $\{f_{k_j}\}_{j=1}^\infty$ uniformly converges in $\overline{B} \subseteq \big(K;V\big)$, say $f_{k_j} \underset{j\to\infty}{\to} f$ uniformly, and $\{x_{k_j}\}_{j=1}^\infty$ converges to $x_0 \in K$.

Since $d(x_k, y_k) < \dfrac{1}{k}$, the corresponding subsequence $\{y_{k_j}\}_{j=1}^\infty$ converges to x_0. Since f is continuous at x_0, for $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in K$ with $d(x, x_0) < \delta$,

$$\|f(x) - f(x_0)\| < \frac{\varepsilon}{4}.$$

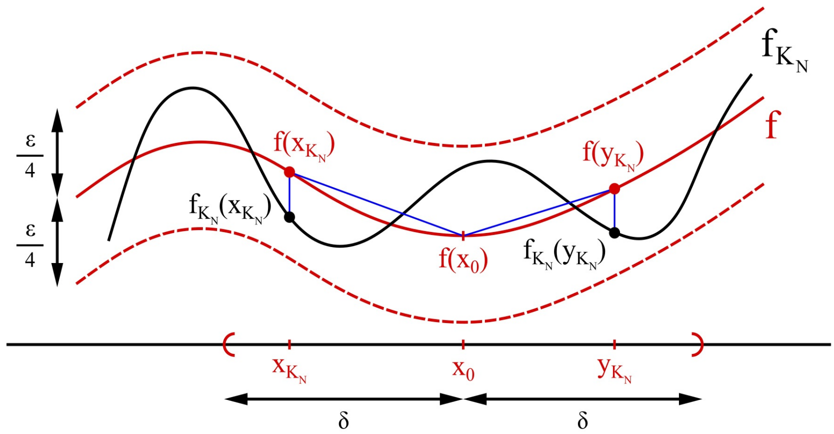
Since $f_{k_j} \to f$ uniformly on K, $x_{k_j} \to x_0$ and $y_{k_j} \to x_0$ as $j \to \infty$, there exists $N \in \mathbb{N}$ such that for every $j \geq N$ and every $x \in K$,

$$\|f_{k_j}(x) - f(x)\| < \frac{\varepsilon}{4}, \quad d(x_{k_j}, x_0) < \delta \text{ and } d(y_{k_j}, x_0) < \delta.$$

We have

$$\begin{aligned} \varepsilon &\leq \|f_{k_N}(x_{k_N}) - f_{k_N}(y_{k_N})\| \\ &\leq \underbrace{\|f_{k_N}(x_{k_N}) - f(x_{k_N})\|}_{\text{unif. conv.}} + \underbrace{\|f(x_{k_N}) - f(x_0)\|}_{\text{continuous}} + \underbrace{\|f(x_0) - f(y_{k_n})\|}_{\text{continuous}} + \underbrace{\|f(y_{k_N}) - f_{k_N}(y_{k_N})\|}_{\text{unif. conv.}} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, we obtain a contradiction.



□

Corollary 4.5.9. *Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed vector space and $K \subseteq M$ be a compact subset. If $\{f_k\}_{k=1}^{\infty} \subseteq C(K; V)$ converges uniformly on K, then $\{f_k \mid k \in \mathbb{N}\}$ is equicontinuous.*

Proof. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K, the set $\{f_k \mid k \in \mathbb{N}\}$ is precompact in $C(K; V)$. By Lemma 4.5.8, $\{f_k \mid k \in \mathbb{N}\}$ is equicontinuous. □

Remark.

(1) The compactness of K is necessary. For example, $f_k(x) = \frac{1}{x}$ on $(0, 1)$ for every $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on $(0, 1)$. But the function $\frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

(2)

$$\{f_k\}_{k=1}^{\infty} \subseteq C(K; V) \text{ converges uniformly on } K \quad \Longrightarrow \quad \{f_k \mid k \in \mathbb{N}\} \text{ is equicontinuouos.}$$

$$\not\Longleftarrow$$

For example, $f_k(x) = k$ for $k \in \mathbb{N}$. Then $\{f_k \mid k \in \mathbb{N}\}$ is equicontinuous. But $\{f_k\}_{k=1}^{\infty}$ does not converges on K.

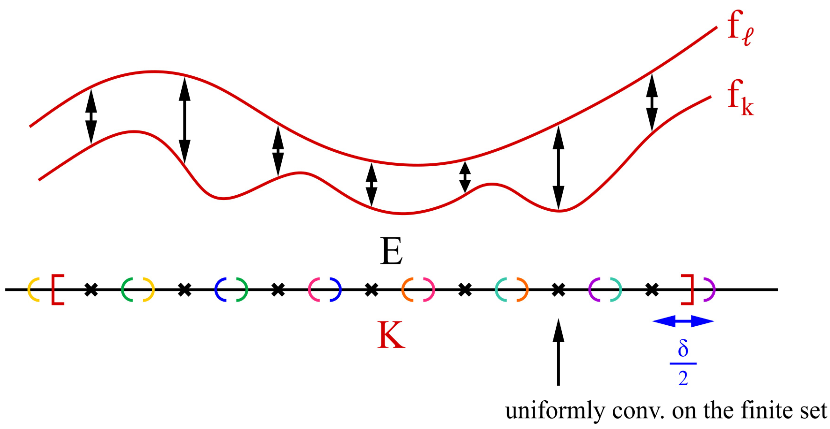
Question: Under what additional conditions does the converse hold?

Guess: There are two possibilities:

(1) "pointwise convergence on K" + "equicontinuous", or

(2) "pointwise convergence on a dense subset E of K" + "$\big(V, \|\cdot\|\big)$ is a Banach space" + "equicontinuous".

Lemma 4.5.10. *Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set and $\{f_k \mid k \in \mathbb{N}\} \subseteq C(K; V)$ be equicontinuous. If $\{f_k\}_{k=1}^{\infty}$ converges pointwise on a dense subset E of K, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K.*



Proof. It suffices to show that $\{f_k\}_{k=1}^{\infty}$ satisfies the Cauchy criterion. That is, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for $k, \ell \geq N$,

$$\|f_k - f_\ell\|_\infty < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\{f_k \mid k \in \mathbb{N}\} \subseteq C(K; V)$ is equicontinuous, there exists $\delta > 0$ such that if $x, y \in K$ with $d(x, y) < \delta$,

$$\|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3} \quad \text{for every } k \in \mathbb{N}.$$

Since K is compact, K is totally bounded. Also, E is dense in K. We can choose $x_1, \cdots, x_L \in E$ such that

$$K \subseteq \bigcup_{i=1}^{L} B\Big(x_i, \frac{\delta}{2}\Big).$$

Since $\{f_k\}_{k=1}^{\infty}$ converges pointwise on E, there exists $N \in \mathbb{N}$ such that for every $k, \ell \geq N$,

$$\big\|f_k(x_i) - f_\ell(x_i)\big\| < \frac{\varepsilon}{3} \quad \text{for every } i = 1, \cdots, N. \tag{4.4}$$

Fix $x \in K$. Since $K \subseteq \bigcup_{i=1}^{L} B\big(x_i, \frac{\delta}{2}\big)$, there exists $1 \leq j \leq L$ such that $x \in B\big(x_j, \frac{\delta}{2}\big)$. For $k, \ell \geq N$,

$$\begin{aligned} \|f_k(x) - f_\ell(x)\| &\leq \underbrace{\|f_k(x) - f_k(x_j)\|}_{\text{equicontinuouos}} + \underbrace{\|f_k(x_j) - f_\ell(x_j)\|}_{(4.4)} + \underbrace{\|f_\ell(x_j) - f_\ell(x)\|}_{\text{equicontinuous}} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since x is an arbitrary point in K and $(V, \|\cdot\|)$ is complete, by Cauchy criterion, there exists $f \in C\big(K; V\big)$ such that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on K. □

Remark. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a Banach space and $K \subseteq M$ be compact. Then $\{f_n\}_{n=1}^{\infty} \subseteq C\big(K; V\big)$ converges uniformly on K if and only if $\{f_n\}_{n=1}^{\infty}$ is equicontinuous and pointwise converges on K.

For the direction "$\Longrightarrow$", the compactness is necessary.
For the direction "$\Longleftarrow$", it only needs totally boundedness.

Heuristically review that for $f_n, f \in C\big(K; V\big)$,

(1) Observe that $f_n \to f$ pointwise on K but not uniformly. It is possible that the functions f_n rapidly increase somewhere (for example, $f_n(x) = x^n$ on $(0, 1)$). In order to exclude this situation, we add the hypothesis of equicontinuity.

(2) Lemma 4.5.8 $\Longrightarrow$ Corollary 4.5.9 $\Longrightarrow$ if $f_n \to f$ uniformly on K, then $\{f_n\}$ is equicontinuous. $\Longrightarrow$ the rapid oscillation cannot happen.

(3) Uniform convergence on $K \Longrightarrow$ equicontinuity. But the converse is false. Under what additional conditions the direction "$\Longleftarrow$" holds?

    Guess:

    (i) pointwise convergence on K + equicontinuous

    (ii) pointwise convergence on E which is dense in K + $(V, \|\cdot\|)$ is complete + equicontinuous.

Recall our questions: let (M, d) be a metric space and $A \subseteq M$.

1. Let $\{f_n\}_{n=1}^{\infty} : A \to \mathbb{R}$ be pointwise bounded. Is there a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges on A?

2. Let $\{f_n\}_{n=1}^{\infty} \subseteq C\big(A; \mathbb{R}\big)$, $f_n \to f$ pointwise on A. Under what additional conditions, the convergence is uniform?

Known facts:

(1) If A is countable and $\{f_k\}_{k=1}^{\infty}$ is pointwise bounded, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges on A.

(2) If $A \subseteq M$ is compact, $\{f_n\}_{n=1}^{\infty}$ is equicontinuous and $f_n \to f$ pointwise on a dense subset of A, then $f_n \to f$ uniformly on A.

(3) Every compact set in a metric space contains a countable dense subset.

Organize the above conditions:

$A \subseteq M$: compact subset (a) $\qquad \{f_n\}_{n=1}^{\infty} : A \to \mathbb{R} \qquad \{f_n\} \subseteq C(A;\mathbb{R})$

(3) $\Downarrow$ $\qquad$ + $\qquad$ +

there exists a countable dense subset E of A $\qquad$ pointwise bounded on A (b) $\qquad$ equicontinuous (c)

(1) $\Downarrow$

there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which convergs on E

(2) $\Downarrow$

Result1: $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly on A.
Result2: $\{f_{n_k}\}_{k=1}^{\infty}$ is uniformly bounded on A.

❑ Arzelà-Ascoli Theorem

In this section, we start with two questions:

Question 1: If $f_n : A \to \mathbb{R}$ is pointwise bounded, is there a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converges on A (pointwise or uniformly)?

Question 2: If $f_n \to f$ pointwise on A, under what additional conditions, we obtain uniform convergence?

Answer 1: Yes, if A is countable; but no if A is uncountable.

Answer 2: Suppose that

(i) K is compact (totally bounded)

(ii) $\{f_n\}$ is equicontinuous

(iii) Either

(a) $f_n \to f$ pointwise on K or

(b) $f_n \to f$ pointwise on a dense subset E of K and $(V, \|\cdot\|)$ is complete.

Then $f_n \to f$ uniformly on K.

Theorem 4.5.11. *(Arzelà-Ascoli Theorem) Let (M, d) be a metric space, $K \subseteq M$ be a compact subset and $f_n : K \to \mathbb{R}$ be a sequence of functions. Suppose that $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded and equicontinuous on K. Then*

(1) $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on K.

(2) $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence.

Proof. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on K, there exists $\delta > 0$ such that if $x, y \in K$ with $d(x, y) < \delta$,

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

Since K is compact, there are finitely many points $x_1, \cdots, x_N \in K$ such that $K \subseteq \bigcup_{i=1}^{N} B(x_i, \delta)$.

(1) Since $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded on K, for $i = 1, \cdots, N$, there exists $M_i > 0$ such that

$$|f_n(x_i)| < M_i \quad \text{for every } n \in \mathbb{N}.$$

Let $M = \max(M_1, \cdots, M_N)$. Since $K \subseteq \bigcup_{i=1}^{N} B(x_i, \delta)$, for $x \in K$, there exists $1 \leq j \leq N$ such that $x \in B(x_j, \delta)$. Hence,

$$|f_n(x)| \leq \underbrace{|f_n(x) - f_n(x_j)|}_{<\varepsilon} + \underbrace{|f_n(x_j)|}_{<M} < M + \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

Therefore, $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on K.

(2) Since K is compact, K contains a countable dense subset, say E. Moreover, we can choose finitely many point $y_1, \cdots, y_r \in E$ such that $K \subseteq \bigcup_{i=1}^{r} B(y_i, \delta)$.

Since $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded on E, there exists a subseqeuence $\{f_{n_k}\}_{k=1}^{\infty}$ converges pointwise on E. Then there exists $N_1 \in \mathbb{N}$ such that if $k, \ell \geq N_1$,

$$|f_{n_k}(y_i) - f_{n_\ell}(y_i)| < \varepsilon \quad \text{for every } i = 1, 2, \cdots, r.$$

If $x \in K$, there exists $1 \leq s \leq r$ such that $x \in B(y_s, \delta)$. Thus for $k, \ell \geq N_1$,

$$\begin{aligned} |f_{n_k}(x) - f_{n_\ell}(x)| &\leq \underbrace{|f_{n_k}(x) - f_{n_k}(y_s)|}_{\text{equi. conti.}} + \underbrace{|f_{n_k}(y_s) - f_{n_\ell}(y_s)|}_{\text{pointwise conv.}} + \underbrace{|f_{n_\ell}(y_s) - f_{n_\ell}(x)|}_{\text{equi.conti.}} \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

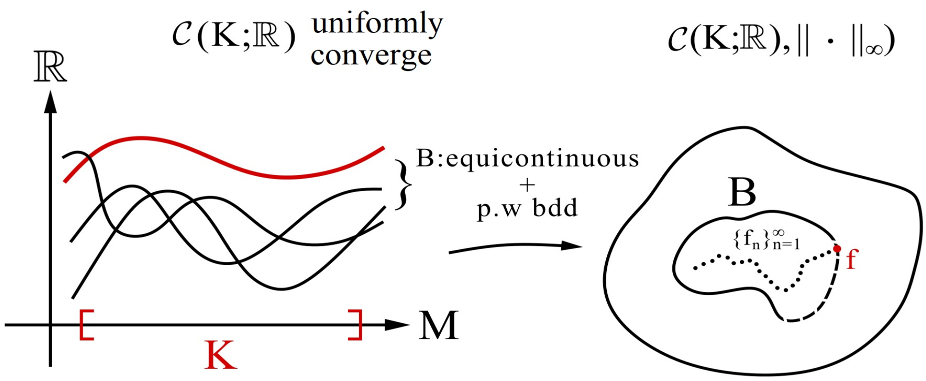
By the Cauchy criterion, $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly on K.

□

Review: Let (M,d) be a metric space and $K \subseteq M$ be compact.

(1) $\{f_n\}_{n=1} \subseteq C(K;\mathbb{R})$ is equicontinuous and pointwise bounded on K if and only if $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

(2) $B \subseteq C(K;\mathbb{R})$ is equicontinuous and pointwise bounded on K if and only if every sequence in B has a uniformly convergent subsequence in $\big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$.
Note that, so far, we do not prove that the set B is compact in $\big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$ yet since we only prove that there exists a subsequence converges in $\big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$ rather than in B.



❑ Compact Sets in $C(K;V)$

Theorem 4.5.12. *Let (M,d) be a metric space, $(V,\|\cdot\|)$ be a Banach space, $K \subseteq M$ be compact. If $B \subseteq \mathbb{C}(K;V)$ is equicontinuous and pointwise precompact, then B is precompact in $\big(C(K;V), \|\cdot\|_\infty\big)$.*

Proof. To prove that every sequence in B has a convergent subseqence in $\big(C(K;V), \|\cdot\|_\infty\big)$. [That is, if $\{f_n\}_{n=1}^{\infty} \subseteq B$, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly.]

Let $\{f_n\}_{n=1}^{\infty} \subseteq B$ be a sequence in B. Then $\{f_n\}_{n=1}^{\infty}$ is pointwise precompact on K. Since K is compact, K contains a countable dense subset $E \subseteq K$. By using the diagonal method, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on E. (Note that we can do this since $\{f_n\}_{n=1}^{\infty}$ is pointwise precompact.)

Since E is dense in K and $\{f_{n_k}\}_{k=1}^{\infty}$ is equicontinuous on K, $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly on K. □

Remark.

(1) A set $B \subseteq C(K;V)$ is precompact if and only if B is equicontinuous and pointwise precompact.

(2) A set $B \subseteq C(K;V)$ is compact if and only if B is closed in $\big(C(K;V), \|\cdot\|_\infty\big)$, equicontinuouos and pointwise compact on K.

Proof. ("$\Longrightarrow$")
Since B is compact in $\big(C(K;V), \|\cdot\|_\infty\big)$, B is closed in $\big(C(K;V), \|\cdot\|_\infty\big)$. By Lemma 4.5.8, B is equicontinuous on K.

It suffices to show that B is pointwise compact on K. For a fixed $x \in K$, let $B_x = \{f(x) \mid f \in B\} \subseteq (V, \|\cdot\|)$. To prove that B_x is compact in $(V, \|\cdot\|)$.

Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence in B_x. Since $\{f_n\}_{n=1}^\infty \subseteq B$ and B is compact in $\big(C(K;V), \|\cdot\|_\infty\big)$, there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges to a function $f \in B$. Then $\{f_{n_k}\}_{k=1}^\infty$ converges uniformly to f on K. Hence, $\{f_{n_k}(x)\}_{k=1}^\infty$ converges to $f(x) \in B_x$. We have B_x is compact in $(V, \|\cdot\|)$.

("$\Longleftarrow$")
By Theorem 4.5.12, B is precompact and B is closed in $\big(C(K;V), \|\cdot\|_\infty\big)$. Thus, B is compact in $\big(C(K;V), \|\cdot\|_\infty\big)$. □

4.6 Stone-Weierstrass Theorem

❑ Introduction

Let $A \subseteq \mathbb{R}$ (or $\mathbb{R}^n$) and $\{p_n\}_{n=1}^\infty : A \to \mathbb{R}$ be a sequence of polynomials. Suppose that $p_n \to f$ uniformly (that is, $\|p_n - f\|_\infty \to 0$ as $n \to \infty$), then f is continuous.

Question: How about the converse? If $f \in C(A;\mathbb{R})$, is there a sequence of polynomials $\{p_n\}_{n=1}^\infty$: $A \to \mathbb{R}$ such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$?

Theorem 4.6.1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function and $\varepsilon > 0$ be given. Then there exists a polynomial $p : [0,1] \to \mathbb{R}$ such that $\|f - p\|_\infty < \varepsilon$.*

Proof. (Probabilistic viewpoint)(Law of large numbers)
Consider the binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^k y^{n-k} \quad \text{where } C_k^n = \frac{n!}{k!(n-k)!} \tag{4.5}$$

Then taking differentiation,

$$n(x+y)^{n-1} = \frac{d}{dx}[(x+y)^n] = \sum_{k=0}^{n} k C_k^n x^{k-1} y^{n-k}$$

and multiplying by x,

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} kC_k^n x^k y^{n-k}. \tag{4.6}$$

Again,

$$n(n-1)x^2(x-y)^{n-2} = \frac{d^2}{dx^2}[(x+y)^n] = \sum_{k=0}^{n} k(k-1)C_k^n x^{k-2} y^{n-k}$$

and

$$n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} k(k-1)C_k^n x^k y^{n-k} \tag{4.7}$$

For $0 \le x \le 1$, taking $y = 1 - x$, then

$$(4.5) \quad \Rightarrow \quad 1^n = \sum_{k=0}^{n} \underbrace{C_k^n x^k (1-x)^{n-k}}_{r_k(x)} \qquad \boxed{1 = \sum_{k=0}^{n} r_k(x)} \tag{4.8}$$

$$(4.6) \quad \Rightarrow \quad nx = \sum_{k=0}^{n} k r_k(x) \qquad \boxed{\text{Expected value}} \tag{4.9}$$

$$(4.7) \quad \Rightarrow \quad n(n-1)x^2 = \sum_{k=0}^{n} k(k-1) r_k(x) \tag{4.10}$$

Then

$$\sum_{k=0}^{n}(k-nx)^2 r_k(x) = \sum_{k=0}^{n} \left[k(k-1) + (1-2nx)k + n^2x^2\right] r_k(x) = \underbrace{nx(1-x)}_{Variance}. \tag{4.11}$$

Since f is continuous on $[0,1]$, f is uniformly continuous on $[0,1]$. Then, for $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in [0,1]$ with $|x-y| < \delta$,

$$|f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Define $P_n(x) = \underbrace{\sum_{k=0}^{n} f(\frac{k}{n}) r_k(x)}_{Berstein's\ polynomial}$. Then for every $x \in [0,1]$,

$$\begin{aligned}
|f(x) - P_n(x)| &\overset{(4.8)}{=} \left|\sum_{k=0}^{n} \left[f(\frac{k}{n}) - f(x)\right] r_k(x)\right| \le \sum_{k=0}^{n} \left|f(\frac{k}{n}) - f(x)\right| r_k(x) \\
&= \sum_{|\frac{k}{n}-x|<\delta} \underbrace{\left|f(\frac{k}{n}) - f(x)\right|}_{<\frac{\varepsilon}{2}} r_k(x) + \sum_{|\frac{k}{n}-x|\ge\delta} \left|f(\frac{k}{n}) - f(x)\right| r_k(x) \\
&\le \frac{\varepsilon}{2} + 2\|f\|_\infty \sum_{|k-nx|\ge n\delta} \frac{(k-nx)^2}{(k-nx)^2} r_k(x) \\
&\le \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n^2\delta^2} \sum_{|k-nx|\ge n\delta} (k-nx)^2 r_k(x) \\
&\overset{(4.11)}{\le} \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2} x(1-x) \\
&\le \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2}.
\end{aligned}$$

Then we can choose N sufficiently large such that $\frac{2\|f\|_\infty}{N\delta^2} < \frac{\varepsilon}{2}$. Hence,

$$|f(x) - P_N(x)| < \varepsilon \quad \text{for every } x \in [0,1].$$

□

Remark. The statement of the above theorem is equivalent to each of the following statement

(1) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$which converges uniformly to f on $[0,1]$.

(2) The collection of all polynomials is dense in $\left(C\left([0,1];\mathbb{R}\right), \|\cdot\|_\infty\right)$.

Corollary 4.6.2. *The collection of polynomials on* $[a,b]$ *is dense in* $\left(C\left([a,b];\mathbb{R}\right), \|\cdot\|_\infty\right)$.

Review of the proof (binomial distribution)

Consider an asymmetric coin with probabilities of head and tail are x and $1-x$ respectively. After n-times tosses,

$$\begin{aligned}
1^n = [x + (1-x)]^n &= \sum_{k=0}^{n} C_k^n x^k (1-x)^k \\
&= C_0^n (1-x)^n + C_1^n x(1-x)^{n-1} + \cdots + \underbrace{C_k^n x^k (1-x)^{n-k}}_{r_k(x)} + \cdots + C_n^n x^n
\end{aligned}$$

where $r_k(x)$ means the probability of exactly k-times head within n-times tosses. Then

$$\sum_{k=0}^{n} r_k(x) = 1. \tag{4.12}$$

The expected valued of head with n tosses is

$$\sum_{k=0}^{n} k r_k(x) = nx. \tag{4.13}$$

Then

$$\sum_{k=0}^{n} \frac{k}{n} r_k(x) = x \quad \text{(投擲 n 次正好為 k 次頭的比例期望值)} \tag{4.14}$$

The variance is $X = X_1 + \cdots + X_n$ where X_i 代表投第 i 次時的 independent Bernoulli distributed random variable. We have

$$Var(X_i) = (1-x)^2 \cdot x + (0-x)^2(1-x) = x(1-x)$$

and

$$Var(X) = Var(X_1) + \cdots + Var(X_n) = nx(1-x).$$

From (4.14), 我們可以想像想投擲 n 次後，出現頭的次數與總擲次數的比值應趨近於出現頭的機率 x, 由大數法則

$$P\big(|\frac{k}{n} - x| > \varepsilon\big) \to 0 \quad \text{as} \quad n \to \infty$$

今假設一賭場公布一賠率計算方式如一函數 $f(x)$, 即 $\frac{k}{n} = \frac{k \text{ times heads}}{n \text{ times tosses}}$, 因此當 n 足夠大時，我們自然會認為賠率應靠近 $f(x)$，則

$$P\big(|f(\frac{k}{n}) - f(x)| > \varepsilon\big) \to 0 \quad \text{as} \quad n \to \infty$$

於此過程中，我們不希望賠率函數 $f(x)$ 是 discontinuous. This suggests that

$$f(x) \approx \underbrace{\sum_{k=0}^{n} f(\frac{k}{n}) r_k(x)}_{\text{expected value of the gambling}} = p_n(x)$$

Bernstein polynomial: $\sum_{k=0}^{n} a_k r_k(x)$.

$$|f(x) - p_n(x)| \le \Big|\sum_{k=0}^{n} \big[f(x - f(\frac{k}{n})\big] r_k(x)\Big| \le \cdots$$

$$P\big(|f(x) - f(\frac{k}{n})| > \delta\big) \underset{\substack{\uparrow \\ \text{Chebyshev inequality}}}{\le} \frac{Var\big(f(\frac{k}{n})\big)}{n^2\delta^2} \le \frac{2\|f\|_\infty}{n^2\delta^2} nx(1-x)$$

Question: If $K \subseteq \mathbb{R}$ (or $\mathbb{R}^n$) is compact, is the collection of all polynomials on K dense in $\big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$?

We will start with discussing some abstract theorems and the answer of the above question will be the application of those theorems.

Let (M, d) be a metric space and $K \subseteq M$ be compact. Consider $X := \big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$. If $f, g \in X$ and $\alpha \in \mathbb{R}$ then

$$(1)\ f \pm g \in X \qquad (2)\ f \cdot g \in X \qquad (3)\ \alpha f \in X$$

Note that f/g may not belong to X.

Question: Is there any set of functions $S = \overset{\text{(ancestor)}}{\{f_1, \cdots, f_n\}}$ such that every function in X can be generated by S under the operators (1), (2) and (3) with finite steps? In other words, is the family (posterity) of S equal to X?

Answer: It seems to be impossible.

Question: If $S = \{f_1, \cdots, f_n\} \subseteq X$, what is the distribution of the family generated by S?

Definition 4.6.3. Let (M, d) be a metric space and $E \subseteq M$ be a subset. A family $\mathscr{A}$ of real-valued functions defined on E is said to be an "*algebra*" if

(i) $f + g \in \mathscr{A}$ for every $f, g \in \mathscr{A}$

(ii) $f \cdot g \in \mathscr{A}$ for every $f, g \in \mathscr{A}$

(iii) $\alpha f \in \mathscr{A}$ for every $f \in \mathscr{A}$ and $\alpha \in \mathbb{R}$

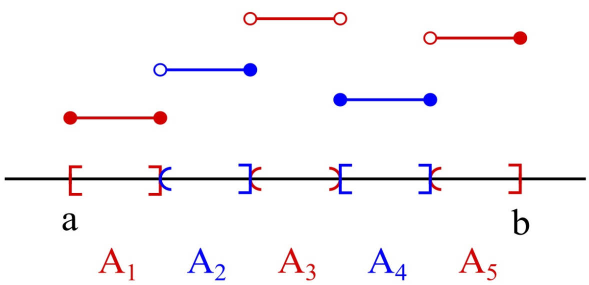
Remark. An algebra $\mathscr{A}$ is closed under addition, multiplication and scalar multiplication.

Example 4.6.4. A function $g : [a, b] \to \mathbb{R}$ is called a "*simple function*" if there exists finitely many subintervals of $[a, b]$, say $A_1, \cdots, A_n$ such that

$$A_i \cap A_j = \emptyset \qquad \text{and} \qquad [a, b] = \bigcup_{i=1}^{n} A_i$$

and real number $a_1, a_2, \cdots, a_n$ such that

$$g(x) = a_i \quad \text{for } x \in A_i,\ i = 1, 2, \cdots, n.$$



Then the collection of all simple function is an algebra. (Exercise)

Example 4.6.5. Let $E \subseteq \mathbb{R}^n$. The collection of all polynomials on E is an algebra.

Example 4.6.6. Let $E \subseteq \mathbb{R}^3$ and the set

$$\mathcal{P}_{even}(E) = \Big\{ p(x,y,z) = \sum_{k=0}^{n} a_{k_1k_2k_3} x^{k_1} y^{k_2} z^{k_3} \quad \text{where } k_1, k_2, k_3 \in \mathbb{N} \cup \{0\},\ k_1 + k_2 + k_3 = 2k \text{ and } \ a_{k_1k_2k_3} \in \mathbb{R} \Big\}$$

is an algebra.

Example 4.6.7. Let $\mathcal{P}_n(\mathbb{T})$ be the collection of all trigonometric polynomials of degree n.

$$\mathcal{P}_n(\mathbb{T}) = \Big\{ \frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx + d_k \sin kx \ \Big| \ \{c_k\}_{k=0}^n,\ \{d_k\}_{k=1}^n \subset \mathbb{R} \Big\}.$$

then $\mathcal{P}(\mathbb{T}) = \bigcup_{n=0}^{\infty} \mathcal{P}_n(\mathbb{T})$ is an algebra.

Example 4.6.8. Let $E \subseteq \mathbb{R}^n$. Then $C\big(E;\mathbb{R}\big)$ is an algebra.

Proposition 4.6.9. *Let (M,d) be a metric space and $E \subseteq M$ be a subset. If $\mathscr{A} \subseteq \Big(C_b\big(E;\mathbb{R}\big), \|\cdot\|_\infty \Big)$ is an algebra, then $\overline{\mathscr{A}}$ is also an algebra.*

Proof. Let $f, g \in \overline{\mathscr{A}}$. Then there are sequences $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $f_n \to f$ and $g_n \to g$ uniformly on E.

since $\mathscr{A}$ is an algebra, $f_n + g_n \in \mathscr{A}$, $f_n \cdot g_n \in \mathscr{A}$ and $\alpha f_n \in \mathscr{A}$. Also,

$$\begin{aligned} f_n + g_n &\to f + g \quad \text{uniformly on } E \\ f_n \cdot g_n &\to f \cdot g \quad \text{uniformly on } E \\ \alpha f_n &\to \alpha f \quad \text{uniformly on } E \end{aligned}$$

(Note that $\|f_n\|_\infty$ and $\|g_n\|_\infty$ are bounded is necessary.) Then $f + g \in \overline{\mathscr{A}}$, $f \cdot g \in \overline{\mathscr{A}}$ and $\alpha f \in \overline{\mathscr{A}}$. Hence, $\overline{\mathscr{A}}$ is also an algebra. □

Remark. $\Big(C_b\big(E;\mathbb{R}\big), \|\cdot\|_\infty \Big)$ is closed in $\big\{ f : E \to \mathbb{R} \ \big| \ \|f\|_\infty < \infty \big\}$

Question: Is it possible to find a set of functions $S = \{f_1, \cdots, f_n\} \subseteq X$ such that the family generated by S dense in X?

Question: If yes, what the sufficient and necessary conditions does the family need to have?

We will rule out some "bad" members of this family. (1) There exists $x \neq y$ such that $f(x) = f(y)$ for every $f \in S$ (2) There exists $x \in K$ such taht $f(x) = 0$ for every $f \in S$.

Guess: If $\mathscr{F} \subseteq X$ is dense, then $\mathscr{F}$ must satisfy

1. *"Separate points on K"*: if for every $x, y \in K$ and $x \neq y$, there exists $f \in \mathscr{F}$ such that $f(x) \neq f(y)$

2. *"Vanish at no point of K"*: if for each $x \in K$, there exists $f \in \mathscr{F}$ such that $f(x) \neq 0$.

Definition 4.6.10. Let (M, d) be a metric space and $E \subseteq M$ be a subset. A family $\mathscr{F}$ of functions defined on E is said to

(1) "*separate points on E*" if for every $x, y \in E$ and $x \neq y$, there exists $f \in \mathscr{F}$ such that $f(x) \neq f(y)$

(2) "*vanish at no point of E*" if for each $x \in E$, there exists $f \in \mathscr{F}$ such that $f(x) \neq 0$.

Example 4.6.11. $\mathcal{P}\big([a, b]\big)$ is the collection of all polynomials on $[a, b]$. Then $\mathcal{P}\big([a, b]\big)$ separates points on $[a, b]$. (e.g. $f(x) = x$) and vanishes at no point of $[a, b]$ (e.g. $f(x) = 1$).

Example 4.6.12. $\mathcal{P}_{even}\big([a, b]\big)$ is the collection of all polynomials of the form $p(x) = \sum_{k=0}^{n} a_k x^{2k}$.
Then
$\mathcal{P}_{even}\big([-1, 1]\big)$ vanishes at no point of $[-1, 1]$, but does not separate points on $[-1, 1]$.
$\mathcal{P}_{even}\big([0, 1]\big)$ vanishes at no point of $[0, 1]$ and separates points on $[0, 1]$.

Lemma 4.6.13. *Let (M, d) be a metric space and $E \subseteq M$ be a subset. Suppose that $\mathscr{A}$ is an algebra of funcrtion on E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants. Then $\mathscr{A}$ contains a function f such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

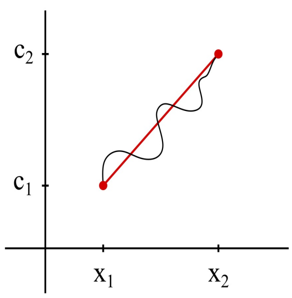
Proof. Since $\mathscr{A}$ separates points on E, there exists $g \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$. Since $\mathscr{A}$ vanishes at no point on E, there exist $h, k \in \mathscr{A}$ such that $h(x_1) \neq 0$ and $k(x_2) \neq 0$. Let

$$f(x) = c_1 \frac{[g(x) - g(x_2)]h(x)}{[g(x_1) - g(x_2)]h(x_1)} + c_2 \frac{[g(x) - g(x_1)]k(x)}{[g(x_2) - g(x_1)]k(x_2)}.$$

Then $f(x_1) = c_1$ and $f(x_2) = c_2$. □

Idea:

We want a function f satisfying $f(a) = c$ and $f(b) = d$. Naturally, we set $f(x) = c\frac{x-b}{a-b} + d\frac{x-a}{b-a}$. But $x - a$ or $x - b$ may not be in $\mathscr{A}$. Hence, we figure out the role of $g(x)$ [separates points on E].



Why do we need "vanish at no point of E"? How about

$$f(x) = c\frac{g(x) - g(b)}{g(a) - g(b)} + d\frac{g(x) - g(a)}{g(b) - g(a)} \tag{4.15}$$

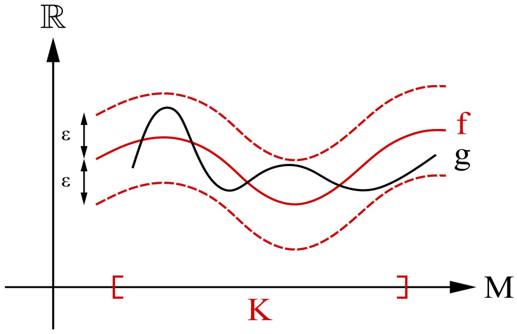
It is possible that

$$f(x) = \underbrace{\frac{c}{g(a)-g(b)}g(x)}_{\in \mathscr{A}} - \underbrace{\frac{cg(b)}{g(a)-g(b)}}_{\notin \mathscr{A}} - \underbrace{\frac{d}{g(b)-g(a)}g(x)}_{\in \mathscr{A}} - \underbrace{\frac{d}{g(b)-g(a)}}_{\notin \mathscr{A}}$$

Remark. (1) If $\mathscr{A}$ contains a nonzero constant function, then (4.15) works. But this implies $\mathscr{A}$ vanishes at no point of E. Hence, the lemma is more general.

(2) If $\mathscr{A}$ separates point on E (or vanishes at no point of E), then so does $\overline{\mathscr{A}}$.

Idea of the proof of Stone-Weierstrass Theorem



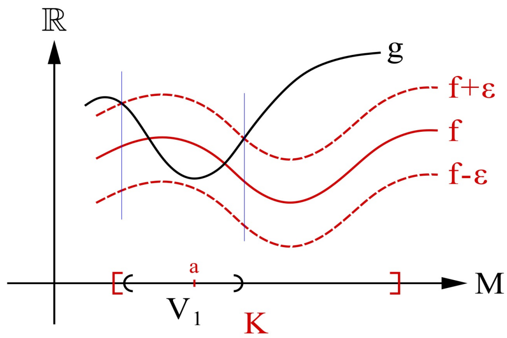
Given $f \in C(K;\mathbb{R})$ and $\varepsilon > 0$. Is there a function $g \in \overline{\mathscr{A}}$ such that $\|f-g\|_\infty < \varepsilon$?

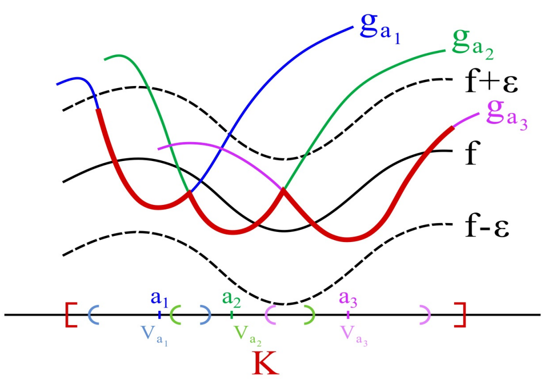
Hope: For $a \in K$, there exists $g_a \in \overline{\mathscr{A}}$ such that

(1) $g_a(x) \geq f(x) - \varepsilon$ and

(2) $|g_a(x) - f(x)| < \varepsilon$ near a

In fact, we could hope $g(x) \geq f(x) - \varepsilon$ for every $x \in K$. Then we can keep part of g, say g_a.





Then we can choose $a_1, a_2, \cdots, a_n \in K$ and set $h = \min(g_{a_1}, \cdots, g_{a_n})$.
Question: Is $\max f, g \in \overline{\mathscr{A}}$ or $\min(f,g) \in \overline{\mathscr{A}}$?
Consider $\max(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$. If we can show $f \in \overline{\mathscr{A}} \Rightarrow |f| \in \overline{\mathscr{A}}$, then we obtain $\max(f,g), \min(f,g) \in \overline{\mathscr{A}}$.

Theorem 4.6.14. *(Stone) Let (M, d) be a metric space, $K \subseteq M$ be a compact set. Let $\mathscr{A} \subseteq \big(C(K;\mathbb{R}), \|\cdot\|_\infty\big)$ be an algebra, separates point on K, and vanishes at no point of K. Then $\mathscr{A}$ is dense in $C(K;\mathbb{R})$.*

Proof. It suffices to show that for $\varepsilon > 0$, there exists $g \in \overline{\mathscr{A}}$ such that $\big\| g - f \big\|_\infty < \varepsilon$.

Step 1: If $f \in \overline{\mathscr{A}}$, then $|f| \in \overline{\mathscr{A}}$.
Proof of Step 1:
Let $a = \sup\limits_{x\in K} |f(x)| < \infty$ and let $\varepsilon > 0$ be given. Since $\phi(y) = |y|$ is continuous on $[-a, a]$, there exists a polynomial $p(y) = \sum\limits_{k=0}^{n} a_k y^k$ on $[-a, a]$ such that $\Big|p(y) - |y|\Big| < \varepsilon$ for every $y \in [-a, a]$.
Then

$$\Big|p\big(f(x)\big) - |f(x)|\Big| < \varepsilon \quad \text{for every } x \in K.$$

Since $\overline{\mathscr{A}}$ is an algebra and $f \in \overline{\mathscr{A}}$, $p\big(f\big) = \sum\limits_{k=0}^{n} a_k f^k \in \overline{\mathscr{A}}$. Hence, $|f| \in \overline{\mathscr{A}}$.

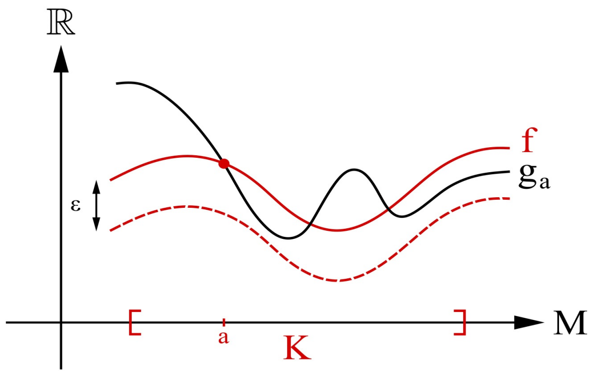
Step 2: If $f, g \in \overline{\mathscr{A}}$, then $\max(f, g) \in \overline{\mathscr{A}}$ and $\min(f, g) \in \overline{\mathscr{A}}$ where

$$\max(f,g) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) \leq g(x) \end{cases} \quad \text{and} \quad \min(f,g) = \begin{cases} g(x) & \text{if } f(x) \geq g(x) \\ f(x) & \text{if } f(x) \leq g(x) \end{cases}$$

Proof of Step 2:
By Step 1 and $\overline{\mathscr{A}}$ is an algebra, $\max(f, g) = \dfrac{f+g}{2} + \dfrac{|f+g|}{2} \in \overline{\mathscr{A}}$ and $\min(f, g) = \dfrac{f+g}{2} - \dfrac{|f+g|}{2} \in \overline{\mathscr{A}}$.

Moreover, by iteration, if $f_1, \cdots, f_n \in \overline{\mathscr{A}}$, then $\max(f_1, \cdots, f_n) \in \overline{\mathscr{A}}$ and $\min(f_1, \cdots, f_n) \in \overline{\mathscr{A}}$.

Step 3: For given $f \in C\big(K;\mathbb{R}\big)$ and $a \in K$, there exists $g_a \in \overline{\mathscr{A}}$ such that $g_a(a) = f(a)$ and $g_a(x) > f(x) - \varepsilon$ for every $x \in K$.

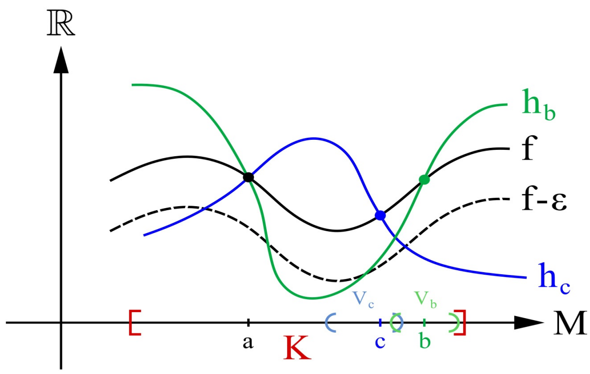


Proof of Step 3:
Since $\mathscr{A}$ separates points on K and vanishes at no point of K, so does $\overline{\mathscr{A}}$. Then for every $y \in K$ and $y \neq a$, there exists $h_y \in \overline{\mathscr{A}}$ such that

$$h_y(a) = f(a) \qquad \text{and} \qquad h_y(y) = f(y).$$

Since h_y is continuouos on K, there exists an open set V_y containing y such that for every $x \in V_y$,

$$h_y(x) > f(x) - \varepsilon.$$

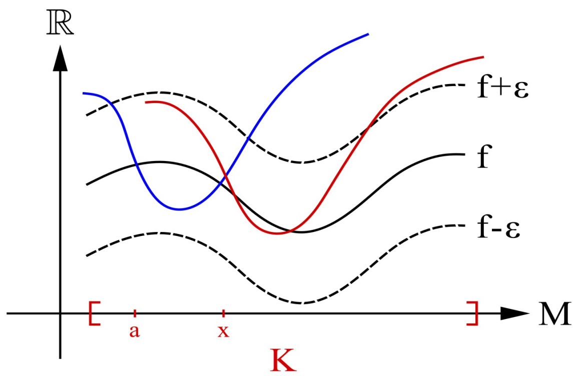


Since K is compact and $K \subseteq \bigcup_{y \in K} V_y$, there exist $y_1, \cdots, y_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} V_{y_i}$.

Let $g_a = \max(h_{y_1}, \cdots, h_{y_n}) \in \overline{\mathscr{A}}$. Then $g_a(a) = f(a)$ and $g_a(x) > f(x) - \varepsilon$ for every $x \in K$.

Step 4: Given $f \in C(K; \mathbb{R})$ and $\varepsilon > 0$, there exists $h \in \overline{\mathscr{A}}$ such that

$$\left|h(x) - f(x)\right| < \varepsilon \quad \text{for every } x \in K.$$



Proof of Step 4:
For every $a \in K$, by Step 3, there exists $g_a \in \overline{\mathscr{A}}$ such that

$$g_a(a) = f(a) \qquad \text{and} \qquad g_a(x) > f(x) - \varepsilon.$$

Since g_a is continuous on K, there exists an open set U_a containing a such that for every $x \in U_a$,

$$g_a(x) < f(x) + \varepsilon.$$

Since K is compact and $K \subseteq \bigcup_{a \in K} U_a$, there exist $a_1, \cdots, a_m \in K$ such that $K \subseteq \bigcup_{i=1}^{m} U_{a_i}$.

Let $h(x) = \min(g_{a_1}, \cdots, g_{a_m})$. Then $h \in \overline{\mathscr{A}}$ and

$$h(x) > f(x) - \varepsilon \qquad \text{and} \qquad h(x) < f(x) + \varepsilon$$

for every $x \in K$.

□

Corollary 4.6.15. *Let (M, d) be a metric space, $K \subseteq M$ be compact and $\mathscr{A} \subseteq C(K; \mathbb{R})$ be an algebra. Then $\mathscr{A}$ is dense in $\left(C(K; \mathbb{R}), \|\cdot\|_\infty\right)$ if and only if $\mathscr{A}$ separates points on K and vanishes at no point of K.*

Corollary 4.6.16. *The set of all polynomials defined on K is dense in $\left(C(K; \mathbb{R}), \|\cdot\|_\infty\right)$.*

Corollary 4.6.17. *Let $K \subseteq \mathbb{R}^n$ be compact and $\mathcal{P}(K)$ be the collection of all polynomials on K. Then $\mathcal{P}(K)$ is dense in $\left(C(K; \mathbb{R}), \|\cdot\|_\infty\right)$*

Example 4.6.18. $\mathcal{P}_{even}([0,1])$ is dense in $\left(C([0,1]; \mathbb{R}), \|\cdot\|_\infty\right)$. But $\mathcal{P}_{even}([-1,1])$ is not dense in $\left(C([-1,1]; \mathbb{R}), \|\cdot\|_\infty\right)$.

Question: Why cannot Taylor series tell us the dense of $\left(C(K; \mathbb{R}), \|\cdot\|_\infty\right)$?

There may have some reasons.

(1) The Taylor polynomial for f of degree n is

$$P_{n,c}(x) = \sum_{k=0}^{n} \frac{f^{(k)(c)}}{k!}(x-c)^k.$$

But a continuous function f needs sufficiently many times derivatives $f^{(k)}(c)$.

(2) Even the Taylor series exists, we cannot say that $P_n(x) \to f(x)$ as $n \to \infty$

(3) Even if $P_n(x) \to f(x)$ as $n \to \infty$ on $(c-R, c+R)$, the interval of convergence may not contain the domain of f.

(4) Even if the Taylor series converges on $\mathbb{R}$, it may not converges to f uniformly on the domain of f. For example, $f(x) = \begin{cases} \cos x & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & x \in [-10, 10] \backslash [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$ Then the Taylor polynomial $P_{n,0}(x)$ conveges to $\cos x$ which will not converge to f on $[-10, 10]$.

Remark. The Stone-Weierstrass Theorem says that for every continuous function defined on a compact set can be approximated (uniformly) by polynomials. Unfortunately, the converging rate of this approximation is too slow.

4.7 Contraction Mappings

Definition 4.7.1. Let (M, d) be a metric space and ϕ maps M into M. We say the map ϕ a "*contraction mapping*" if there is a constant $0 \le c < 1$ such that for every $x, y \in M$

$$d\big(\phi(x), \phi(y)\big) \le c d(x, y).$$

Remark. (1) A contraction mapping is uniformly continuous.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $|f'(x)| < c < 1$. Then f is a contraction mapping on $\mathbb{R}$.

Example 4.7.2. Let $f(x) = x^2$ on $[0, r]$. For $0 \le x < y \le r$, there exists $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) = 2c(y - x)$. Hence, if $r < \frac{1}{2}$ then $2c < 2r < 1$ and this implies that f is a contraction mapping on $[0, r]$.

Question: Is $f(x) = x^2$ a contraction mapping on $[0, \frac{1}{2}]$?
Assume that f is a contraction mapping on $[0, \frac{1}{2}]$. There exists $0 \le c < 1$ such that

$$\big|f(x) - f(y)\big| \le c|x - y| \quad \text{for every } x, y \in [0, \frac{1}{2}].$$

Let $x_n = \frac{1}{2} - \frac{1}{n}$. Then

$$\big|f(\frac{1}{2}) - f(x_n)\big| = \frac{1}{4} - (\frac{1}{2} - \frac{1}{n})^2 = (1 - \frac{1}{n})|\frac{1}{2} - x_n| > c|\frac{1}{2} - x_n| \quad \text{as } n \text{ as sufficiently large.}$$

Hence, f is not a contraction mapping on $[0, \frac{1}{2}]$.

■ **Fixed point**

Definition 4.7.3. Let (M, d) be a metric space and $\phi : M \to M$ be a mapping. We call a point $x_0 \in M$ a fixed point for ϕ if $\phi(x_0) = x_0$.

Theorem 4.7.4. *(Contraction Mapping Theorem)(Banach Fixed Point Theorem)*
Let (M, d) be a complete metric space and $\phi : M \to M$ be a contraction mapping. Then there exists a unique fixed point for ϕ.

Proof. Since ϕ is a contraction mapping on M, there exists a constant $0 \le c < 1$ such that for every $x, y \in M$,

$$d\big(\phi(x), \phi(y)\big) \le cd(x, y).$$

(**Existence**) Taking arbitrarily a point $x_0 \in M$ and define $x_{n+1} = \phi(x_n)$ for $x = 0, 1, 2, \cdots$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(\phi(x_n), \phi(x_{n-1})\\ &\le cd(x_n, x_{n-1}) = cd\big(\phi(x_{n-1}), \phi(x_{n-2})\big)\\ &\le c^2 d(x_{n-1}, x_{n-2})\\ &\le \cdots\\ &\le c^n d(x_1, x_0). \end{aligned}$$

If $n > m$,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-}) + \cdots + d(x_{m+1}, x_m)\\ &\le (c^{n-1} + c^{n-2} + \cdots + c^m)d(x_1, x_0)\\ &= c^m(1 + c + \cdots + c^{n-m-1})d(x_1, x_0)\\ &\le c^m(1 + c + c^2 + \cdots)d(x_1, x_0) \end{aligned}$$

Since $0 \le c < 1$,

$$d(x_n, x_m) \le c^m \cdot \frac{1}{1 - c} d(x_1, x_0).$$

Hence, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m \geq N$, $d(x_n, x_m) < \varepsilon$. That is, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in M. Since M is complete, there exists $x \in M$ such that $\lim_{n\to\infty} x_n = x$.

Check that x is a fixed point for ϕ. Since $\lim_{n\to\infty} x_n = x$ and ϕ is continuous,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \phi(x_{n-1}) = \phi(\lim_{n\to\infty} x_{n-1}) = \phi(x).$$

(**Uniqueness**) Assume $y \in M$ is also a fixed point for ϕ. Then

$$d(x, y) = d\big(\phi(x), \phi(y)\big) \leq cd(x, y).$$

Thus, $d(x, y) = 0$ and this implies $x = y$. □

Remark. (1) Let x be the fixed point for ϕ in the above theorem. For any starting point $x_0 \in M$, $x = \lim_{n\to\infty} \phi^n(x_0)$.

(2) The condition $c < 1$ is necessary. For example, $M = \mathbb{R}$ and $\phi(x) = x + 1$ $(c = 1)$. Then

$$d\big(\phi(x), \phi(y)\big) = d(x + 1, y + 1) = d(x, y)$$

But there exists no fixed point for ϕ.

(3) Even if ϕ satisfies $d\big(\phi(x), \phi(y)\big) < d(x, y)$, there may not exist a fixed point for ϕ. Fox example $M = [1, \infty)$ and $\phi(x) = x + \frac{1}{x}$. For $x > y$,

$$\big|\phi(x) - \phi(y)\big| = x - y + \underbrace{(\frac{1}{x} - \frac{1}{y})}_{<0} < |x - y|.$$

But there exists no fixed point for ϕ.

(4) If M is compact and ϕ satisfies $d\big(\phi(x), \phi(y)\big) < d(x, y)$, then there exists a fixed point for ϕ. Consider $g(x) = d\big(\phi(x), x\big)$. Then g has minimum x_0 which is a fixed point for ϕ.

Example 4.7.5. Let $f(x, y) = \big(\frac{1}{4}x + \frac{1}{3}y - 2, \frac{1}{5}x - \frac{1}{3}y + 3\big)$. Determine whether there exists a fixed point for f. Consider

$$f(x_1, y_1) - f(x_2, y_2) = \Big(\frac{1}{4}(x_1 - x_2) + \frac{1}{3}(y_1 - y_2), \frac{1}{5}(x_1 - x_2) - \frac{1}{3}(y_1 - y_2)\Big).$$

Then

$$\begin{aligned}\big\|f(x_1, y_1) - f(x_2, y_2)\big\|^2 &= (\frac{1}{16} + \frac{1}{25})(x_1 - x_2)^2 + \frac{1}{30}(x_1 - x_2)(y_1 - y_2) + \frac{2}{9}(y_1 - y_2)^2 \\ &\leq \frac{1}{2}[(x_1 - x_2)^2 + (y_1 - y_2)^2].\end{aligned}$$

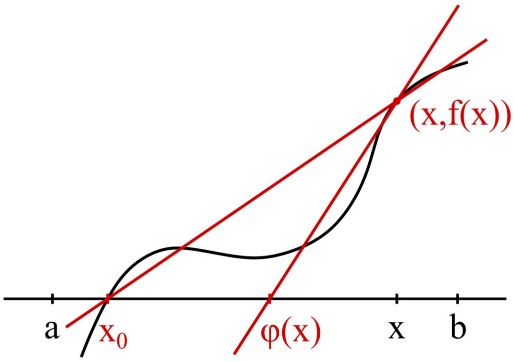
Hence, f is a contraction mapping on $\mathbb{R}^2$ and there exists a fixed point.

Exercise. Determine whther the functon $f(x, y) = \big(\frac{1}{3}\sin x - \frac{1}{3}\cos y + 2, \frac{1}{6}\cos x + \frac{1}{2}\sin y - 1\big)$ has a fixed point.

❏ Application

■ The Secant Method

Let f be a continuously differentiable on $[a,b]$, $f'(x) > 0$ on (a,b) and $f(a)f(b) < 0$. Then f is strictly increasing. By I.V.T, there exists a unique zero of f.
Question: How to find the zero of f?



Let y_0 be the zero of f. For $x \in [a,b]$, we want to define $\phi : [a,b] \to [a,b]$ such that $\phi(x)$ is between x and the zero of f (located on the same side of zero as x). By M.V.T,

$$\frac{f(x)-0}{x-\phi(x)} \overset{\substack{x \text{ stays on} \\ \text{the same side}}}{>} \frac{f(x)}{x-y_0} \overset{M.V.T}{=} f'(\xi)$$

Then $f(x) > f'(\xi)\big(x-\phi(x)\big)$ and hence $\phi(x) > x - \dfrac{f(x)}{f'(\xi)}$. Assume $\sup\limits_{x\in[a,b]} f'(x) < \infty$. Let $M = \sup\limits_{x\in[a,b]} f'(x) + 1$. Define $\phi(x) = x - \dfrac{f(x)}{M}$. Hence, $\phi(x) = x \Longleftrightarrow f(x) = 0$. Consider

$$\phi'(x) = 1 - \frac{f'(x)}{M} > 0.$$

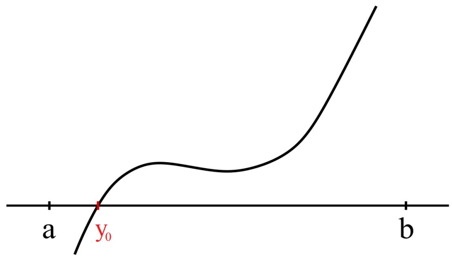
By M.V.T,

$$\big|\phi(x)-\phi(y)\big| = |\phi'(\xi)||x-y| \le \underbrace{\Big(1 - \min_{\xi\in[a,b]} \frac{f'(\xi)}{M}\Big)}_{c} |x-y|$$

for every $x, y \in [a,b]$. Since $\phi'(x) > 0$, ϕ is strictly increasing on $[a,b]$. For $x \in [a,b]$, $a < \phi(a) \le \phi(x) \le \phi(b) < b$. Then ϕ maps from $[a,b]$ to $[a,b]$ and thus ϕ is a contraction mapping on $[a,b]$. There exists $y_0 \in [a,b]$ such that $\phi(y_0) = y_0$. Choose an arbitrary point $x_1 \in [a,b]$ and define $x_{n+1} = \phi(x_n)$ and we obtain $\lim\limits_{n\to\infty} x_n = y_0$.

■ The Newton's Method (Newton-Raphson Iteration)

Let f be a continuously differentiable on $[a,b]$, $f(a) < 0$ and $f(b) > 0$. By I.V.T, there exists $y_0 \in [a,b]$ such that $f(y_0) = 0$.

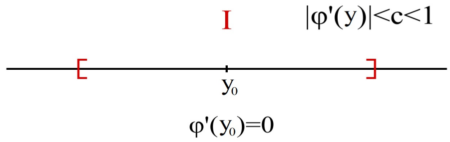


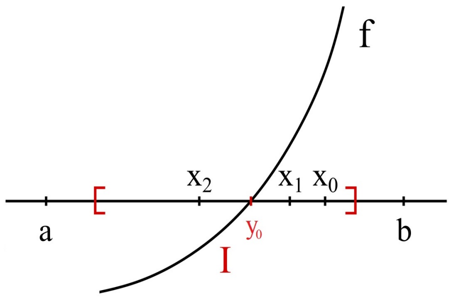
Choose $\phi(x) = x - c(x)f(x)$ where $c(x)$ is a nonvanishing function. Then $f(x) = 0$ if and only if $\phi(x) = x$. Hence, $\phi(y_0) = y_0$.

Observation:

Suppose that there exists an interval I of y_0 such that

$$|\phi'(x)| \le c < 1 \quad \text{for every } x \in I.$$





Choose $x_0 \in I$ and $x_{n+1} = \phi(x_n)$. Then

$$\begin{aligned}
|x_1 - y_0| &= \left|\phi(x_0) - \phi(y_0)\right| \overset{M.V.T}{=} |\phi'(t_0)||x_0 - y_0| < c|x_0 - y_0| \\
|x_2 - y_0| &< c|x_1 - y_0| < c^2|x_0 - y_0| \\
&\vdots \\
|x_n - y_0| &< c^n|x_0 - y_0| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, we will choose a suitable $c(x)$ such that $\phi(x)$ is continuously differentiable and there exists an interval I of y_0 such that $|\phi'(x)| \le c < 1$ on I. Then we can choose an initial point $x_0 \in I$ such that $y_0 = \lim\limits_{n\to\infty} \phi^{(}x_0)$.

Now, let's choose a nonvanishing function $c(x)$. Observe that

$$\begin{aligned}
\phi(x) &= x - c(x)f(x) \\
\Rightarrow \quad \phi'(x) &= 1 - c'(x)f(x) - c(x)f'(x) \\
\Rightarrow \quad \phi'(y_0) &= 1 - c(y_0)f'(y_0)
\end{aligned}$$

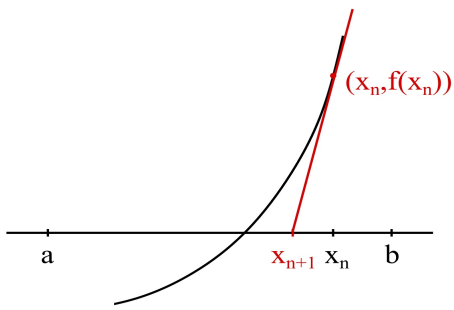
Choose $c(x) = \frac{1}{f'(x)}$ $\left(|\phi'(y_0)| = 0 < 1\right)$. Since ϕ is continuously differentiable, there exists an interval I of y_0 such that $|\phi'(x)| \le c < 1$ on I.

Then

$$\phi(x) = x - c(x)f(x) = x - \frac{f(x)}{f'(x)}.$$

Choose $x_1 \in I$ and $x_{n+1} = \phi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

This method is called "*Newton's iteration*".



Example 4.7.6. Find a positive root of the equation

$$2x^4 + 2x^3 - 3x^2 - 5x - 5 = 0$$

with an accuracy of three decimal places.

Proof. Let $f(x) = 2x^4 + 2x^3 - 3x^2 - 5x - 5$. Then $f'(x) = 8x^3 + 6x^2 - 6x - 5$, $f(1) = -9$ and $f(2) = 21$. Choose $x_1 = 1.6$.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.6 - \frac{0.6193}{33.528} = 1.5815 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5815 - \frac{0.114}{32.1623} = 1.5780 \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 1.5780 + 0.0031 = 1.5811 \end{aligned}$$

□

(**Sufficient condition:**) Let f be a twice continuously differentiable function on $[a, b]$ with $f(a)f(b) < 0$. (By I.V.T, there exists $y_0 \in [a, b]$ such that $f(y_0) = 0$.). Suppose that $f'(x) \neq 0$ for every $x \in [a, b]$. Then $\phi'(x) = \dfrac{f(x)f''(x)}{\left(f'(x)\right)^2}$. We have

$$|\phi'(x)| \leq c < 1 \quad \text{as } x \text{ is sufficiently close to } y_0.$$

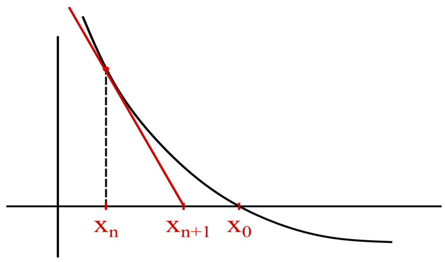
Also, we want to find an interval I of y_0 such that ϕ maps from I into I.

Exercise. Problem 5.29

Exercise. Suppose that f is continuously differentiable, $f' > 0$ on $[a, b]$ and $f(a)f(b) < 0$. Then there exists $y_0 \in [a, b]$ such that $f(y_0) = 0$. If $f'(y_0) > 0$ then there exists an interval I of y_0 such that $\phi(x) = x - \dfrac{f(x)}{f'(x)}$ is a contraction mapping on I.

■ **Error of Newton's Method (Newton-Raphson Iteration)**

Assume $\underbrace{|f'(x)| \geq \frac{1}{M}}_{(1)}$ and $\underbrace{|f''(x)| \leq 2M}_{(2)}$ for every $x \in I$.



From $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

$$|x_{n+1} - x_n| = \left|\frac{f(x_n)}{f'(x_n)}\right| \overset{(1)}{\leq} M|f(x_n)|.$$

By Taylor's theorem,

$$f(x_{n+1}) = \underbrace{f(x_n) + f'(x_n)(x_{n+1} - x_n)}_{=0} + \frac{1}{2}f''(\xi)(x_{n+1} - x_n)^2. \tag{4.16}$$

Then

$$|f(x_{n+1})| = \frac{1}{2}|f''(\xi)|(x_{n+1} - x_n)^2 \overset{(2)}{\leq} M|x_{n+1} - x_n|^2. \tag{4.17}$$

By (4.16), (4.17),

$$|x_{n+1} - x_n| \leq M|f(x_n)| \leq M^2|x_n - x_{n-1}|^2.$$

Hence, if $|f(x_1)| < 1$ and $M < 1$ then $f(x_n) \to 0$ by (4.17) and $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|^2$.

Remark. Let f be a twice continuously differentiable function. Suppose that there exists y_0 such that $f(y_0) = 0$ and $f'(y_0) > 0$ (or < 0). Then there exists a neighborhood I of y_0 such that $\phi(x) = x - \dfrac{f(x)}{f'(x)}$ is a contraction mapping on I.

Proof. Since f is twice continuously differentiable and $f'(y_0) > 0$, there exists $\delta_1 > 0$ such that $f'(x) > 0$ for $x \in (y_0 - \delta_1, y_0 + \delta_1)$ and

$$\phi'(x) = \frac{f(x)f''(x)}{\big(f'(x)\big)^2}$$

is continuous on $(y_0 - \delta_1, y_0 + \delta_1)$.

Since $f(y_0) = 0$ and $f'(x) > 0$, $\phi'(y_0) = 0$ and $f(x)$ is increasing on $(y_0 - \delta_1, y_0 + \delta_1)$ and $\phi(y_0) = y_0$. Then there exists $0 < \delta < \delta_1$ such that for $x \in (y_0 - \delta, y_0 + \delta)$,

$$|\phi'(x)| < 1 \tag{4.18}$$

and $f(x) \begin{cases} < 0 & x \in [y_0 - \delta, y_0) \\ > 0 & x \in (y_0, y_0 + \delta] \end{cases}$ for every $x \in [y_0 - \delta, y_0 + \delta]$. Thus,

$$|\phi(x) - y_0| = |\phi(x) - \phi(y_0)| = |\phi'(\xi)||x - y_0| \overset{(4.18)}{<} |x - y_0| \le \delta.$$

This implies that

$$\phi(x) \in [y_0 - \delta, y_0 + \delta]. \tag{4.19}$$

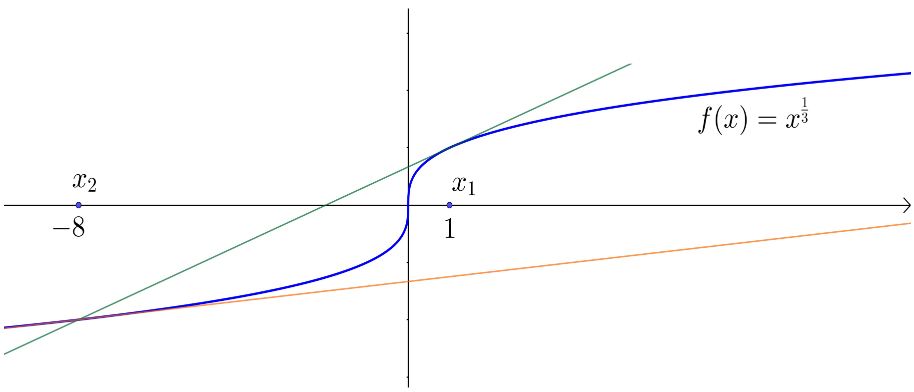
By (4.18) and (4.19), ϕ is a contraction mapping on $[y_0 - \delta, y_0 + \delta]$. □

Remark. Under the above assumption, if $x_1 \in [y_0 - \delta, y_0 + \delta]$ and $x_{n+1} = \phi(x_n)$, then $\{x_n\}_{n=1}^{\infty}$ converges to y_0 which is the fixed point for $\phi(x)$. Also, x_0 is the zero of $f(x)$.

Remark. The Newton's method might fail.

$$\phi(x) = x - \frac{f(x)}{f'(x)} \quad \text{and} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

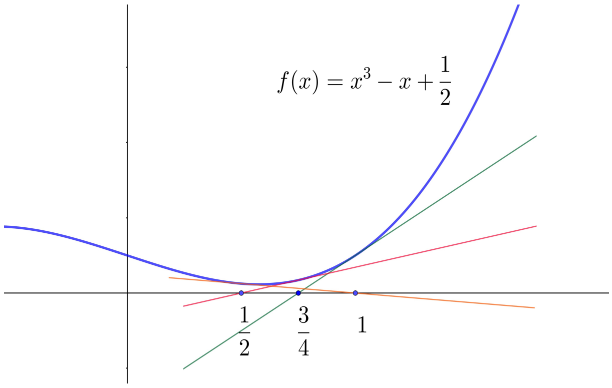
Case 1: $|x_n| \to \infty$ as $n \to \infty$. For example $f(x) = x^{1/3}$ and $x_1 = 1$.



Case 2: $x_1 = x_3 = x_5 = \cdots = x_{2n+1} = \cdots$ and $x_2 = x_4 = x_6 = \cdots = x_{2n} = \cdots$.

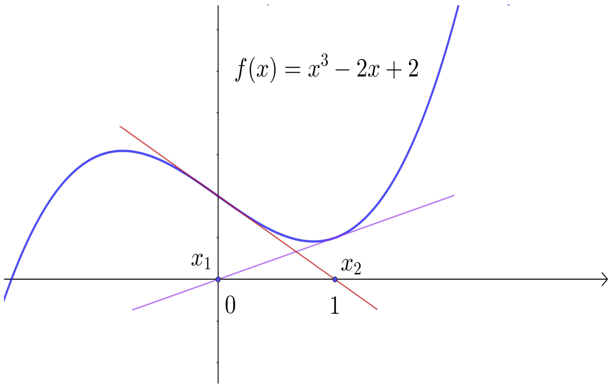
For example $f(x) = x^3 - x + \frac{1}{2}$ and

$$x_1 = x_4 = x_7 = \cdots = 1,$$
$$x_2 = x_5 = x_8 = \cdots = \frac{3}{4},$$
$$x_3 = x_6 = x_9 = \cdots = \frac{1}{2}.$$

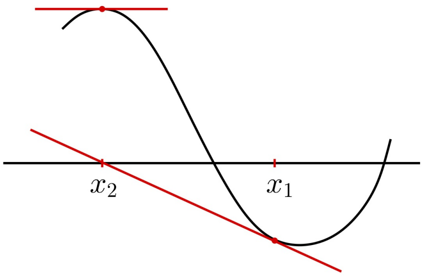


For example, $f(x) = x^2 - 2x + 2$ and

$$x_1 = x_3 = x_5 = \cdots = 0,$$
$$x_2 = x_4 = x_6 = \cdots = 1.$$



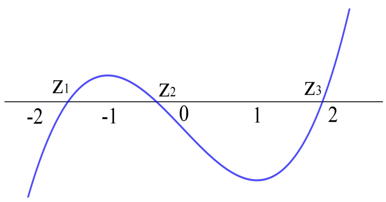
Case 3: $f'(x_n) = 0$.



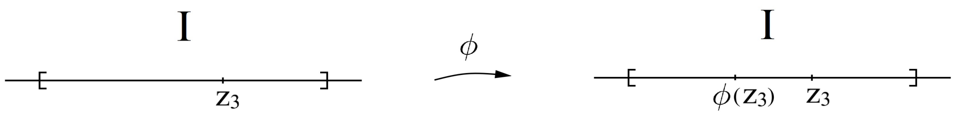
Case 4: $|x_n - y_0| \gg 1$

Example 4.7.7. Consider $x^3 - 3x - 1 = 0$. Let $f(x) = x^3 - 3x - 1$ and the three zero of $f(x)$ be $z_1 < z_2 < z_3$.

We can check that $z_3 \in [1, 2]$. Find an interval I of z_3 such that the Newton's iteration $\{x_n\}_{n=1}^{\infty}$ converges to z_3 if we choose any initial point $x_1 \in I$.



Strategy: Let $\phi(x) = x - \dfrac{f(x)}{f'(x)}$. To find an interval I of z_3 such that $\phi(x)$ is a contraction mapping on I. To prove (i) $\phi : I \to I$ and (ii) $|\phi(x) - \phi(y)| < c|x - y|$ for some $0 \le c < 1$ and for every $x, y \in I$.



Consider $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$ and $f''(x) = 6x$. Then $f'(x) > 0$ and $f''(x) > 0$ for every $x \in [1, 2]$. Hence, f is increasing on $[1, 2]$ and $f(x) \begin{cases} < 0 & x \in [1, z_3) \\ > 0 & x \in (z_3, 2] \end{cases}$. Consider $\phi'(x) = \dfrac{f(x)f''(x)}{\left(f'(x)\right)^2}$. For $x \in [\frac{3}{2}, 2]$, $\frac{15}{4} \le f'(x) \le 9$ and $f''(x) < 12$. If $|x - z_3| < \delta$,

$$|f(x) - 0| = |f(x) - f(z_3)| < |f'(\xi)||x - z_3| \le 9|x - z_3| < 9\delta.$$

Therefore,

$$|\phi'(x)| < \frac{12 \cdot 9\delta}{(15/4)^2} = \frac{192}{25}\delta.$$

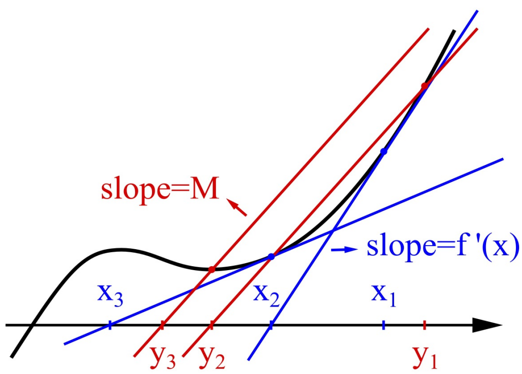
Then we can choose δ sufficiently small such that (i) and (ii) hold.

■ **Compare the secant method and the Newton's method**

• **The Secant Method**

$$\phi(x) = x - \frac{f(x)}{M} \quad \text{where } M = \sup_{x \in [a,b]} |f'(x)| + 1.$$

The slope is never zero. But the rate of convergence is slow and this method needs to detect the sign of $f'(x)$ in advance.

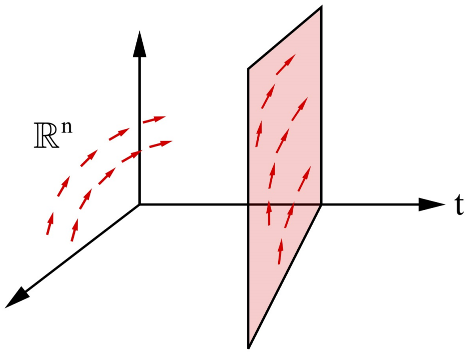


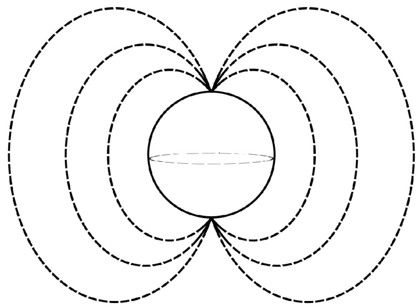
• **The Newton's Method**

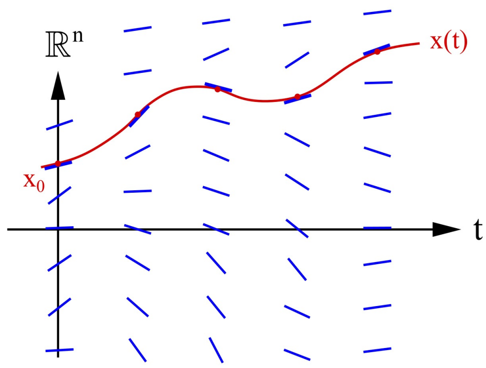
$$\phi(x) = x - \frac{f(x)}{f'(x)} \quad \text{where } M = \sup_{x \in [a,b]} |f'(x)| + 1.$$

The slope $f'(x)$ may be zero. But the rate of convergence is faster and $\frac{*}{f'(x)}$ will automatically detect the sign of $f'(x)$.

4.8 The existence and uniqueness of the solutions to ODE's







$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)$

direction field

■ Ordinary Differential Equations (ODE)

Let's consider the initial value problem (I.V. P) for the ODE.

$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}\big(\mathbf{x}(t), t\big) \quad \text{for } t \in [t_0, t_0 + \triangle t] & (4.20) \\ \mathbf{x}(t_0) = \mathbf{x}_0 & (4.21) \end{cases}$$

Definition 4.8.1. Let I be an interval and $t_0 \in I$. A function $\mathbf{x}(t) : I \to \mathbb{R}^n$ is called "*a solution of the ODE* (4.20) *with initial condition* (4.21) *on* I" if $\mathbf{x}'(t)$ exists on I, and $\mathbf{x}(t), \mathbf{x}'(t)$ satisfy (4.20) and (4.21).

Question: For given $\mathbf{f}(\mathbf{x}, t) : \mathbb{R}^n \times I \to \mathbb{R}^n$ and $\mathbf{x}_0 \in \mathbb{R}^n$, is there a solution to (4.20) and (4.21)?

Question: For what sufficient condition of f, there exists a solution to (4.20) and (4.21)?

Question: If the solution exists, is it unique? That is, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions to (4.20) and (4.21), are they equal for every $t \in I$?

Heuristic Idea: Suppose $\mathbf{g}(t) : I \to \mathbb{R}^n$, $\mathbf{g}(t) = \big(g_1(t), \cdots, g_n(t)\big)$, $g_i(t) : I \to \mathbb{R}$, $\mathbf{x}(t) = \big(x_1(t), \cdots, x_n(t)\big)$ and $\mathbf{x}_0 = (x_0^1, \cdots, x_0^n) \in \mathbb{R}^n$ such that

$$\begin{cases} \mathbf{x}'(t) &= \mathbf{g}(t) \\ \mathbf{x}_0(t_0) &= \mathbf{x}_0 \end{cases} \Longleftrightarrow \begin{cases} \big(x_1'(t), \cdots, x_n'(t)\big) &= \big(g_1(t), \cdots, g_n(t)\big) \\ \big(x_1(t_0), \cdots, x_n(t_0)\big) &= (x_0^1, \cdots, x_0^n) \end{cases}$$

By the Fundamental Theorem of Calculus,

$$x_1(t) = x_0^1 + \int_{t_0}^{t} g_1(s)\, ds, \cdots, x_n(t) = x_0^n + \int_{t_0}^{t} g_n(s)\, ds.$$

Denote

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{g}(s)\, ds.$$

In our case, $\mathbf{f}(\mathbf{x}, t) : B(\mathbf{x}_0, r) \times [t_0, T] \to \mathbb{R}^n$, $\mathbf{x}(t)$ satisfies

$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}\big(\mathbf{x}(t), t\big) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

Then $\mathbf{x}(t) = \mathbf{x}_0 + \displaystyle\int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds$.

Theorem 4.8.2. *(Fundamental Theorem of ODE) Suppose that for some* $r > 0$, $T > t_0$, $f : \underbrace{B(x_0, r)}_{\subseteq \mathbb{R}^n} \times [t_0, T] \to \mathbb{R}^n$ *is continuous in* (x, t) *and is Lipschitz in* x*; that is, there exists* $K > 0$ *such that for every* $x, y \in B(x_0, r)$ *and* $t \in [t_0, T]$*,*

$$\big\| f(x, t) - f(y, t) \big\| \le K \|x - y\|.$$

Then there exists $0 < \triangle < R$ *such that there exists a unique solution to* (4.20) *and* (4.21).

Observe that our goal is to find an element $\mathbf{x} = \mathbf{x}(t) \in C\big([t_0, t_0 + \triangle], \mathbb{R}^n\big)$ such that

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds \quad \text{for } t \in [t_0, t_0 + \triangle].$$

Hence, if $M \subseteq C\big([t_0, t_0 + \triangle]; \mathbb{R}^n\big)$ is certian subset and $\Phi : M \to M$ is defined by

$$\Phi\big(\mathbf{x}(t)\big) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds.$$

Then our targent function (the solution) is a fixed point for Φ. That is $\mathbf{x}(t)$ satisfies $\mathbf{x}(t) = \Phi\big(\mathbf{x}(t)\big) = \mathbf{x}_0 + \displaystyle\int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds$.

Proof. Note that we will use the notationi $\| \cdot \| = \| \cdot \|_{\mathbb{R}^n}$ and abuse $\mathbf{x}_0 \in \mathbb{R}^n$ or $\mathbf{x}_0 = \mathbf{x}_0(t)$ as a constant function in the proof. Also, we use $\|\mathbf{x} - \mathbf{x}\|_\infty = \sup\limits_{t \in [t_0, t_0 + \triangle]} \|\mathbf{x}(t) - \mathbf{x}_0(t)\|$ and $\mathbf{f}(\mathbf{x}_0, \cdot) = \mathbf{f}\big(\mathbf{x}_0(\cdot), \cdot\big)$.

Let

$$M = \Big\{ \mathbf{x}(t) \in C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n \,\Big|\, \sup_{t \in [t_0, t_0 + \triangle]} \underbrace{\big\| \mathbf{x}(t) - \mathbf{x}_0 \big\|}_{\mathbf{x}_0 \text{ is a constant function}} \le \frac{r}{2} \Big\}.$$

Define a mapping Φ on M by

$$\Phi\big(\mathbf{x}(t)\big) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, dx \quad \text{for every } t \in [t_0, t_0 \triangle t].$$

> The number $\triangle$ is to be determined later. That is, we will choose $\triangle t$ such that Φ has a fixed point in $\big(M, \|\cdot\|_\infty\big)$ by the following steps: (1) $\Phi : M \to M$, (2) Φ is a contraction on M, and (3) M is complete.

(1) (i) **To check that $\Phi \in C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n\big)$ for every $\mathbf{x} \in M$**

For $\mathbf{x} \in M$, given $\varepsilon > 0$, to find $\delta > 0$ such that if $t_1, t_2 \in [t_0, t_0 + \triangle t]$ and $|t_1 - t_2| < \delta$

$$\begin{aligned}
\Big\|\Phi(\mathbf{x})(t_1) - \Phi(\mathbf{x}(t_1))\Big\| &= \Big\|\int_{t_0}^{t_2} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds - \int_{t_0}^{t_1} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds\Big\| \\
&= \Big\|\int_{t_1}^{t_2} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds\Big\| \le \int_{t_1}^{t_2} \Big\|\mathbf{f}\big(\mathbf{x}(s), s\big)\Big\|\, ds \\
&\le A|t_1 - t_2| < K\delta
\end{aligned}$$

for some large number $A > 0$ satisfying $\Big\|\mathbf{f}\big(\mathbf{x}(s), s\big)\Big\| < A$ for every $s \in [t_0, t_0 + \triangle t]$.

We can choose $\delta = \dfrac{\varepsilon}{A}$ and hence Φ is continuous in t. That is, $\Phi \in C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n\big)$.

(ii) **To show that $\Phi : M \to M$.**

For $t \in [t_0, t_0 + \triangle]$ and $\mathbf{x} \in M$,

$$\begin{aligned}
\Big\|\Phi(\mathbf{x})(t) - \mathbf{x}_0(t)\Big\| &= \Big\|\int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, dx\Big\| \\
&\le \Big\|\int_{t_0}^{t} \big[\mathbf{f}\big(\mathbf{x}(s), s\big) - \mathbf{f}\big(\mathbf{x}_0(s), s\big)\big]\, ds + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}_0(s), s\big)\, ds\Big\| \\
&\le \Big\|\int_{t_0}^{t} \big[\mathbf{f}\big(\mathbf{x}(s), s\big) - \mathbf{f}\big(\mathbf{x}_0(s), s\big)\big]\, ds\Big\| + \Big\|\int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}_0(s), s\big)\, ds\Big\| \\
\text{(Lipschitz in } x) \quad &\le \int_{t_0}^{t_0+\triangle t} K\Big\|\mathbf{x}(s) - \mathbf{x}_0(t)\Big\| ds + \int_{t_0}^{t_0+\triangle t} \Big\|\mathbf{f}\big(\mathbf{x}_0(s), s\big)\Big|\, ds \\
&\le K\int_{t_0}^{t_0+\triangle t} K\Big\|\mathbf{x}(s) - \mathbf{x}_0(s)\Big\|\, ds + \int_{t_0}^{t_0+\triangle t} \Big\|\mathbf{f}\big(\mathbf{x}_0(s), s\big)\Big\|\, ds \\
&\le K\int_{t_0}^{t_0+\triangle t} \underbrace{\sup_{s\in t, t_0+\triangle t} \|\mathbf{x}(s) - \mathbf{x}_0(s)\|}_{=\|\mathbf{x}-\mathbf{x}_0\|_\infty}\, ds + \int_{t_0}^{t_0+\triangle t} \underbrace{\sup_{t\in[t_0, t_0+\triangle t]} \Big\|\mathbf{f}\big(\mathbf{x}_0(s), s\big)\Big\|}_{=\|\mathbf{f}(\mathbf{x}_0(\cdot),\cdot)\|_\infty}\, ds \\
&\le \triangle t\Big[K\underbrace{\|\mathbf{x} - \mathbf{x}_0\|_\infty}_{\le \frac{r}{2}} + \underbrace{\Big\|\mathbf{f}\big(\mathbf{x}_0(\cdot), \cdot\big)\Big\|}_{\textit{fixed number}}\Big]
\end{aligned}$$

Hence,

$$\Big\|\Phi(\mathbf{x}) - \mathbf{x}_0\Big\|_\infty = \sup_{t\in[t_0, t_0+\triangle t]} \Big\|\Phi\big(\mathbf{x}\big)(t) - \mathbf{x}_0(t)\Big\| \le \triangle t\Big[K\underbrace{\Big\|\mathbf{x} - \mathbf{x}_0\Big\|_\infty}_{\le \frac{r}{2}} + \underbrace{\Big\|\mathbf{f}\big(\mathbf{x}_0(\cdot), \cdot\big)\Big\|_\infty}_{\textit{fixed number}}\Big].$$

Choose $0 <\le \dfrac{r}{Kr + 2\|\mathbf{f}\big(\mathbf{x}_0(\cdot), \cdot\big)\|_\infty}$. Then $\Big\|\Phi(\mathbf{x}) - \mathbf{x}_0\Big\| \le \dfrac{r}{2}$ and hence $\Phi(\mathbf{x}) \in M$.

(2) For $\mathbf{x}, \mathbf{y} \in M$, to prove that when $\triangle t$ is sufficiently small (TBD),

$$\big|\Phi(\mathbf{x}) - \Phi(y)\|_\infty = \sup_{t\in[t_0,t_0+\triangle]} \big\|\Phi\big(\mathbf{x}\big)(t) - \Phi\big(\mathbf{y}\big)(t)\big\| \le c\big\|\mathbf{x}-\mathbf{y}\big\|_\infty.$$

for some $0 \le c < 1$.

For $t \in [t_0, t_0 + \triangle t]$,

$$\begin{aligned}
\big\|\Phi\big(\mathbf{x}\big)(t) - \Phi\big(\mathbf{y}\big)(t)\big\| &= \Big\|\int_{t_0}^t \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds - \int_{t_0}^t \mathbf{f}\big(\mathbf{y}(s), s\big)\, ds\Big\| \\
&= \Big\|\int_{t_0}^t \big[\mathbf{f}\big(\mathbf{x}(s), s\big) - \mathbf{f}\big(\mathbf{y}(s), s\big)\big]\, ds\Big\| \\
&\le \int_{t_0}^t \big\|\mathbf{f}\big(\mathbf{x}(s), s\big) - \mathbf{f}\big(\mathbf{y}(s), s\big)\big\|\, ds \\
&\le \int_{t_0}^{t_0+\Delta t} K\big\|\mathbf{x}(s) - \mathbf{y}(s)\big\|\, ds \\
&\le \int_{t_0}^{t_0+\Delta t} K\big\|\mathbf{x}-\mathbf{y}\big\|_\infty\, ds \\
&= \triangle t K\big\|\mathbf{x}-\mathbf{y}\big\|_\infty.
\end{aligned}$$

Then

$$\big\|\Phi\big(\mathbf{x}\big) - \Phi\big(\mathbf{y}\big)\big\|_\infty = \sup_{t\in[t_0,t_0+\Delta t]} \big\|\Phi\big(\mathbf{x}\big)(t) - \Phi\big(\mathbf{y}\big)(t)\big\| \le \underbrace{\triangle t K}_{c}\big\|\mathbf{x}-\mathbf{y}\big\|_\infty$$

Choose $0 < \triangle$ such that $\triangle t K < 1$, say $\triangle \le \dfrac{1}{2K}$. Then $\Phi : M \to M$ is a contraction mapping.

(3) **To prove that M is complete in the norm $\|\cdot\|_\infty$**
Since $\Big(C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n\big), \|\cdot\|_\infty\Big)$ is complete, if suffices to show that M is closed in $\Big(C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n\big), \|\cdot\|_\infty\Big)$.

Let $\{\mathbf{x}_n\}_{n=1}^\infty \subseteq M$ and $\mathbf{x} \in \Big(C\big([t_0, t_0 + \triangle t]; \mathbb{R}^n\big), \|\cdot\|_\infty\Big)$ such that

$$\mathbf{x}_n \to \mathbf{x} \quad \text{as } n \to \infty.$$

That is, $\big\|\mathbf{x}_n - \mathbf{x}\|_\infty \to 0$ as $n \to \infty$. We will prove that $\mathbf{x} \in M$.

For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that if $n \ge N$, $\big\|\mathbf{x}_n - \mathbf{x}\big\|_\infty < \varepsilon$.

Since $\mathbf{x}_n \in M$ for every $n \in \mathbb{N}$, $\big\|\mathbf{x}_N - \mathbf{x}_0\big\|_\infty \le \dfrac{r}{2}$. Then

$$\big\|\mathbf{x} - \mathbf{x}_0\big\|_\infty \le \big\|\mathbf{x} - \mathbf{x}_N\big\|_\infty + \big\|\mathbf{x}_N - \mathbf{x}_0\big\|_\infty \le \varepsilon + \frac{r}{2}.$$

Since ε is arbitrary, taking $\varepsilon \searrow 0$, we have $\big\|\mathbf{x} - \mathbf{x}_0\big\|_\infty \le \dfrac{r}{2}$. Thus, $\mathbf{x} \in M$.

By (1), (2) and (3), for $0 < \triangle t \leq \min\Big(T - t_0, \dfrac{r}{Kr + 2\big\|\mathbf{f}(\mathbf{x}_0, \cdot)\big\|_\infty}, \dfrac{1}{2K}\Big)$, there exists a unique fixed point $\mathbf{x} = \mathbf{x}(t) \in M$. That is ,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds \quad \text{for every } t \in [t_0, t_0 + \triangle t].$$

(**Uniqueness**) Let $\mathbf{x}, \mathbf{y}$ be two solution of (4.20) and (4.21) on $[t_0, t_0 + \triangle t]$. Then

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds \quad \text{and} \quad \mathbf{y}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\big(\mathbf{y}(s), s\big)\, ds.$$

Then

$$\begin{aligned}
\big\|\mathbf{x} - \mathbf{y}\big\|_\infty &= \sup_{t \in [t_0, t_0 + \triangle t]} \Big\| \int_{t_0}^{t} \mathbf{f}\big(\mathbf{x}(s), s\big)\, ds - \int_{t_0}^{t} \mathbf{f}\big(\mathbf{y}(s), s\big)\, ds \Big\| \\
&\leq \int_{t_0}^{t} \sup_{t \in [t_0, t_0 + \triangle t]} \Big\| \big(\mathbf{x}(s), s\big) - \mathbf{f}\big(\mathbf{y}(s), s\big) \Big\|\, ds \\
&\leq \triangle t K \big\|\mathbf{x} - \mathbf{y}\big\|_\infty \\
&\leq \frac{1}{2}\big\|\mathbf{x} - \mathbf{y}\big\|_\infty.
\end{aligned}$$

Hence, $\big\|\mathbf{x} - \mathbf{y}\|_\infty = 0$ and we have $\mathbf{x}(t) = \mathbf{y}(t)$ for every $t \in [t_0, t_0 + \triangle t]$.

In fact, it is not necessary to prove this part since fixed point theorem already gives the uniqueness.

□

Example 4.8.3. Find a function $x(t) : [0, T] \to \mathbb{R}$ such that

$$\begin{cases} x'(t) = x(t) \\ x(0) = 1. \end{cases} \tag{4.22}$$

Proof. Define

$$\Phi\big(\mathbf{x}\big)(t) = 1 + \int_0^t x(s)\, ds, \quad , x_0(t) = 1$$

and $x_{n+1}(t) = \Phi\big(x_n\big)(t)$. Then

$$\begin{aligned}
x_1(t) &= 1 + \int_0^t 1\, ds = 1 + t \\
x_2(t) &= 1 + \int_0^t 1 + s\, ds = 1 + t + \frac{t^2}{2} \\
x_3(t) &= 1 + \int_0^t 1 + s + \frac{s^2}{2}\, ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} \\
&\vdots \\
x_k(t) &= = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^k}{k!}.
\end{aligned}$$

Then $\{x_k\}_{k=0}^{\infty}$ converges to $x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$ which is the solution of I.V. P for (4.22). □

Example 4.8.4. Find a function $x(t)$ such that

$$\begin{cases} x'(t) &= tx(t) \\ x(0) &= 3 \end{cases} \tag{4.23}$$

Proof. Define

$$\Phi\big(x\big)(t) = 3 + \int_0^t sx(s)\,dx, \quad x_0(t) = 3$$

and $x_{n+1}(t) = \Phi\big(x_n\big)(t)$. Then

$$\begin{aligned} x_1(t) &= 3 + \int_0^t sx_0(s)\,ds = 3 + \int_0^t 3s\,ds = 3 + \frac{3t^2}{2} \\ x_2(t) &= 3 + \int_0^t sx_1(s)\,ds + 3 + \int_0^t 3 + \frac{3}{2}s^2 = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} \\ &\vdots \\ x_k(t) &= 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} + \cdots + \frac{3t^{2k}}{2\cdot 4\cdots(2k)} \end{aligned}$$

We have $x_k(t) \to x(t) = 3 + 3\sum_{k=1}^{\infty} \frac{t^{2k}}{2\cdot 4\cdot(2k)} = 3e^{\frac{t^2}{2}}$ which is the solution of the I.V. P for (4.23).

□

Remark. This process is called the "*Picard iteration*".

Example 4.8.5. Let $x_c(t) = \begin{cases} 0 & \text{if } 0 \le t < c \\ \frac{1}{4}(t-c)^2 & \text{if } t \ge c \end{cases}$. Then

$$\begin{cases} x_c'(t) &= \big(x(t)\big)^{1/2} \\ x_c(0) &= 0 \end{cases} \quad \text{for all } c > 0.$$

Hence, this initial value problem has infinitely many solution. Why?
$f(x_0, t) = \sqrt{x}$ is not Lipschitz near 0. That is, no matter what $K > 0$ is, there exists $x, y \in (-\delta, \delta)$ such that

$$\big|f(x,t) - f(y,t)\big| > K|x - y|.$$

Chapter 5

Differentiation of Maps

5.1 Bounded Linear Maps

Definition 5.1.1. (1) Let X and Y be vector spaces. A mapping $L : X \to Y$ is said to be "*linear*" if

$$L(cx_1 + x_2) = cL(x_1) + L(x_2) \quad \text{for every } c \in \mathbb{R} \text{ and } x_1, x_2 \in X.$$

We usually write "Lx" instead of $L(x)$. Denote the collection of all linear maps from X to Y by $\mathcal{L}(X;Y)$. Note that $\mathcal{L}(X;Y)$ is a vector space.

(2) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. A linear map $L : X \to Y$ is said to be "*bounded*" if

$$\sup_{\substack{x\in X\\ \|x\|_X=1}} \|Lx\|_Y < \infty.$$

(3) The collection of all bounded linear maps from X to Y is denoted by $\mathcal{B}(X;Y)$ and the number $\sup\limits_{\substack{x\in X\\ \|x\|_X=1}} \|Lx\|_Y$ is denoted by $\|L\|_{\mathcal{B}(X;Y)}$.

Example 5.1.2. Let $A \in M_{m\times n}(\mathbb{R})$ be a $m \times n$ matrix. For $x \in \mathbb{R}^n$, define

$$Lx = Ax.$$

That is,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$Lx = Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^m$$

Consider

$$\frac{\|Ax\|^2_{\mathbb{R}^m}}{\|x\|^2_{\mathbb{R}^n}} = \frac{\langle Ax, Ax\rangle_{\mathbb{R}^m}}{\langle x, x\rangle_{\mathbb{R}^n}}.$$

Since

$$\langle Ax, Ax\rangle_{\mathbb{R}^m} = (Ax)^T Ax = x^T A^T Ax = \langle x, A^T Ax\rangle_{\mathbb{R}^n} = \langle A^T Ax, x\rangle_{\mathbb{R}^n} \le \|A^T A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \|x\|_{\mathbb{R}^n},$$

we have

$$\frac{\|Ax\|^2_{\mathbb{R}^m}}{\|x\|^2_{\mathbb{R}^n}} \le \|A^T A\|^2_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}$$

Therefore, $L \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^m)$ and $\|L\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)}$ is equal to the square root of the largest eigenvalue of $A^T A$.

Proposition 5.1.3. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces and* $L \in \mathcal{B}(X; Y)$. *Then*

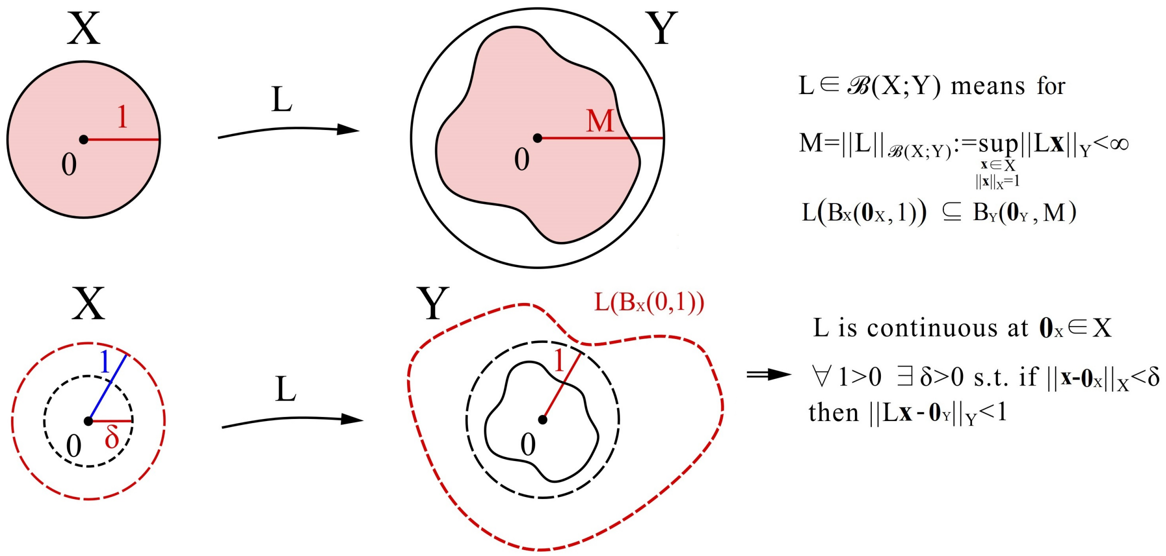
$$\|L\|_{\mathcal{B}(X;Y)} = \sup_{\substack{x \in X \\ x \ne 0}} \frac{\|Lx\|_Y}{\|x\|_X} = \inf\left\{M > 0 \;\middle|\; \|Lx\|_Y \le M\|x\|_X\right\}.$$

Proof. (Exercise) □

Remark. $\|Lx\|_Y \le \|L\|_{\mathcal{B}(X;Y)} \|x\|_X$.

Proposition 5.1.4. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces and* $L \in \mathcal{L}(X; Y)$. *Then* L *is continuous on* X *if and only if* $L \in \mathcal{B}(X; Y)$.

Proof. ($\Longrightarrow$)



Since L is linear, $L0_X = 0_Y$. Since L is continuous at 0_X, for $0 < \varepsilon = 1$, there exists $\delta > 0$ such that if $\|x - 0_X\|_X < \delta$, then $\|Lx\|_Y = \|Lx - L0_X\| < 1$. Thus, for $x \in X$ with $\|x\|_X = \frac{\delta}{2}$, $\|Lx\|_Y < 1$.

Since L is linear, for $x \in X$ with $\|x\|_X = 1$, $\|Lx\|_Y < \frac{2}{\delta}$. Then we have

$$\sup_{\substack{x\in X\\ \|x\|_X=1}} \|Lx\|_Y \leq \frac{2}{\delta}$$

and hence $L \in \mathcal{B}\big(X; Y\big)$.

($\Longleftarrow$)
If $L \in \mathcal{B}\big(X; Y\big)$, then $M = \|L\|_{\mathcal{B}(X;Y)} < \infty$. Then

$$\|Lx_1 - Lx_2\|_Y = \|L(x_1 - x_2)\|_Y \leq M\|x_1 - x_2\|_X \quad \text{for every } x_1, x_2 \in X.$$

Hence, L is (uniformly) continuous on X. □

Remark. A linear map L is continuouos on X if and only if L is continuous at $\mathbf{0}_X$.

Proposition 5.1.5. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Then*

(1) $\Big(\mathcal{B}\big(X; Y\big), \|\cdot\|_{\mathcal{B}(X;Y)}\Big)$ is a normed space.

(2) Moreover, if $(Y, \|\cdot\|_Y)$ is a Banach space, then $\Big(\mathcal{B}\big(X; Y\big), \|\cdot\|_{\mathcal{B}(X;Y)}\Big)$ is a Banach space.

Proof. (1) (Exercise)

(2) Let $\{L_k\}_{k=1}^{\infty} \subseteq \mathcal{B}\big(X; Y\big)$ be a Cauchy sequence. Then $\|L_m - L_n\|_{\mathcal{B}(X;Y)} \to 0$ as $m, n \to \infty$ and there exists $M > 0$ such that $\|L_k\|_{\mathcal{B}(X;Y)} < M$ for every $k \in \mathbb{N}$.

To prove that there exists $L \in \mathcal{B}(X; Y)$ such that $L_k \to L$ in $\Big(\mathcal{B}\big(X; Y\big), \|\cdot\|_{\mathcal{B}(X;Y)}\Big)$.

For $x \in X$,

$$\|L_kx - L_nx\|_Y = \|(L_k - L_n)x\|_Y \leq \underbrace{\|L_k - L_n\|_Y}_{\to\, 0 \text{ as } k,n\to 0} \|x\|_X \to 0 \quad \text{as} \quad k, n \to \infty.$$

Since $(Y, \|\cdot\|_Y)$ is a Banach space, there exists $y = y(x) \in Y$ such that $L_kx \to y$ in $(Y, \|\cdot\|_Y)$ as $k \to \infty$.

Note that for every $x \in X$, there exists a correspoiding $y \in Y$ such that $L_kx \to y$.

Define a map $L : X \to Y$ by $Lx := \lim_{k\to\infty} L_kx$. To check that

$$L_k \to L \quad \text{in } \Big(\mathcal{B}\big(X; Y\big), \|\cdot\|_{\mathcal{B}(X;Y)}\Big).$$

That is, to check

(i) $L \in \mathcal{B}(X;Y)$

(ii) $\|L_k - L\|_{\mathcal{B}(X;Y)} \to 0$ as $k \to \infty$.

(i) For $x_1, x_2 \in X$ and $0 \neq c \in \mathbb{R}$, since $L_k \in \mathcal{B}(X;Y)$ for every $k \in \mathbb{N}$, $Lx_1 = \lim_{k\to\infty} L_k x_1$ and $Lx_2 = \lim_{k\to\infty} L_k x_2$, we have

$$L(cx_1 + x_2) = \lim_{k\to\infty} L_k(cx_1 + x_2) = c \lim_{k\to\infty} L_k(x_1) + \lim_{k\to\infty} L_k(x_2) = cLx_1 + Lx_2.$$

Thus, $L \in \mathcal{L}(X;Y)$. Moreover, for $\|x\|_X = 1$, there exists $N_x \in \mathbb{N}$ such that if $k \geq N_x$, $\|Lx - L_k x\|_Y < 1$. Then

$$\|Lx\|_Y \leq \|Lx - L_k x\|_Y + \|L_k x\|_Y \leq \underbrace{\|Lx - L_k x\|_Y}_{<1} + \underbrace{\|L_k\|_{\mathcal{B}(X;Y)}}_{<M} \underbrace{\|x\|_X}_{=1} < M + 1.$$

Since x is an arbitrary element in X with $\|x\|_X = 1$, we have

$$\|L\|_{\mathcal{B}(X;Y)} = \sup_{\substack{x\in X \\ \|x\|_X=1}} \frac{\|Lx\|_Y}{\|x\|_X} < M + 1$$

and hence $L \in \mathcal{B}(X;Y)$.

(ii) Since $\{L_k\}_{k=1}^{\infty}$ is Cauchy in $\left(\mathcal{B}(X;Y), \|\cdot\|_{\mathcal{B}(X;Y)}\right)$, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $k, n \geq N$,

$$\|L_k - L_n\|_{\mathcal{B}(X;Y)} < \frac{\varepsilon}{2}.$$

For $x \in X$, $\|x\|_X = 1$, since $\lim_{n\to\infty} L_n x = Lx$, there exists $N_1 = N_1(x) \in \mathbb{N}$ such that if $n \geq N_1$,

$$\|L_n x - Lx\|_Y < \frac{\varepsilon}{2}.$$

Hence, for $k \geq N$, we choose $n \geq \max(N, N_1)$ and then

$$\|L_k x - Lx\|_Y \leq \|L_k x - L_n x\|_Y + \|L_n x - Lx\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since x is arbitrary, $\|L_k - L\|_{\mathcal{B}(X;Y)} = \sup_{\|x\|_X=1} \|(L_k - L)x\|_Y < \varepsilon$ whenever $k \geq N$. Therefore, $L_k \to L$ in $\left(\mathcal{B}(X;Y), \|\cdot\|_{\mathcal{B}(X;Y)}\right)$.

□

Proposition 5.1.6. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces, $L \in \mathcal{B}(X;Y)$ and $K \in \mathcal{B}(Y;Z)$. Then $K \circ L \in \mathcal{B}(X;Z)$ and*

$$\|K \circ L\|_{\mathcal{B}(X;Z)} \leq \|K\|_{\mathcal{B}(Y;Z)} \|L\|_{\mathcal{B}(X;Y)}$$

Note that we often write $K \circ L$ as KL if K and L are linear.

Proof. Check $K \circ L$ is linear (exercise).
Check $K \circ L$ is bounded.

$$\|(K \circ L)x\|_Z = \|K \underbrace{(Lx)}_{\in Y}\|_Z \leq \|K\|_{\mathcal{B}(Y;Z)} \|Lx\|_Y \leq \|K\|_{\mathcal{B}(Y;Z)} \|L\|_{\mathcal{B}(X;Y)} \|x\|_X.$$

□

Example 5.1.7. If $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$, then

$$A \in \mathcal{B}\big(\mathbb{R}^n, \mathbb{R}^m\big) \quad \text{and} \quad B \in \mathcal{B}\big(\mathbb{R}^k, \mathbb{R}^n\big)$$

and

$$AB \in M_{m\times k}(\mathbb{R}) \quad \text{and} \quad AB \in \mathcal{B}\big(\mathbb{R}^k; \mathbb{R}^m\big).$$

Lemma 5.1.8. *Let X be a finite dimensional vector space. Then every norm on X is equivalent. That is, if* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are two norms on X, then there exists* $\alpha, \beta > 0$ *such that for every* $x \in X$,

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_2.$$

Proof. (Exercise) □

Theorem 5.1.9. *Let* $(X, \|\cdot\|)_X$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces and X be finite dimensional. Then every linear map from X to Y is bounded. That is,* $\mathcal{L}(X;Y) = \mathcal{B}(X;Y)$.

Proof. Let $T \in \mathcal{L}(X;Y)$. Since X is finite dimensional, say $\dim X = n$, all norms on X are equivalent.

Let $\{e_1, \cdots, e_n\}$ be a basis of X. For $x \in X$, there exist $c_1, \cdots, c_n \in \mathbb{R}$ such that $x = c_1e_1 + \cdots + c_ne_n$. Define a norm

$$|||x||| := \max_{1\le i\le n} |c_i| \quad \text{(Check that } |||\cdot||| \text{ is a norm on } X\text{)}$$

Then there exists $M_1 > 0$ such that $|||x||| \le M_1\|x\|_X$ for every $x \in X$. Let $M_2 = \sum_{i=1}^{n} \|Te_i\|_Y$. Then for every $x = c_1e_1 + \cdots + c_ne_n \in X$,

$$\begin{aligned} \|Tx\|_Y &= \|T(\sum_{i=1}^{n} c_ie_i)\|_Y = \|\sum_{i=1}^{n} c_iTe_i\|_Y \le \sum_{i=1}^{n} |c_i|\ \|Te_i\|_Y \\ &\le \max_{1\le i\le n}|c_i| \sum_{i=1}^{n} \|Te_i\|_Y \le M_2\ |||\ x\ |||\ \le M_1M_2\|x\|_X. \end{aligned}$$

Hence, $T \in \mathcal{B}\big(X;Y\big)$. □

Theorem 5.1.10. *Let* $GL(n)$ *be the set of all invertible linear maps on* $\mathbb{R}^n$. *That is,*

$$GL(n) = \big\{L \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n) \ \big|\ L \text{ is one-to one (and hence onto).}\big\}$$

(1) If $L \in GL(n)$ *and* $K \in \mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^n\big)$ *satisfying* $\|K - L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} < 1$, *then* $K \in GL(n)$.

(2) $GL(n)$ *is an open set of* $\mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^n\big)$.

(3) The mapping $L \to L^{-1}$ *is continuous on* $GL(n)$.

■ ***Heuristical Ideas:***

(1) First of all, consider a function $f : \mathbb{R} \to \mathbb{R}$ and $f(a) = b$. Imagine that L is the derivative of f at a. That is, $L = f'(a)$. Then

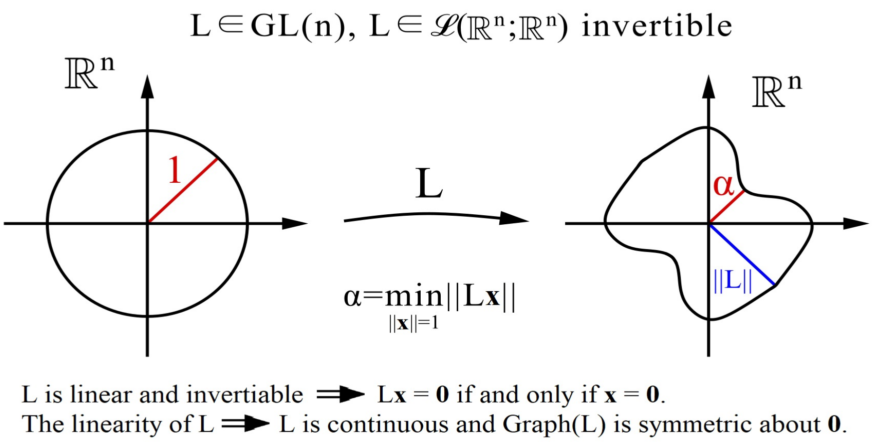
$$\begin{aligned} L \text{ is invertible.} &\iff f'(a) \neq 0 \text{ (say } f'(a) > 0) \\ &\implies f \text{ is increasing near } a. \\ &\implies f \text{ is one-to-one near } a. \\ &\implies f^{-1} \text{ exists near } a. \\ &\implies \left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = L^{-1}. \end{aligned}$$

Suppose K is the derivative of g at a. That is, $K = g'(a)$. If $|K - L| = |f'(a) - g'(a)| \geq |f'(a)|$, then $g'(a)$ could be 0 and hence it is possible that g is not invertible near a.

In order to hope g is invertible near a, we hope

$$|f'(a) - g'(a)| < |f'(a)| \iff \frac{|f'(a) - g'(a)|}{|f'(a)|} = |L - K| \cdot \frac{1}{|L|} < 1$$

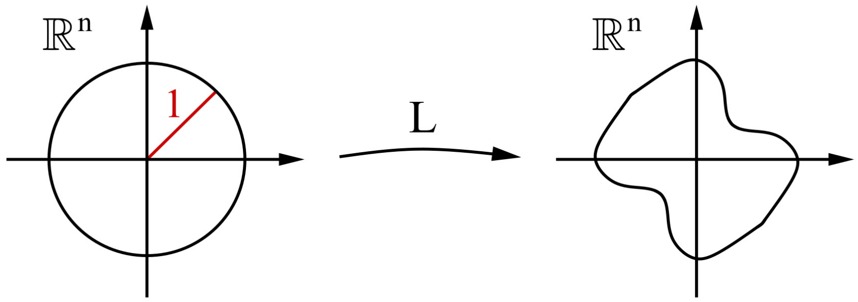
Moreover, consider $L \in GL(n)$. Then $L \in \mathcal{L}\left(\mathbb{R}^n; \mathbb{R}^n\right)$ is invertible. Recall that L is linear and invertible. Then $Lx = 0$ if and only if $x = 0$. Also, the linearity implies that L is continuous at 0 and the graph of L is symmetric about the origin. Imagine the Graph(L)



Hence, L is linear and invertible if and only if the graph of $L\left(\partial B(0,1)\right)$ is a distorted sphere with center at the origin.

Let $K \in \mathcal{B}\left(\mathbb{R}^n; \mathbb{R}^n\right)$. If we hope that K is invertible, we want the graph of $K\left(\partial B(0,1)\right)$ is a certain "distored sphere" with center 0. (That is, the graph $K\left(\partial B(0,1)\right)$ does not touch the origin.)

L ∈ GL(n), L ∈ $\mathscr{L}(\mathbb{R}^n;\mathbb{R}^n)$ invertible



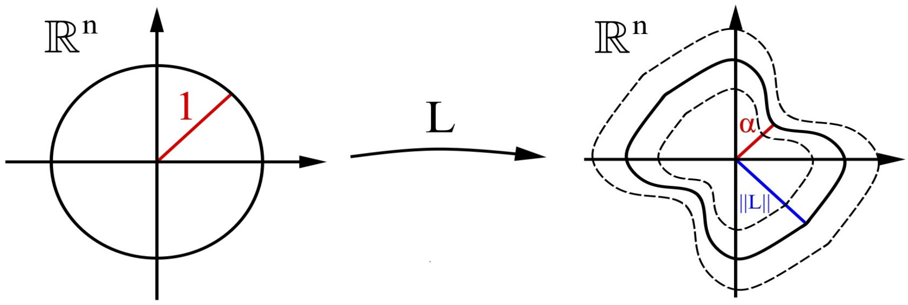
Question: *If $L \in GL(n)$, how to achieve that $K \in GL(n)$?*

Heuristically, we may achieve this goal as long as the distance of the farthest point on $(K-L)\big(\partial B(0,1)\big)$ is less than the distance of the nearest point on $L\big(\partial B(0,1)\big)$.

Let $\alpha = \min_{\|x\|=1} \|Lx\|$. Suppose that for $x \in \partial B(0,1)$, $\|Kx - Lx\| \le \|K-L\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \alpha$. Then $K\big(\partial B(0,1)\big)$ will not attain the origin. Moreover, it works if

$$\|K-L\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \alpha \iff \|K-L\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \cdot \frac{1}{\alpha} < 1$$

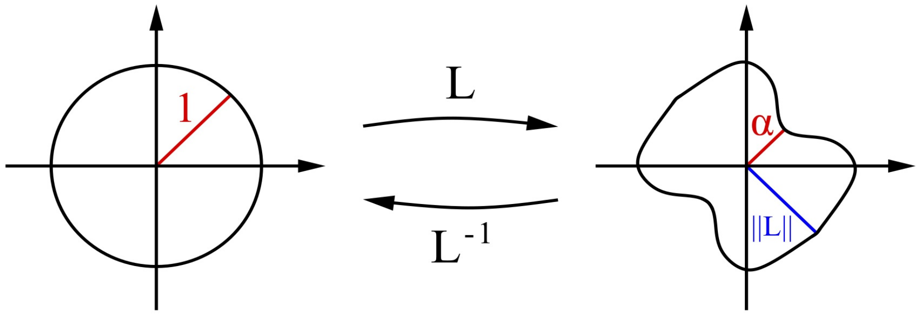
Question: *How large is $\frac{1}{\alpha}$? Is $\|L^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = \frac{1}{\alpha}$?*



Let $\alpha = \min_{\|\mathbf{x}\|=1} \|L\mathbf{x}\|$. Choose $\delta = \frac{1}{2}\alpha$ and ask that for every $\mathbf{x} \in \mathbb{R}^n$, $\|K\mathbf{x} - L\mathbf{x}\|_{\mathbb{R}^n} < \frac{\alpha}{2}$ (hence $\|K-L\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^n)} < \frac{\alpha}{2}$). Then $K\big(\partial B(\mathbf{0},1)\big)$ will not atain $\mathbf{0}$.

Moverover, it works if $\|K-L\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^n)} < \alpha \iff \|K-L\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^n)} \cdot \frac{1}{\alpha} < 1$. Heuristically,

$$\|L^{-1}\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^n)} \overset{??}{=} \frac{1}{\alpha}.$$



$$\|L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} = \sup_{\|x\|=1}\|Lx\| = \sup_{\|x\|=1}\frac{\|Lx\|}{\|x\|}$$
$$\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} = \sup_{\|y\|=1}\frac{\|L^{-1}y\|}{\|y\|} = 1\Big/\inf_{\|x\|=1}\frac{\|Lx\|}{\|x\|} = \frac{1}{\alpha}$$

Remark. $L \in \mathcal{L}\big(\mathbb{R}^n;\mathbb{R}^n\big)$. Then L is invertible if and only if the graph of $L(S^{n-1})$ must be a distorted sphere, centered at 0 and symmetric about 0 as well as every line through 0 intersect $L(S^{n-1})$ exact two times, say y and $-y$.
($\Longrightarrow$) By linearity and invertibility.
($\Longleftarrow$) Let $\{e_1,\cdots,e_n\}$ be a basis. Check that $\{Le_1,\cdots,Le_n\}$ are linearly independent. Suppose not, there exists $y \neq 0$ but $y \notin \text{Span}\{Le_1,\cdots,Le_n\}$. Choose a line ℓ passing 0 and y. By the hypothesis, there exists $z \in \ell$ and $z \in L(S^{n-1})$. Then $z = cy$ for some $c \neq 0$ and $z \in \text{Span}\{Le_1,\cdots,Le_n\}$. Hence, $y \in \text{Span}\{Le_1,\cdots,Le_n\}$ and we obtain a contradiction.

■ ***Heuristical Ideas:***

(2) We imagine that $L \in \mathcal{L}\big(\mathbb{R}^n,\mathbb{R}^n\big)$. Then $L \overset{corresp.}{\longleftrightarrow} A \in M_{n\times n}(\mathbb{R})$.

$$\begin{array}{ccc} L \in GL(n) & & A \text{ is invertible} \\ \Updownarrow & \longleftrightarrow & \Updownarrow \\ L \text{ is invertible} & & \det A \neq 0 \end{array}$$

If perturbing L a little bit, the determinant of the corresponding matrix is still nonzero. Hence, L is an interior point of $GL(n)$.

Proof. (1) Let $\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} = \dfrac{1}{\alpha}$ and $\|K - L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} = \beta$. Then $\beta < \alpha$. Hence, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \alpha\|x\| &= \alpha\|L^{-1}Lx\| \leq \underbrace{\alpha\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}_{=1}\|Lx\| \\ &\leq \|(K-L)x\| + \|Kx\| \leq \|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|x\| + \|Kx\|. \end{aligned}$$

Then $(\alpha - \beta)\|x\| \leq \|Kx\|$. We have $Kx = 0$ if and only if $x = 0$. Therefore, K is invertible.

(2) Let $L \in GL(n)$. Then $L^{-1} \in GL(n)$ and $L^{-1} \in \mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^n\big)$. Choose $\delta = \dfrac{1}{\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}$. By (1),

$$B(L,\delta) = \left\{K \in \mathcal{B}(\mathbb{R}^n,\mathbb{R}^n) \;\middle|\; \|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} < \frac{1}{\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}\right\} \subseteq GL(n)$$

Hence, $GL(n)$ is open.

(3) Given $\varepsilon > 0$, choose $\delta = \min\left(\dfrac{1}{2\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}, \dfrac{\varepsilon}{2\|L^{-1}\|^2_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}\right)$. For $\|K - L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} < \delta$,

$$\begin{aligned} \|L^{-1} - K^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} &= \|K^{-1}(K-L)L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} \\ &\leq \|K^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} \\ &\leq \Big(\|K^{-1} - L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} + \|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\Big)\|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} \\ &\leq \|K^{-1} - L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\underbrace{\|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}_{<\frac{1}{2}} + \underbrace{\|K-L\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}\|L^{-1}\|^2_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)}}_{<\frac{\varepsilon}{2}} \\ &\leq \frac{1}{2}\|K^{-1} - L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} + \frac{\varepsilon}{2}. \end{aligned}$$

Hence, $\|K^{-1} - L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} < \varepsilon$.

□

❑ Matrix Representation

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be finite dimensional vector spaces with $\dim X = n$ and $\dim Y = m$. Let $\mathscr{B} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\tilde{\mathscr{B}} = \{\tilde{\mathbf{e}}_1, \cdots, \tilde{\mathbf{e}}_m\}$ be basis of X and Y respectively.

For $x \in X$, there exists unique n-tuple $(c_1, \cdots, c_n)$, $c_i \in \mathbb{R}$ such that

$$x = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n.$$

Similarly, for $y \in Y$, there exists unique m-tuple $(d_1, \cdots, d_m)$, $d_i \in \mathbb{R}$ such that

$$y = d_1\tilde{\mathbf{e}}_1 + \cdots + d_m\tilde{\mathbf{e}}_m.$$

We denote

$$[x]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [y]_{\tilde{\mathscr{B}}} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{R})$ and $x = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Define

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{1k}c_k \\ \vdots \\ \sum_{k=1}^{n} a_{mk}c_k \end{bmatrix}.$$

Then $Ax = y$ where $y = \sum_{k=1}^{n} a_{1k}c_k\tilde{\mathbf{e}}_1 + \cdots + \sum_{k=1}^{n} a_{mk}c_k\tilde{\mathbf{e}}_m = \sum_{\ell=1}^{m}\Big(\sum_{k=1}^{n} a_{\ell k}c_k\Big)\tilde{\mathbf{e}}_\ell$. Therefore, $A \in \mathcal{B}\big(X;Y\big)$ is a bounded linear map from X to Y. (Check)!

Question: A $m \times n$ matrix $A \in \mathcal{L}\big(X;Y\big)$, how about the converse?

Let $L \in \mathcal{L}\big(X;Y\big)$. Consider $L\mathbf{e}_1, L\mathbf{e}_2, \cdots, L\mathbf{e}_n \in Y$. Then there exist $d_{ij} \in \mathbb{R}$, $1 \le i \le m$ and $1 \le j \le n$ such that

$$\begin{aligned} L\mathbf{e}_1 &= d_{11}\tilde{\mathbf{e}}_1 + d_{21}\tilde{\mathbf{e}}_2 + \cdots + d_{m1}\tilde{\mathbf{e}}_m \rightsquigarrow \begin{bmatrix} d_{11} \\ \vdots \\ d_{m1} \end{bmatrix}_{\tilde{\mathscr{B}}} \\ L\mathbf{e}_2 &= d_{12}\tilde{\mathbf{e}}_1 + d_{22}\tilde{\mathbf{e}}_2 + \cdots + d_{m2}\tilde{\mathbf{e}}_m \rightsquigarrow \begin{bmatrix} d_{12} \\ \vdots \\ d_{m2} \end{bmatrix}_{\tilde{\mathscr{B}}} \\ &\vdots \\ L\mathbf{e}_n &= d_{1n}\tilde{\mathbf{e}}_1 + d_{2n}\tilde{\mathbf{e}}_2 + \cdots + d_{mn}\tilde{\mathbf{e}}_m \rightsquigarrow \begin{bmatrix} d_{1n} \\ \vdots \\ d_{mn} \end{bmatrix}_{\tilde{\mathscr{B}}} \end{aligned}$$

Then

$$\big[L\big]_{\mathscr{B}\tilde{\mathscr{B}}} = \big[L\mathbf{e}_1 \quad L\mathbf{e}_2 \quad \cdots \quad L\mathbf{e}_n\big] = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix}.$$

For $x = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$,

$$\big[Lx\big]_{\tilde{\mathscr{B}}} = \big[L\big]_{\mathscr{B}\tilde{\mathscr{B}}}\big[x\big]_{\mathscr{B}} = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & & \vdots \\ d_{m1} & \cdots & d_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Remark. For given basis $\mathscr{B}$ and $\tilde{\mathscr{B}}$ of X and Y respectively, the space of linear maps from X to Y is one-to-one corresponding to the space of $m \times n$ matrics.That is,

$$\mathcal{L}(X;Y) \overset{1-1\ corresp.}{\longleftrightarrow} M_{m\times n}(\mathbb{R})$$

Note. In our class, we use the stardard basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ on $\mathbb{R}^n$ where $\mathbf{e}_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T$.

5.2 Definition of Derivatives and the Matrix Representation of Derivatives

Goal: Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, $f : \mathcal{U} \to \mathbb{R}^m$ and $\mathbf{a} \in \mathcal{U}$. To define the derivative of f at $\mathbf{a}$.

Recall: Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ and $a \in I$. We say that f is differentiable at a if the limit $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists and denote the limit $f'(a)$.

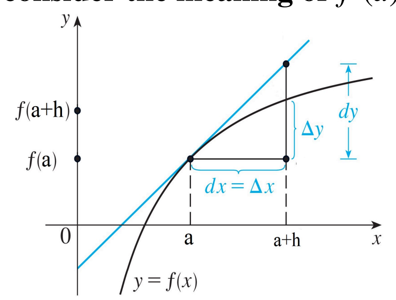
Question: How about the derivative of a function on higher dimensions?

Guess: We try to find the derivative by similar way. For $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{a} \in \mathcal{U}$, consider

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a})}{\mathbf{h}}$$

It does not make sense since the denominator is a "vector" rather than a number . Thus, we need a new definition of derivative.

Reconsider the meaning of $f'(a)$



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

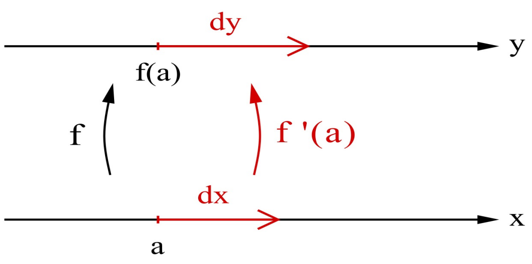
means the slope of the tangent line of f at a. Then

$$dy = f'(a)\, dx.$$

where

dx is the instantaneous displacement in the x-direction

dy is the instantaneous displacement in the y-direction



$f'(a)$ maps the vector $\overrightarrow{dx}$ to the vector $\overrightarrow{dy}$ $(= f'(x)\overrightarrow{dx})$

Remark. $f'(a)$ is not only a number, but also a "map" from the vector space $\mathbb{R}$ to the vector space $\mathbb{R}$. This map, $f'(a)$, sends a vector $\overrightarrow{dx}$ to a vector $\overrightarrow{dy}(= f'(a)\overrightarrow{dx})$. For $\overrightarrow{v} \in \mathbb{R}$,

$$f'(a) : \overrightarrow{v} \longrightarrow f'(a)\overrightarrow{v}$$

Check that for $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}$ and $c \in \mathbb{R}$,

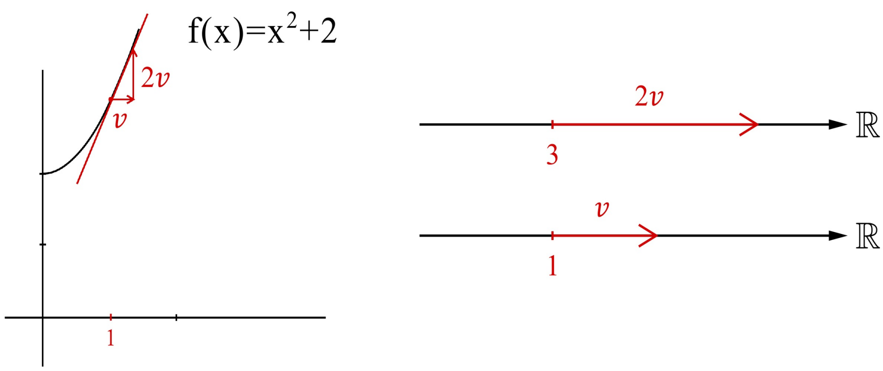
$$\lim_{h\to 0} \frac{f\big(a + h(cv + w)\big) - f(a)}{h} = (cv + w) \lim_{h\to 0} \frac{f\big(a + h(cv + w)\big) - f(a)}{h(cv + w)} = f'(a)(cv + w).$$

Hence,

$$f'(a)(c\overrightarrow{v} + \overrightarrow{w}) = cf'(x)\overrightarrow{v} + f'(a)\overrightarrow{w}$$

and the map $f'(a)$ is linear. That is, $f'(a) \in \mathcal{B}\big(\mathbb{R}; \mathbb{R}\big)$.

Example 5.2.1.
Let $f(x) = x^2 + 2$. Then $f'(1) = 2$. We can regard "2" as a map which sends every vector $\overrightarrow{v}$ to $2\overrightarrow{v}$.



■ **Rewrite the definition of the derivative of f at a**

If the limit $\displaystyle\lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$ exists, then

$$\begin{aligned} &\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = f'(a) \\ \Longleftrightarrow\quad &\lim_{h\to 0} \frac{f(a + h) - f(a) - f'(a)h}{h} = 0 \\ \Longleftrightarrow\quad &\lim_{|h|\to 0} \frac{|f(a + h) - f(a) - f'(a)h|}{|h|} = 0. \end{aligned}$$

The derivative $f'(a)$ reflects the instantaneous change of $f(a + h) - f(a)$ satisfying

$$f(a + h) - f(a) \approx f'(a)h + \text{error}$$

where

$$\frac{|\text{error}|}{|h|} \to 0 \quad \text{as} \quad h \to 0.$$

Therefore, we can rephrase the definition of derivative. If there exists a map (number) $L \in \mathcal{B}(\mathbb{R};\mathbb{R})$ (or $\in \mathbb{R}$) which sends h to Lh such that

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - Lh|}{|h|} = 0$$

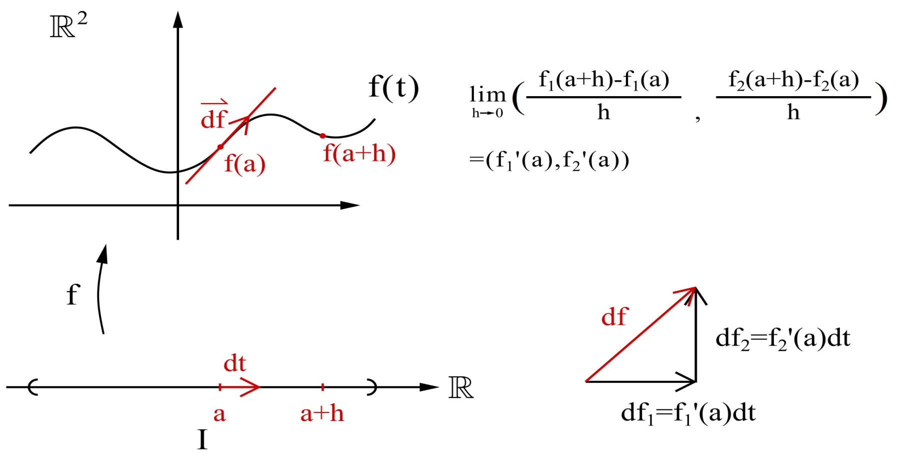
then we say that f is differentiable at a and denote the number L by $f'(a)$.

■ $\mathbf{f} : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$

Look at the case $\mathbf{f} : I \subseteq \mathbb{R} \to \mathbb{R}^n$, $\mathbf{f}(t) = \big(f_1(t), \cdots, f_n(t)\big)$ and $a \in I$. As we know, $\mathbf{f}'(a) = \big(f_1'(a), \cdots, f_n'(a)\big)$.

Consider the case $n = 2$.

$$\lim_{h\to 0}\Big(\frac{f_1(a+h) - f_1(a)}{h}, \frac{f_2(a+h) - f_2(a)}{h}\Big) = \big(f_1'(a), f_2'(a)\big).$$



$\mathbf{f}'(a)$ maps the vector $\overrightarrow{dt}$ to the vector $\big\langle f_1'(a)dt, f_2'(a)dt\big\rangle$.

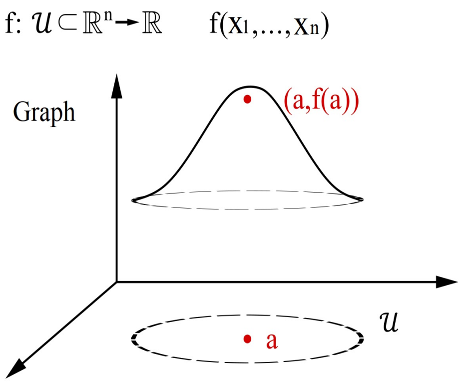
Remark. $\mathbf{f}'(a) = \big(f_1'(a), f_2'(a)\big)$ is not only a vector, but also a map from the vector space $\mathbb{R}$ to the vector space $\mathbb{R}^2$. The map

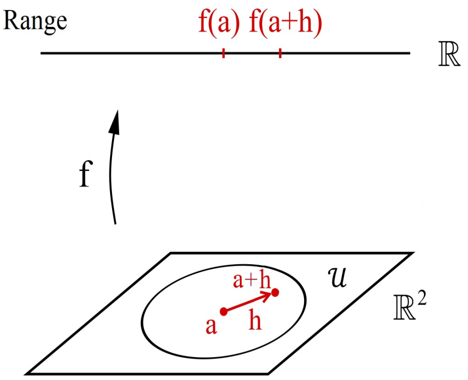
$$\mathbf{f}'(a) : \underbrace{\vec{v}}_{\in\,\mathbb{R}} \longrightarrow \underbrace{\langle f_1'(a)v, f_2'(a)v\rangle}_{\in\,\mathbb{R}^2}$$

Check that $\mathbf{f}'(a)$ is linear. Hence $\mathbf{f}'(a) \in \mathcal{L}\big(\mathbb{R}, \mathbb{R}^2\big)$.

■ $f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$**:** $f(x_1, \cdots, x_n)$

Consider the case $n = 2$. Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{e}_1 = \langle 1, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1\rangle$, $\mathbf{h} = \langle h_1, h_2\rangle = h_1\mathbf{e}_1 + h_2\mathbf{e}_2$.





We hope to find some quantity which can reflect the rate of change of the value with respect to the variables in every direction

(i) $\mathbf{h} = h_1\mathbf{e}_1$,

$$\lim_{h_1\to 0}\frac{f(\mathbf{a}+h_1\mathbf{e}_1)-f(\mathbf{a})}{h_1} = \frac{\partial f}{\partial x_1}(\mathbf{a}) \Rightarrow f(a_1+h_1,a_2)-f(a_1,a_2) \approx \frac{\partial f}{\partial x_1}(\mathbf{a})h_1.$$

(ii) $\mathbf{h} = h_2\mathbf{e}_2$,

$$\lim_{h_2\to 0}\frac{f(\mathbf{a}+h_2\mathbf{e}_2)-f(\mathbf{a})}{h_2} = \frac{\partial f}{\partial x_2}(\mathbf{a}) \Rightarrow f(a_1,a_2+h_2)-f(a_1,a_2) \approx \frac{\partial f}{\partial x_2}(\mathbf{a})h_2.$$

(iii) $\mathbf{h} = h_1\mathbf{e}_1 + h_2\mathbf{e}_2$,

$$\begin{aligned} df &= f(\mathbf{a}+\mathbf{h})-f(\mathbf{a}) = f(a_1+h_1,a_2+h_2) \\ &= f(a_1+h_1,a_2+h_2)-f(a_1,a_2+h_2)+f(a_1,a_2+h_2)-f(a_1,a_2) \\ &\approx \frac{\partial f}{\partial x_1}(\mathbf{a})h_1 + \frac{\partial f}{\partial x_2}(\mathbf{a})h_2 \\ &= \underbrace{\langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a})\rangle}_{=\nabla f(\mathbf{a})} \cdot \underbrace{\langle h_1, h_2\rangle}_{=\mathbf{h}} \end{aligned}$$

$\nabla f(\mathbf{a})$ maps the vector $\underbrace{\mathbf{h}}_{\in\mathbb{R}^2}$ to the vector $\underbrace{\nabla f(\mathbf{a})\cdot\mathbf{h}}_{\in\mathbb{R}}$

Remark. $\nabla f(\mathbf{a}) = \langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a})\rangle$ is not only a vector, but also a map from the vector space $\mathbb{R}^2$ into the vector space $\mathbb{R}$. The map

$$\nabla f(\mathbf{a}) : \underbrace{\mathbf{v}}_{\in\,\mathbb{R}^2} \longrightarrow \underbrace{\nabla f(\mathbf{a})\cdot\mathbf{v}}_{\in\,\mathbb{R}}$$

We hope to find some "object" which can reflect the rate of change of the value with respect to the variable in every direction (like the gradient of multi-variable real-valued function ∇f).

Definition 5.2.2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, $\mathcal{U} \subseteq X$ be open.

(1) A map $f : \mathcal{U} \to Y$ is said to be differentiable at $a \in \mathcal{U}$ if there exists a bounded linear map $L \in \mathcal{B}\big(X; Y\big)$ such that the limit

$$\lim_{\substack{x\to a\\ x\in\mathcal{U}}} \frac{\|f(x) - f(a) - L(x-a)\|_Y}{\|x-a\|_X} = 0$$

We denote this bounded linear map $Df(a)$ and call it the "*derivative of f at a*".

(2) If $f : \mathcal{U} \to Y$ is differentiable at every point in $\mathcal{U}$, we say that f is differentiable on $\mathcal{U}$. Hence, $Df : \mathcal{U} \to \mathcal{B}\big(X; Y\big)$ is a map from $\mathcal{U}$ into $\Big(\mathcal{B}\big(X; Y\big), \|\cdot\|_{\mathcal{B}(X;Y)}\Big)$.

Remark. If $f : \mathcal{U} \to Y$ is differentiable at $a \in \mathcal{U}$, then

(1)
$$\lim_{\substack{x\to a\\ x\in\mathcal{U}}} \frac{\|f(x) - f(a) - Df(a)(x-a)\|_Y}{\|x-a\|_X} = 0$$

or taking $\mathbf{h} = \mathbf{x} - \mathbf{a}$,

$$\lim_{\substack{\mathbf{h}\to\mathbf{0}\\ \mathbf{a}+\mathbf{h}\in\mathcal{U}}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})\mathbf{h}\|_Y}{\|\mathbf{h}\|_X} = 0$$

Note that "$Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$" or "$Df(\mathbf{a})\mathbf{h}$" is a linear operator $Df(\mathbf{a})$ applying on the vector $\mathbf{x} - \mathbf{a}$ or $\mathbf{h}$, but not the product of $Df(\mathbf{a})$ and $\mathbf{x} - \mathbf{a}$ (or $\mathbf{h}$).

(2) $Df(a) \in \mathcal{B}\big(X; Y\big)$ maps a vector $h \in X$ to $Df(a)h \in Y$.

(3) For given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in B(a, \delta) \cap \mathcal{U}$, then

$$\|f(x) - f(a) - Df(a)(x-a)\|_Y < \varepsilon\|x-a\|_X.$$

Definition 5.2.3. For $a \in \mathcal{U}$, if there exists a bounded linear map $T \in \mathcal{B}\big(X; \mathcal{B}(X; Y)\big)$ such that

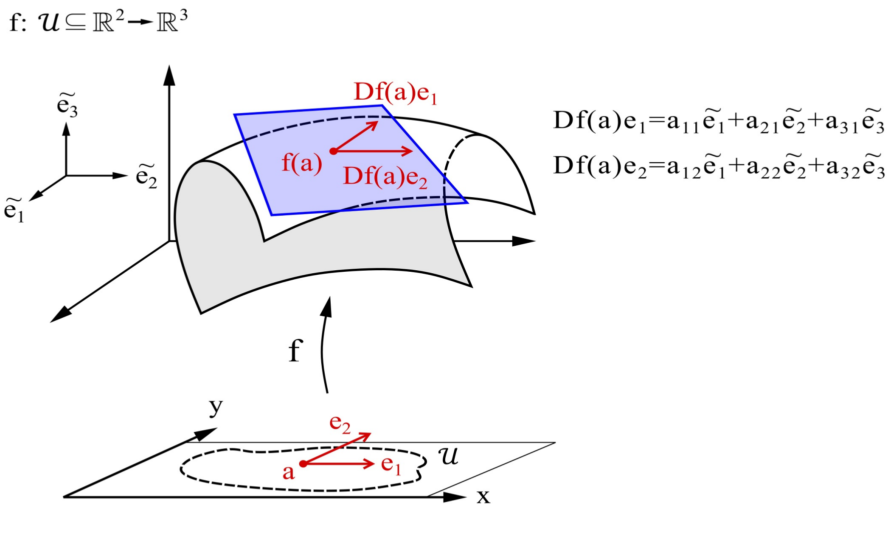
$$\lim_{\substack{x\to a\\ x\in\mathcal{U}}} \frac{\|Df(x) - Df(a) - T(x-a)\|_{\mathcal{B}(X;Y)}}{\|x-a\|_X}$$

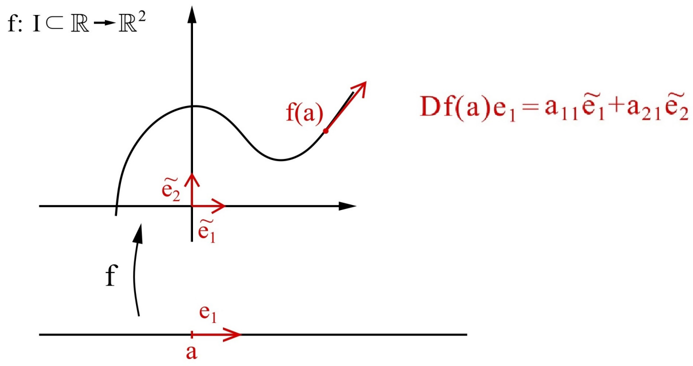
exists, we denote the linear map T by $D^2 f(a)$

Remark.

$$\begin{array}{lll} D^2 f(a) & \in \mathcal{B}\big(X; \mathcal{B}(X; Y)\big) & \\ D^2 f(a)(x) & \in \mathcal{B}\big(X; Y\big) & \text{for every } x \in X \\ D^2 f(a)(x)(z) & \in Y & \text{for every } z \in X. \end{array}$$

❑ Geometric Meaning



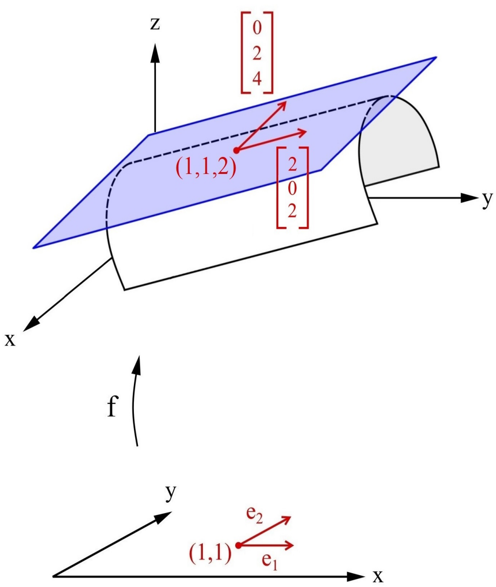


Let $f(x, y) = (x, y^2, x^2 + y^4) : \mathbb{R}^2 \to \mathbb{R}^3$.

$$\left[Df(1,1)\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}_{(x,y)=(1,1)}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2y \\ 2x & 4y^3 \end{bmatrix}_{(x,y)=(1,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 4 \end{bmatrix} = A.$$

$$Ae_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1\tilde{e}_1 + 0\tilde{e}_2 + 2\tilde{e}_3 \text{ and } Ae_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$



Example 5.2.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Then every $L \in \mathcal{B}\big(X;Y\big)$ is differentiable on X and $DL(a) = L$ for every $x \in X$ since

$$\lim_{x\to a} \frac{\|Lx - La - L(x-a)\|_Y}{\|x-a\|_X} = 0$$

Example 5.2.5. Define $L \in \mathcal{B}\big(\mathbb{R};\mathbb{R}\big)$ by $\underbrace{Lx = 2x}_{x\to 2x}$. Find $T \in \mathcal{B}\big(\mathbb{R};\mathbb{R}\big)$ such that

$$\lim_{x\to a} \frac{|2x - 2a - T(x-a)|}{|x-a|} = 0$$

For $T \in \mathcal{B}\big(\mathbb{R};\mathbb{R}\big)$, let $T(1) = c$. Then $Tx = cx$ for every $x \in \mathbb{R}$. Suppose that

$$\lim_{x\to a} \frac{|2x - 2a - c(x-a)|}{|x-a|} = 0$$

then $c = 2$ and we have $Tx = 2x$. Hence, for $f(x) = 2x$, $f'(x) = 2 \in \mathcal{B}\big(\mathbb{R};\mathbb{R}\big)$.

Example 5.2.6. Let $f(x) = x^3$.

$$\lim_{x\to a} \frac{|x^3 - a^3 - 3a^2(x-a)|}{|x-a|} = 0$$

Then $Df(a) \in \mathcal{B}\big(\mathbb{R};\mathbb{R}\big)$ defined by

$$Df(a)x = 3ax.$$

Example 5.2.7. Let $f(t) = (t, t^2)$. Find $L \in \mathcal{B}\big(\mathbb{R};\mathbb{R}^2\big)$ such that

$$\lim_{t\to t_0} \frac{\|(t,t^2) - (t_0, t_0^2) - L(t-t_0)\|_{\mathbb{R}^2}}{|t-t_0|} = 0$$

Define $[L(t_0)](s) = (s, 2t_0 s)$. Then

$$\lim_{t\to t_0} \frac{\|(t,t^2) - (t_0, t_0^2) - L(t-t_0)\|_{\mathbb{R}^2}}{|t-t_0|} = \lim_{t\to t_0} \frac{\|(t-t_0, t^2 - t_0^2) - (t-t_0, 2t_0(t-t_0))\|_{\mathbb{R}^2}}{|t-t_0|} = 0$$

Theorem 5.2.8. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed space, $\mathcal{U} \subseteq X$ be open and $f : \mathcal{U} \to Y$ be differentiable at $a \in \mathcal{U}$. Then $\big(Df\big)(a)$ is uniquely determined by f.*

Proof. Let $L_1, L_2 \in \mathcal{B}\big(X;Y\big)$ such that

$$\lim_{\substack{x\to a\\ x\in\mathcal{U}}} \frac{\|f(x) - f(a) - L_1(x-a)\|_Y}{\|x-a\|_X} = 0 = \lim_{\substack{x\to a\\ x\in\mathcal{U}}} \frac{\|f(x) - f(a) - L_2(x-a)\|_Y}{\|x-a\|_X} \tag{5.1}$$

It sufficies to show that for every $z \in X$ and $\|z\|_X = 1$, $L_1 z = L_2 z$.

By (5.1), given $\varepsilon > 0$ choose $\delta > 0$ such that $B(a,\delta) \subseteq \mathcal{U}$ and if $x \in B(a,\delta)$ then

$$\|f(x) - f(a) - L_1(x-a)\|_Y < \frac{\varepsilon}{2}\|x-a\|_X$$

and

$$\|f(x) - f(a) - L_2(x-a)\|_Y < \frac{\varepsilon}{2}\|x-a\|_X.$$

Fix $z \in X$ and $\|z\|_X = 1$, choose $0 < r < \delta$. Let $x = a + rz$. Then

$$\begin{aligned} r\|L_1z - L_2z\|_Y &= \|L_1(rz) - L_2(rz)\|_Y \\ &\le \|f(a+rz) - f(a) - L_1(rz)\|_Y + \|f(a+rz) - f(a) - L_2(rz)\|_Y \\ &< \frac{\varepsilon}{2}\|rz\|_X + \frac{\varepsilon}{2}\|rz\|_X \\ &= \varepsilon r\|z\|_X \\ &= \varepsilon r. \end{aligned}$$

Hence, $\|L_1z - L_2z\|_Y < \varepsilon$. Since ε is arbitrary, $L_1z = L_2z$. □

From now on, we will consider the function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and we assume $\mathbb{R}^n$ (or $\mathbb{R}^m$) is a vector space with the standard basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ (or $\{\tilde{\mathbf{e}}_1, \cdots, \tilde{\mathbf{e}}_m\}$).

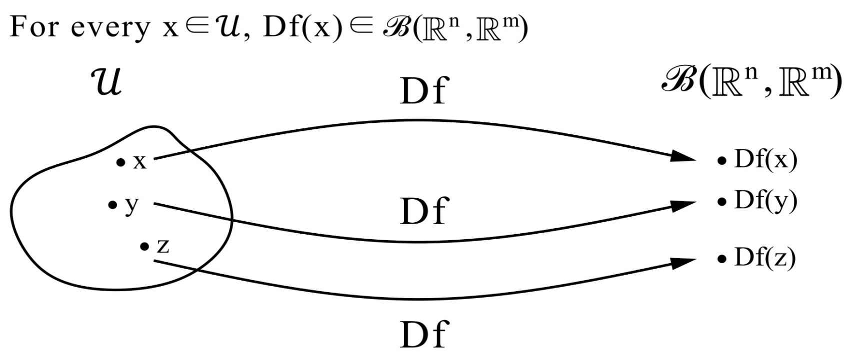
Remark. (1) $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a} \in \mathcal{U}$. Rewrite $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = Df(a)\mathbf{h} + r(\mathbf{h})$. Then

$$\lim_{\|\mathbf{h}\|_{\mathbb{R}^n} \to 0} \frac{\|r(\mathbf{h})\|_{\mathbb{R}^m}}{\|\mathbf{h}\|_{\mathbb{R}^n}} = 0$$

This represents that $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \approx Df(\mathbf{a})\mathbf{h}$ (or $f(\mathbf{a}+\mathbf{h}) \approx f(\mathbf{a}) + D(\mathbf{a})\mathbf{h}$) as $\|\mathbf{h}\|_{\mathbb{R}^n}$ is sufficiently small. This suggests that if f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

(2) The derivative of f at $\mathbf{a}$, $Df(\mathbf{a})$, is also called "*the total derivative of f at* **a**", to distinguish it from the partial derivative.

(3) For every $\mathbf{x} \in \mathcal{U}$, $Df(\mathbf{x}) \in \mathcal{B}\big(\mathbb{R}^n, \mathbb{R}^m\big)$ and $Df : \mathcal{U} \to \mathcal{B}\big(\mathbb{R}^n, \mathbb{R}^m\big)$ is a map from $\mathcal{U}$ to $\mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^m\big)$.



Definition 5.2.9. (1) Let $\mathcal{U} \subseteq \mathbb{R}^n$, $f : \mathcal{U} \to \mathbb{R}$ and $\mathbf{a} \in \mathcal{U}$. If the limit

$$\lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_j) - f(\mathbf{a})}{h}$$

exists, we call the limit " the partial derivative of f at $\mathbf{a}$ in the direction $\mathbf{e}_j$ and denote the limit $\dfrac{\partial f}{\partial x_j}(\mathbf{a})$.

(2) Let $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ and $\mathbf{a} \in \mathcal{U}$. Then $\mathbf{f} = (f_1, f_2, \cdots, f_m) = \sum_{i=1}^{m} f_i(\mathbf{x})\tilde{\mathbf{e}}_i$ where $\{\tilde{\mathbf{e}}_1, \cdots, \tilde{\mathbf{e}}_m\}$ is the standard basis of $\mathbb{R}^m$. We obatin $f_i(\mathbf{x}) = \mathbf{f} \cdot \tilde{\mathbf{e}}_i$. Then

$$\frac{\partial f_i}{\partial x_j}(\mathbf{a}) = \lim_{h\to 0} \frac{f_i(\mathbf{a} + h\mathbf{e}_j) - f_i(\mathbf{a})}{h}$$

for $1 \le i \le m$ and $1 \le j \le n$ provided the limit exists.

Remark. We want to determine whether a function f is differentiable at a point. For a single variable function, the existence of derivative is sufficient. But for several variables functions, the continuity or at least boundedness of the partial derivatives is needed.

Let $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in \mathcal{U}$, $\mathbf{f} = (f_1, \cdots, f_m)$.

To guess what the form of $D\mathbf{f}(\mathbf{a})$.

Since $D\mathbf{f}(\mathbf{a}) \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^m)$, there exists $A \in M_{m\times n}(\mathbb{R})$ such that

$$D\mathbf{f}(\mathbf{a})\mathbf{x} = A\mathbf{x} \quad \text{for every } \mathbf{x} \in \mathbb{R}^n.$$

We can wrtie

$$A = \begin{bmatrix} D\mathbf{f}(\mathbf{a})\mathbf{e}_1 & D\mathbf{f}(\mathbf{a})\mathbf{e}_2 & \cdots & D\mathbf{f}(\mathbf{a})\mathbf{e}_n \end{bmatrix}$$

Find $D\mathbf{f}(\mathbf{a})\mathbf{e}_j = \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m$. By the definition of $D\mathbf{f}(\mathbf{a})$,

$$0 = \lim_{h\to 0} \frac{\|\mathbf{f}(a + h\mathbf{e}_j) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(h\mathbf{e}_j)\|_{\mathbb{R}^m}}{\|h\mathbf{e}_j\|_{\mathbb{R}^n}} = \lim_{h\to 0} \frac{\left\| \begin{bmatrix} f_1(\mathbf{a} + h\mathbf{e}_j) \\ \vdots \\ f_m(\mathbf{a} + h\mathbf{e}_j) \end{bmatrix} - \begin{bmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_m(\mathbf{a}) \end{bmatrix} - h \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \right\|_{\mathbb{R}^m}}{|h|}.$$

Hence,

$$\lim_{h\to 0} \frac{|f_i(\mathbf{a} + h\mathbf{e}_j) - f_i(\mathbf{a}) - hv_i|}{|h|} = 0.$$

We have

$$v_i = \frac{\partial f_i}{\partial x_j}(\mathbf{a}) \quad \text{and then } D\mathbf{f}(\mathbf{a})\mathbf{e}_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_j}(\mathbf{a}) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(\mathbf{a}) \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} D\mathbf{f}(\mathbf{a}) \end{bmatrix}$$

or

$$\big(D\mathbf{f}(\mathbf{a})\big)_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{a}) \quad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

Hence, for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

$$\big(D\mathbf{f}(\mathbf{a})\big)\mathbf{x} = \Big[\big(D\mathbf{f}(\mathbf{a})\big)_{ij}\Big] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem 5.2.10. *Suppose* $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* $\mathbf{a} \in \mathcal{U}$*. Then the partial derivative* $\frac{\partial f_i}{\partial x_j}(\mathbf{a})$ *exist for* $i = 1, \cdots, m,\ j = 1, \cdots, n$ *and*

$$D\mathbf{f}(\mathbf{a})\mathbf{e}_j = \sum_{i=1}^{m} \Big(\frac{\partial f_i}{\partial x_j}\Big)(\mathbf{a})\tilde{\mathbf{e}}_{\mathbf{i}} \qquad \textit{for } 1 \leq j \leq n.$$

Proof. Fix j, since $\mathbf{f}$ is differentiable at $\mathbf{a}$,

$$\begin{aligned} 0 &= \lim_{h\to 0} \frac{\|\mathbf{f}(\mathbf{a}+h\mathbf{e}_j) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(h\mathbf{e}_j)\|_{\mathbb{R}^m}}{\|h\mathbf{e}_j\|_{\mathbb{R}^n}} = \lim_{h\to 0} \frac{\|\mathbf{f}(\mathbf{a}+h\mathbf{e}_j) - \mathbf{f}(\mathbf{a}) - hD\mathbf{f}(\mathbf{a})(\mathbf{e}_j)\|_{\mathbb{R}^m}}{|h|} \\ &\left(= \lim_{h\to 0} \frac{\left\| \begin{bmatrix} f_1(\mathbf{a}+h\mathbf{e}_j) \\ \vdots \\ f_m(\mathbf{a}+h\mathbf{e}_j) \end{bmatrix} - \begin{bmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_m(\mathbf{a}) \end{bmatrix} - h \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \right\|_{\mathbb{R}^m}}{|h|} \right) \end{aligned}$$

For each component of $\mathbf{f}$,

$$\lim_{h\to 0} \frac{|f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a}) - hDf_i(\mathbf{a})\mathbf{e}_j|}{|h|} = 0.$$

By the definition of partial derivative,

$$Df_i(\mathbf{a})\mathbf{e}_j = \frac{\partial f_i}{\partial x_j}(\mathbf{a}).$$

That is,

$$\begin{aligned} D\mathbf{f}(\mathbf{a})\mathbf{e}_j = \begin{bmatrix} Df_1(\mathbf{a})\mathbf{e}_j \\ \vdots \\ Df_m(\mathbf{a})\mathbf{e}_j \end{bmatrix} &= Df_1(\mathbf{a})\mathbf{e}_j \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + Df_m(\mathbf{a})\mathbf{e}_j \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\mathbf{a})\tilde{\mathbf{e}}_{\mathbf{i}} \end{aligned}$$

□

Definition 5.2.11. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$. The matrix

$$J\mathbf{f}(\mathbf{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$$

is called the "*Jacobian matrix of* **f** *at* **x**".

Remark. The Jacobian matrix of a function **f** might exist even if **f** is not differentiable.

If **f** is differentiable at **x**, then the Jacobian matrix must exist and

$$\left[D\mathbf{f}(\mathbf{a})\right] = J\mathbf{f}(\mathbf{a})$$

$$D\mathbf{f}(\mathbf{x}) \text{ exists} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad J\mathbf{f}(\mathbf{x}) \text{ exists and } \left[D\mathbf{f}(\mathbf{x})\right] = J\mathbf{f}(\mathbf{x})$$

> Importance: In the future, we will prove some functions are differentiable at **a**. For example, product rule, quotient rule, chain rule. We have to guess a linear map first. The first and the only guess must be $[J\mathbf{f}(\mathbf{a})]$.

For example, $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$. Then $\dfrac{\partial f}{\partial x}(0, 0) = 0 = \dfrac{\partial f}{\partial y}(0, 0)$. But f is not continuous at $(0, 0)$ and hence f is not differentiable at $(0, 0)$.

Assume that f is differentiable at $(0, 0)$. Then $Df(0, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$. But

$$\frac{\left| f(x, y) - f(0, 0) - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right|}{\left| \begin{bmatrix} x \\ y \end{bmatrix} \right|} = \frac{|xy|}{(x^2 + y^2)^{1/2}} \nrightarrow 0 \quad \text{along the direction } x = y.$$

❑ Compute $J\mathbf{f}(\mathbf{x})$ and $D\mathbf{f}(\mathbf{a})$

Example 5.2.12. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(x, y) = (\underbrace{x^2}_{f_1}, \underbrace{y^2}_{f_2}, \underbrace{x^4 y^2}_{f_3})$.

$$Jf(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 0 & 2y \\ 4x^3 y^2 & 2x^4 y \end{bmatrix}.$$

Suppose that f is differentiable at (x, y), then

$$\left[Df(x, y)\right] = Jf(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \\ 4x^3 y^2 & 2x^4 y \end{bmatrix}.$$

Check that for $\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2$.

$$\frac{\|f(x+h_1, y+h_2) - f(x,y) - \big[Jf(x,y)\big]\,\mathbf{h}\|_{\mathbb{R}^3}}{\|\mathbf{h}\|_{\mathbb{R}^2}}$$
$$= \frac{\left\| \Big((x+h_1)^2, (y+h_2)^2, (x+h_1)^4(y+h_2)^2\Big) - (x^2, y^2, x^4y^2) - \begin{bmatrix} 2x & 0 \\ 0 & 2y \\ 4x^3y^2 & 2x^4y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|_{\mathbb{R}^3}}{\|(h_1, h_2)\|_{\mathbb{R}^2}}$$
$$\to \quad 0 \quad \text{as } (h_1, h_2) \to (0,0).$$

Hence, f is differentiable at (x, y).

Definition 5.2.13. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\mathbf{f} : \mathcal{U} \to \mathbb{R}$ and $\mathbf{a} \in \mathcal{U}$.

$$\big[D\mathbf{f}(\mathbf{a})\big] = \Big[\tfrac{\partial f}{\partial x_1}(\mathbf{a}) \quad \tfrac{\partial f}{\partial x_2}(\mathbf{a}) \quad \cdots \quad \tfrac{\partial f}{\partial x_n}(\mathbf{a})\Big] =: \nabla\mathbf{f}(\mathbf{a})$$

The derivative of $\mathbf{f}$ at $\mathbf{a}$ is called the "*gradient of* $\mathbf{f}$ *at* $\mathbf{a}$".

5.3 Continuity of Differentiable Maps

Theorem 5.3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, $\mathcal{U} \subseteq X$ be open, and $f : \mathcal{U} \to Y$ be differentiable at $a \in \mathcal{U}$. Then f is continuous at a.*

Proof. Since f is differentiable at a, there exists $L \in \mathcal{B}\big(X; Y\big)$ such that

$$\frac{\|f(x) - f(a) - L(x-a)\|_Y}{\|x-a\|_X} \to 0 \quad \text{as } \|x-a\|_X \to 0.$$

Then for $0 < \varepsilon < 1$, there exists $\delta_1 > 0$,

$$\|f(x) - f(a) - L(x-a)\|_Y < \varepsilon\|x-a\|_X$$

whenever $\|x-a\|_X < \delta_1$. Choose $0 < \delta < \min(\delta_1, \dfrac{\varepsilon}{\|L\|_{\mathcal{B}(X;Y)} + 1})$. If $\|x-a\|_X < \delta$,

$$\|f(x) - f(a)\|_Y \le \|L\|_{\mathcal{B}(X;Y)}\|x-a\|_X + \varepsilon\|x-a\|_X = \Big(\|L\|_{\mathcal{B}(X;Y)} + \varepsilon\Big)\|x-a\|_X < \varepsilon.$$

Hence, f is continuous at a. □

Remark.

$$f \text{ is differentiable at } a \quad \Longrightarrow \quad f \text{ is continuous at } a.$$
$$\not\Longleftarrow$$

Example 5.3.2. Let $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Then f is continuous at $(0,0)$ (Check).

$f_x(0,0) = \frac{\partial f}{\partial x}(0,0) = 1$ and $f_y(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Assume that f is differentiable at $(0,0)$, then $\big[Df(0,0)\big] = \big[1 \quad 0\big]$. But

$$\frac{\left| f(x,y) - f(0,0) - \big[1 \quad 0\big] \begin{bmatrix} x \\ y \end{bmatrix} \right|}{\|(x,y)\|_{\mathbb{R}^2}} = \frac{|x|y^2}{(x^2+y^2)^{3/2}} \nrightarrow 0 \quad \text{as} \quad (x,y) \to (0,0) \text{ along } x = y.$$

Hence, f is not differentiable at $(0,0)$.

5.4 Conditions for Differentiability

Proposition 5.4.1. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $\mathbf{f} = (f_1, \cdots, f_m) : \mathcal{U} \to \mathbb{R}^m$. Then $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if f_i is differentiable at $\mathbf{a}$ for $i = 1, \cdots, m$.*

Proof. ($\Longrightarrow$)
Since $\mathbf{f}$ is differentiable at $\mathbf{a}$, $\left[D\mathbf{f}(\mathbf{a})\right] = J\mathbf{f}(\mathbf{a})$ and for $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta$,

$$\|\mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{a}) - \big(D(\mathbf{a})\big)(\mathbf{x} - \mathbf{a})\|_{\mathbb{R}^m} < \varepsilon\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n}.$$

Let $\{\mathbf{e}_i\}_{i=1}^m$ be the standard basis of $\mathbb{R}^m$. Define $L_i \in \mathcal{L}\big(\mathbb{R}^n;\mathbb{R}\big)$ such that for $\mathbf{h} \in \mathbb{R}^n$,

$$L_i(\mathbf{h}) = \mathbf{e}_i^T \left[D\mathbf{f}(\mathbf{a})\right] \mathbf{h}.$$

$$\left[D\mathbf{f}(\mathbf{a})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & & & & \vdots \\ \frac{\partial f_i}{\partial x_1}(\mathbf{a}) & \cdots & \cdots & \cdots & \frac{\partial f_i}{\partial x_n}(\mathbf{a}) \\ \vdots & & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} Df_1(\mathbf{a}) \\ \vdots \\ Df_i(\mathbf{a}) \\ \vdots \\ Df_m(\mathbf{a}) \end{bmatrix} = \left[L\mathbf{e}_1 \quad \cdots \quad L\mathbf{e}_n\right]$$

Then $L_i \in \mathcal{B}\big(\mathbb{R}^n;\mathbb{R}\big)$ and for $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta$,

$$\left|f_i(\mathbf{x}) - f_i(\mathbf{a}) - L_i(\mathbf{x} - \mathbf{a})\right| \leq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})\|_{\mathbb{R}^m} < \varepsilon\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n}.$$

Thus, f_i is differentiable at $\mathbf{a}$ and $Df_i(\mathbf{a}) = L_i$.

($\Longleftarrow$)

Since f_i is differentiable at $\mathbf{a}$ for $i = 1, \cdots, n$, there exist $L_1, \cdots, L_m \in \mathcal{B}\big(\mathbb{R}^m;\mathbb{R}\big)$ and for $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta$,

$$\left|f_i(\mathbf{x}) - f_i(\mathbf{a}) - L_i(\mathbf{x} - \mathbf{a})\right| < \frac{\varepsilon}{m}\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} \quad \text{for } i = 1, \cdots, m.$$

Define $L \in \mathcal{L}\big(\mathbb{R}^n;\mathbb{R}^m\big)$ by $L\mathbf{x} = (L_1\mathbf{x}, \cdots, L_m\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$. Then $L \in \mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^m\big)$ and if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta$,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - L(\mathbf{x} - \mathbf{a})\|_{\mathbb{R}^m} \leq \sum_{i=1}^m \left|f_i(\mathbf{x}) - f_i(\mathbf{a}) - L_i(\mathbf{x} - \mathbf{a})\right| < \varepsilon\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n}.$$

Hence, $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $D\mathbf{f}(\mathbf{a}) = L$. □

Remark. (1) For a vector-valued function defined on an open subset of $\mathbb{R}^n$.

$$\text{Componentwise differentiable} \Longleftrightarrow \text{Differentiable}$$

(2)

$$\begin{aligned} J\mathbf{f}(\mathbf{a}) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} Df_1(\mathbf{a}) \\ \vdots \\ Df_m(\mathbf{a}) \end{bmatrix} \quad \text{(if } \mathbf{f} \text{ is differentiable, then } Df_i \text{ exists)} \\ &= \begin{bmatrix} L\mathbf{e}_1 & \cdots & L\mathbf{e}_n \end{bmatrix} \quad \text{(if } \mathbf{f} \text{ if differentiable, then } L \text{ exists.)} \end{aligned}$$

The proposition does not mean that if $\dfrac{\partial f_i}{\partial x_j}$ exists at $\mathbf{a}$ for every i, j, then $\mathbf{f}$ is differentiable at $\mathbf{a}$ since $\dfrac{\partial f_i}{\partial x_j}$ exists for all $1 \le j \le m$ does not imply f_i is differentiable at $\mathbf{a}$.

The proposition means that $f_1, \cdots, f_m$ are differentiable at $\mathbf{a}$ if and only if $\mathbf{f}$ is differentiable at $\mathbf{a}$. Hence $Df_1, \cdots, Df_m$ exist at $\mathbf{a}$ if and only if $\mathbf{f}$ is differentiable at $\mathbf{a}$ and

$$\begin{bmatrix} D\mathbf{f}(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} Df_1(\mathbf{a}) \\ \vdots \\ Df_m(\mathbf{a}) \end{bmatrix} \quad \Rightarrow \quad \frac{\partial f_i}{\partial x_j}(\mathbf{a}) \text{ exists for every } 1 \le i \le m,\ 1 \le j \le n$$

$\not\Leftarrow$

Question: In what conditions on $\dfrac{\partial f_i}{\partial x_j}(\mathbf{a})$ (or $J\mathbf{f}(\mathbf{a})$), we can say $\mathbf{f}$ is differentiable at $\mathbf{a}$?

Theorem 5.4.2. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$. If*

(1) $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ exist in a neighborhood of $\mathbf{a}$, and

(2) $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ are continuous at $\mathbf{a}$ (except possibly one of them). That is, at most one of $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ is discontinuous at $\mathbf{a}$.

then f is differentiable at $\mathbf{a}$.

Proof. W.L.O.G, we may assume $n = 2$, $\dfrac{\partial f}{\partial x_1}$ is continuous at $\mathbf{a}$ (and $\dfrac{\partial f}{\partial x_2}$ may or may not be continuous at $\mathbf{a}$).

Since $\dfrac{\partial f}{\partial x_1}$ is continuous at $\mathbf{a}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^2} < \delta$,

$$\left| \frac{\partial f}{\partial x_1}(\mathbf{x}) - \frac{\partial f}{\partial x_1}(\mathbf{a}) \right| < \frac{\varepsilon}{2} \tag{5.2}$$

Since $\dfrac{\partial f}{\partial x_1}(\mathbf{a})$ and $\dfrac{\partial f}{\partial x_2}(\mathbf{a})$ exist, there are $\delta_1, \delta_2 > 0$ such that if $|h| < \delta_1$,

$$\left| f(\mathbf{a} + h\mathbf{e}_1) - f(\mathbf{a}) - \frac{\partial f}{\partial x_1}(\mathbf{a})h \right| < \frac{\varepsilon}{2}|h|$$

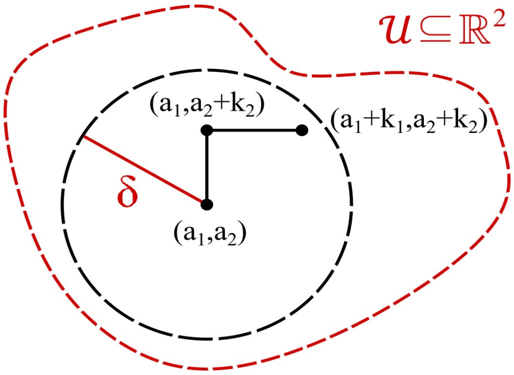
and if $|h| < \delta_2$,

$$\left| f(\mathbf{a} + h\mathbf{e}_2) - f(\mathbf{a}) - \frac{\partial f}{\partial x_2}(\mathbf{a})h \right| < \frac{\varepsilon}{2}|h|. \tag{5.3}$$

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{k} = \mathbf{x} - \mathbf{a} = (x_1 - a_1, x_2 - a_2) = (k_1, k_2)$. Consider

$$\begin{aligned} &\left| f(\mathbf{x}) - f(\mathbf{a}) - \left[\frac{\partial f}{\partial x_1}(\mathbf{a}) \underbrace{(x_1 - a_1)}_{k_1} + \frac{\partial f}{\partial x_2}(\mathbf{a}) \underbrace{(x_2 - a_2)}_{k_2} \right] \right| \\ = \quad & \left| \left[f(a_1 + k_1, a_2 + k_2) - f(a_1, a_2 + k_2) - \frac{\partial f}{\partial x_1}(\mathbf{a})k_1 \right] + \left[f(a_1, a_2 + k_2) - f(a_1, a_2) - \frac{\partial f}{\partial x_2}(\mathbf{a})k_2 \right] \right| \\ \overset{\text{M.V.T}}{=} \quad & \left| \left[\frac{\partial f}{\partial x_1}(a_1 + \theta_1, a_2 + k_2)k_1 - \frac{\partial f}{\partial x_1}(\mathbf{a})k_1 \right] + \left[f(a_1, a_2 + k_2) - f(a_1, a_2) - \frac{\partial f}{\partial x_2}(\mathbf{a})k_2 \right] \right| \end{aligned} \tag{5.4}$$

for some $\theta_1 \in (0, k_1)$.



Choose $\|\mathbf{k}\|_{\mathbb{R}^2} < \min(\delta, \delta_1, \delta_2)$. Then

$$(5.4) \underset{(5.2),(5.3)}{\leq} \frac{\varepsilon}{2}|k_1| + \frac{\varepsilon}{2}|k_2| < \varepsilon\|\mathbf{k}\|_{\mathbb{R}^2} = \varepsilon\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^2}.$$

Hence f is differentiable at $\mathbf{a}$. □

Remark. If two or more of $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ are discontinuous at $\mathbf{a}$, then f could be not differentiable. For example, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$. The function f is not differentiable at $(0, 0)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Definition 5.4.3. (1) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ be differentiable on $\mathcal{U}$. We say that $\mathbf{f}$ is continuously differentiable on $\mathcal{U}$ if $D\mathbf{f} : \mathcal{U} \to \mathcal{B}(\mathbb{R}^n; \mathbb{R}^m)$ is continuous on $\mathcal{U}$.

(2) The collection of all continuously differentiable functions from $\mathcal{U}$ to $\mathbb{R}^m$ is denoted by

$$C^1(\mathcal{U}; \mathbb{R}^m) = \{\mathbf{f} : \mathcal{U} \to \mathbb{R}^m \text{ is differentiable} \mid D\mathbf{f} : \mathcal{U} \to \mathcal{B}(\mathbb{R}^n; \mathbb{R}^m) \text{ is continuous.}\}$$

(3) The collection of all bounded differentiable functions from $\mathcal{U}$ to $\mathbb{R}^m$ is denoted by

$$C_b^1\big(\mathcal{U};\mathbb{R}^m\big) = \Big\{\mathbf{f} : \mathcal{U} \to \mathbb{R}^m \text{ is differentiable.} \;\Big|\; \sup_{\mathbf{x}\in\mathcal{U}} \|\mathbf{f}(\mathbf{x})\|_{\mathbb{R}^m} + \sup_{\mathbf{x}\in\mathcal{U}} \|D\mathbf{f}(\mathbf{x})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)} < \infty\Big\}.$$

Example 5.4.4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Then

$$C_b\big(I;\mathbb{R}\big) = \Big\{f : I \to \mathbb{R} \text{ is differentiable.} \;\Big|\; \sup_{x\in I} |f(x)| + \sup_{x\in I} |f'(x)| < \infty\Big\}.$$

Corollary 5.4.5. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$. Then $\mathbf{f} \in C^1\big(\mathcal{U};\mathbb{R}^m\big)$ if and only if all $\dfrac{\partial f_i}{\partial x_j}$ exist and are continuous for $1 \le i \le m$ and $1 \le j \le n$.*

Proof. For a matrix $A = \big[a_{ij}\big] \in M_{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$,

$$\|A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)}\|\mathbf{x}\|_{\mathbb{R}^n} \le \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|\Big)\|\mathbf{x}\|_{\mathbb{R}^n} \le mn\|A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)}\|\mathbf{x}\|_{\mathbb{R}^n}$$

($\Longrightarrow$)

Since $\mathbf{f}$ is differentiable on $\mathcal{U}$, $\dfrac{\partial f_i}{\partial x_j}$ exist for $i = 1, \cdots, m$ and $j = 1, \cdots, n$ (by Proposition 5.4.1). Since $D\mathbf{f}$ is continuous on $\mathcal{U}$, for $\mathbf{a} \in \mathcal{U}$ and for given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta$,

$$\|D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)} < \varepsilon.$$

Then, for $i = 1, \cdots, n$ and $j = 1, \cdots, m$,

$$\Big|\frac{\partial f_i}{\partial x_j}(\mathbf{x}) - \frac{\partial f_i}{\partial x_j}(\mathbf{a})\Big| \le \|D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)} < \varepsilon.$$

Therefore, $\dfrac{\partial f_i}{\partial x_j}$ is continuous at $\mathbf{a}$. Since $\mathbf{a}$ is arbitrary in $\mathcal{U}$, $\dfrac{\partial f_i}{\partial x_j}$ is continuous on $\mathcal{U}$.

($\Longleftarrow$)

Since all partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist and are continuous on $\mathcal{U}$, by Theorem 5.4.2, $\mathbf{f}$ is differentiable on $\mathbf{U}$. Since $\dfrac{\partial f_i}{\partial x_j}$ is continuous at $\mathbf{a} \in \mathcal{U}$, for $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta_1$,

$$\Big|\frac{\partial f_i}{\partial x_j}(\mathbf{x}) - \frac{\partial f_i}{\partial x_j}(\mathbf{a})\Big| < \frac{\varepsilon}{mn}$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Hence,

$$\|D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)} \le \sum_{i=1}^{n}\sum_{j=1}^{m} \Big|\frac{\partial f_i}{\partial x_j}(\mathbf{x}) - \frac{\partial f_i}{\partial x_j}(\mathbf{a})\Big| < \varepsilon.$$

and we have $D\mathbf{f}$ is continuous at $\mathbf{a}$.

Since $\mathbf{a}$ is arbitrary in $\mathcal{U}$, $D\mathbf{f}$ is continuous on $\mathcal{U}$. □

Example 5.4.6. Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ then $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.
Therefore, f is differentiable, but f' is not continuous at 0.

Definition 5.4.7. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open. We define a norm on $C_b^1\big(\mathcal{U};\mathbb{R}\big)$ by

$$\|\mathbf{f}\|_{C_b^1(\mathcal{U};\mathbb{R}^m)} := \sup_{\mathbf{x}\in\mathcal{U}} \Big[\|\mathbf{f}(\mathbf{x})\|_{\mathbb{R}^m} + \sum_{i=1}^{m}\sum_{j=1}^{n} \Big|\frac{\partial f_i}{\partial x_j}(\mathbf{x})\Big|\Big].$$

Proposition 5.4.8. $\Big(C_b^1\big(\mathcal{U};\mathbb{R}^m\big); \|\cdot\|_{C_b^1(\mathcal{U};\mathbb{R}^m)}\Big)$ *is a Banach space.*

Definition 5.4.9. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$. The derivative of f is called "*the gradient of f*" and denoted by "*grad f*" or "∇f". That is, $Df = \nabla f$ and $Df(\mathbf{a}) = \nabla f(\mathbf{a})$.

Definition 5.4.10. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open , $\mathbf{a} \in \mathcal{U}$, $f : \mathcal{U} \to \mathbb{R}$. Let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. Then

$$\big(D_{\mathbf{v}}f\big)(\mathbf{a}) := \frac{d}{dt}\Big|_{t=0} f(\mathbf{a}+t\mathbf{v}) = \lim_{t\to 0}\frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t}$$

is called "*the directional derivative of f at* **a** *in the direction* **v**".

Remark. Let $\mathbf{e}_j = \langle 0, \cdots, \underset{ith}{1}, \cdots, 0\rangle$. Then $\frac{\partial f}{\partial x_j}(\mathbf{a}) = D_{\mathbf{e_j}}f(\mathbf{a})$ is the directional derivative of f at **a** in the direction $\mathbf{e}_j$.

5.5 The Product Rules and Chain Rule

❑ Proerties of Differentiation

Theorem 5.5.1. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, $\mathbf{a} \in \mathcal{U}$, $\mathbf{f}, \mathbf{g} : \mathcal{U} \to \mathbb{R}^n$ be differentiable at* **a***, $h : \mathcal{U} \to \mathbb{R}$ be differentiable at* **a***, $\alpha \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ be a vector. Then*

(1) $\mathbf{f} \pm \mathbf{g}$ *is differentiable at* **a** *and*

$$D(\mathbf{f} \pm \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) \pm D\mathbf{g}(\mathbf{a}).$$

(2) $\alpha\mathbf{f}$ *is differentiable at* **a** *and*

$$D(\alpha\mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}).$$

(3) $h\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ *is differentiable at* **a** *and*

$$\underbrace{\overbrace{D(h\mathbf{f})(\mathbf{a})}^{\in\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)}\mathbf{v}}_{\in\mathbb{R}^m} = \underbrace{h(\mathbf{a})}_{\in\mathbb{R}}\underbrace{\overbrace{D\mathbf{f}(\mathbf{a})}^{\in\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)}\mathbf{v}}_{\in\mathbb{R}^m} + \underbrace{\overbrace{(Dh)(\mathbf{a})}^{\in\mathscr{B}(\mathbb{R}^n;\mathbb{R})}\mathbf{v}}_{\in\mathbb{R}}\underbrace{\mathbf{f}(\mathbf{a})}_{\in\mathbb{R}^m}.$$

(4) If $h(\mathbf{a}) \neq 0$, then $\frac{\mathbf{f}}{h} : \mathcal{U} \to \mathbb{R}^m$ is differentiable at **a** *and*

$$D\Big(\frac{\mathbf{f}}{h}\Big)(\mathbf{a})\mathbf{v} = \frac{h(\mathbf{a})\big(D\mathbf{f}(\mathbf{a})\mathbf{v}\big) - \big(Dh\big)(\mathbf{a})\mathbf{v}\mathbf{f}(\mathbf{a})}{h^2(\mathbf{a})}$$

Proof. We only prove (3) here. Let $\mathbf{f} = (f_1, \cdots, f_n)$. Consider the Jacobian matrix of $A = \left[J(h\mathbf{f})\right](\mathbf{a})$,

$$A_{ij} = \frac{\partial[(h\mathbf{f})_i]}{\partial x_j}(\mathbf{a}) = \frac{\partial(hf_i)}{\partial x_j}(\mathbf{a}) = h(\mathbf{a})\frac{\partial f_i}{\partial x_j}(\mathbf{a}) + \frac{\partial h}{\partial x_j}(\mathbf{a})f_i(\mathbf{a}).$$

For $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = h(\mathbf{a})D\mathbf{f}(\mathbf{a})\mathbf{v} + Dh(\mathbf{a})\mathbf{v}\mathbf{f}(\mathbf{a})$. Consider

$$\begin{aligned}(h\mathbf{f})(\mathbf{x}) - (h\mathbf{f})(\mathbf{a}) - A(\mathbf{x}-\mathbf{a}) &= \underbrace{h(\mathbf{a})\left[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})\right]}_{(I)} \\ &+ \underbrace{\left[h(\mathbf{x}) - h(\mathbf{a}) - Dh(\mathbf{a})(\mathbf{x}-\mathbf{a})\right]\mathbf{f}(\mathbf{x})}_{(II)} + \underbrace{\left[Dh(\mathbf{a})(\mathbf{x}-\mathbf{a})\right]\left[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\right]}_{(III)}\end{aligned}$$

(i) Since $\mathbf{f}$ is differentiable at $\mathbf{a}$,

$$0 \le \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|(I)\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \le |h(\mathbf{a})| \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} = 0$$

(ii) Since $\mathbf{f}$ is differentiable at $\mathbf{a}$, $\mathbf{f}$ is continuous at $\mathbf{a}$. Then there exists $K > 0$ such that $\|\mathbf{f}(\mathbf{x})\|_{\mathbb{R}^m} \le K$ as $\mathbf{x}$ is near $\mathbf{a}$.

Since h is differentiable at $\mathbf{a}$,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|(II)\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \le K \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|h(\mathbf{x}) - h(\mathbf{a}) - Dh(\mathbf{a})(\mathbf{x}-\mathbf{a})\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} = 0$$

(iii) Since $Dh(\mathbf{a}) \in \mathcal{B}\left(\mathbb{R}^n;\mathbb{R}\right)$ and $\mathbf{f}$ is continuous at $\mathbf{a}$, $\|Dh(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R})} < \infty$ and $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|_{\mathbb{R}^m} = 0$. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|(III)\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \le \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|Dh(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R})}\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|_{\mathbb{R}^m}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} = 0$$

Hence, $h\mathbf{f}$ is differentiable at $\mathbf{a}$ and $D(h\mathbf{f})(\mathbf{a}) = A$. □

■ Matrix Representation

Let $\mathbf{f}, \mathbf{g} : \mathcal{U} \to \mathbb{R}^m$, $h : \mathcal{U} \to \mathbb{R}$ are differentiable on $\mathcal{U}$. The matrix representation of the derivatives of $\mathbf{f}$ and $\mathbf{g}$ are

$$\left[D\mathbf{f}\right](\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix}(\mathbf{x}),\ \left[D\mathbf{g}\right](\mathbf{x}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n}\end{bmatrix}(\mathbf{x}) \text{ and } \left[Dh\right](\mathbf{x}) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n}\end{bmatrix}(\mathbf{x}).$$

Then

$$\left[D(\mathbf{f} \pm \mathbf{g})\right](\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \pm \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \pm \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} \pm \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \pm \frac{\partial g_m}{\partial x_n}\end{bmatrix}(\mathbf{x})$$

$$\begin{aligned}
\left[D(h\mathbf{f})\right](\mathbf{x}) &= \begin{bmatrix} \frac{\partial(hf_1)}{\partial x_1} & \cdots & \frac{\partial(hf_1)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(hf_m)}{\partial x_1} & \cdots & \frac{\partial(hf_m)}{\partial x_n} \end{bmatrix}(\mathbf{x}) \\
&= \begin{bmatrix} h\frac{\partial(f_1)}{\partial x_1} + \frac{\partial h}{\partial x_1}f_1 & \cdots & h\frac{\partial(f_1)}{\partial x_n} + \frac{\partial h}{\partial x_n}f_1 \\ \vdots & & \vdots \\ h\frac{\partial(f_m)}{\partial x_1} + \frac{\partial h}{\partial x_1}f_m & \cdots & h\frac{\partial(f_m)}{\partial x_n} + \frac{\partial h}{\partial x_n}f_m \end{bmatrix}(\mathbf{x}) \\
&= h(\mathbf{x})\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(\mathbf{x}) + \begin{bmatrix} \frac{\partial h}{\partial x_1}f_1 & \cdots & \frac{\partial h}{\partial x_n}f_1 \\ \vdots & & \vdots \\ \frac{\partial h}{\partial x_1}f_m & \cdots & \frac{\partial h}{\partial x_n}f_m \end{bmatrix}(\mathbf{x})
\end{aligned}$$

❑ Chain Rule

Theorem 5.5.2. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in \mathcal{U}$, $\mathbf{g} : \mathbf{f}(\mathcal{U}) \to \mathbb{R}^\ell$ be differentiable $\mathbf{f}(\mathbf{a})$. Then $\mathbf{F} = \mathbf{g} \circ \mathbf{f} : \mathcal{U} \to \mathbb{R}^\ell$ is differentiable at $\mathbf{a}$ and for a vector $\mathbf{h} \in \mathbb{R}^n$,*

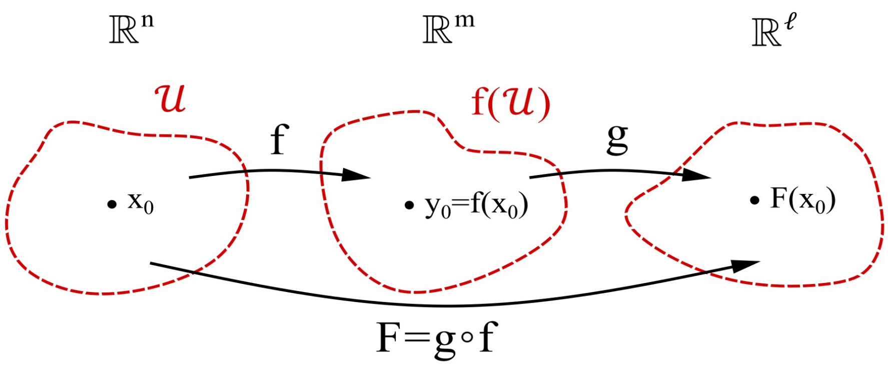
$$\underbrace{\left[D\mathbf{F}(\mathbf{a})\right](\mathbf{h})}_{\in \mathcal{B}(\mathbb{R}^n;\mathbb{R}^\ell)} = \underbrace{\left[D\mathbf{g}\right]\left(\mathbf{f}(\mathbf{a})\right)}_{\in \mathcal{B}(\mathbb{R}^m;\mathbb{R}^\ell)} \underbrace{\left[(D\mathbf{f})(\mathbf{a})\mathbf{h}\right]}_{\in \mathcal{B}(\mathbb{R}^n;\mathbb{R}^m)}$$

Moreover, let $\mathbf{f} = \mathbf{f}(x_1, \cdots, x_n)$ and $\mathbf{g} = \mathbf{g}(y_1, \cdots y_m)$ then

$$\left(D\mathbf{F}(\mathbf{a})\right)_{ij} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\left(\mathbf{f}(\mathbf{a})\right)\frac{\partial f_k}{\partial x_j}(\mathbf{a})$$

and

$$\left[(D\mathbf{F}(\mathbf{a}))\right]_{\ell\times n} = \left[D\mathbf{g}\left(\mathbf{f}(\mathbf{a})\right)\right]_{\ell\times m}\left[D\mathbf{f}(\mathbf{a})\right]_{m\times n}$$



$$Df(\mathbf{a}) : \quad \mathbb{R}^n \to \mathbb{R}^m \qquad \mathbf{v} \to Df(\mathbf{a})\mathbf{v}$$

$$Dg(\mathbf{b}) : \quad \mathbb{R}^m \to \mathbb{R}^\ell \qquad \mathbf{u} \to Dg(\mathbf{b})\mathbf{u}$$

$$\begin{gathered}
DF(\mathbf{a}) : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell \\
DF(\mathbf{a})\mathbf{v} = Dg(\mathbf{b})\left(Df(\mathbf{a})\mathbf{v}\right) \\
\left[DF(\mathbf{a})\right] = \left[Dg(\mathbf{b})\right]\left[Df(\mathbf{a})\right]
\end{gathered}$$

Proof. Let $\mathbf{b} = \mathbf{f}(\mathbf{a})$, $A = D\mathbf{f}(\mathbf{a}) \in \mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)$ and $B = D\mathbf{g}(\mathbf{b}) \in \mathscr{B}(\mathbb{R}^m;\mathbb{R}^\ell)$. To prove $\left[D\mathbf{F}(\mathbf{a})\right] = BA$.

Let $\varepsilon > 0$ be given. Since $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{b} = \mathbf{f}(\mathbf{a})$, there exists $\delta_1, \delta_2 > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta_1$,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - A(\mathbf{x} - \mathbf{a})\|_{\mathbb{R}^m} < \min\left(1, \frac{\varepsilon}{2\|B\|_{\mathscr{B}(\mathbb{R}^m;\mathbb{R}^\ell)} + 1}\right)\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} \tag{5.5}$$

and if $\|\mathbf{y} - \mathbf{b}\|_{\mathbb{R}^m} < \delta_2$,

$$\|\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{b}) - B(\mathbf{y} - \mathbf{b})\|_{\mathbb{R}^\ell} < \min\left(1 + \frac{\varepsilon}{2\left[\|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)} + 1\right]}\right)\|\mathbf{y} - \mathbf{b}\|_{\mathbb{R}^m}. \tag{5.6}$$

Since $\mathbf{f}$ is continuous at $\mathbf{a}$, there exists $\delta_3 > 0$ such that if $\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n} < \delta_3$, then

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|_{\mathbb{R}^m} < \delta_2. \tag{5.7}$$

Let $\mathbf{h} \in \mathbb{R}^n$ such that $\|\mathbf{h}\|_{\mathbb{R}^n} < \min(\delta_1, \delta_3)$. Then

$$\|\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) - A\mathbf{h}\|_{\mathbb{R}^m} \overset{(5.5)}{\leq} \frac{\varepsilon}{2\|B\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^\ell)}}\|\mathbf{h}\|_{\mathbb{R}^n} \tag{5.8}$$

and

$$\|\mathbf{g}\big(\mathbf{f}(\mathbf{a} + \mathbf{h})\big) - \mathbf{g}\big(\mathbf{f}(\mathbf{a})\big) - B\big(\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a})\big)\|_{\mathbb{R}^\ell} \overset{(5.6)(5.7)}{\leq} \frac{\varepsilon}{2\|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)}}\|\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a})\|_{\mathbb{R}^m}. \tag{5.9}$$

Hence,

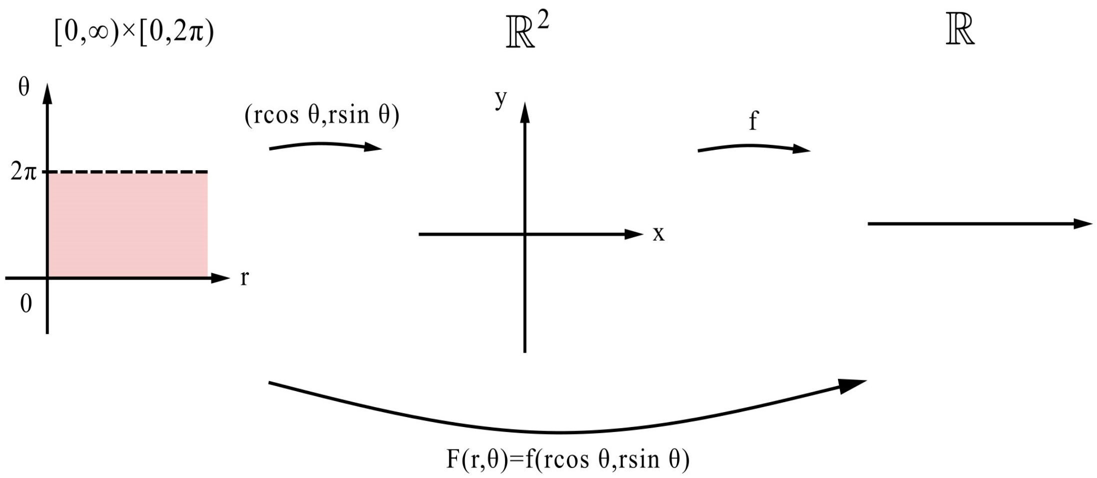
$$\begin{aligned}
& \|\mathbf{F}(\mathbf{a} + \mathbf{h}) - \mathbf{F}(\mathbf{a}) - BA\mathbf{h}\|_{\mathbb{R}^\ell} \\
\leq\ & \|\mathbf{F}(\mathbf{a} + \mathbf{h}) - \mathbf{F}(\mathbf{a}) - B\left[\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a})\right]\|_{\mathbb{R}^\ell} + \|B\left[\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a})\right] - BA\mathbf{h}\|_{\mathbb{R}^\ell} \\
\overset{(5.9)}{\leq}\ & \frac{\varepsilon}{2\left[\|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)} + 1\right]}\|\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a})\|_{\mathbb{R}^m} + \|B\|_{\mathscr{B}(\mathbb{R}^m;\mathbb{R}^\ell)}\|\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) - A\mathbf{h}\|_{\mathbb{R}^n} \\
\overset{(5.5)}{\leq}\ & \frac{\varepsilon}{2\left[\|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)} + 1\right]}\left[\|\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) - A\mathbf{h}\|_{\mathbb{R}^m} + \|A\mathbf{h}\|_{\mathbb{R}^m}\right] + \frac{\varepsilon\|B\|_{\mathscr{B}(\mathbb{R}^m;\mathbb{R}^\ell)}}{2\|B\|_{\mathscr{B}(\mathbb{R}^m;\mathbb{R}^\ell)} + 1}\|\mathbf{h}\|_{\mathbb{R}^n} \\
\overset{(5.5)}{\leq}\ & \frac{\varepsilon}{2\left[\|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)} + 1\right]}\left[\|\mathbf{h}\|_{\mathbb{R}^n} + \|A\|_{\mathscr{B}(\mathbb{R}^n;\mathbb{R}^m)}\|\mathbf{h}\|_{\mathbb{R}^n}\right] + \frac{1}{2}\|\mathbf{h}\|_{\mathbb{R}^n} \\
\leq\ & \varepsilon\|\mathbf{h}\|_{\mathbb{R}^n}.
\end{aligned}$$

Therefore, $\mathbf{F}$ is differentiable at $\mathbf{a}$ and $D\mathbf{F}(\mathbf{a}) = BA$. □

Example 5.5.3. Let $x = r\cos\theta$, $y = r\sin\theta$, $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Let

$$F(r, \theta) = f(r\cos\theta, r\sin\theta) : [0, \infty) \times [0, 2\pi] \to \mathbb{R}.$$

$$\left[DF\right](r, \theta) = \begin{bmatrix} \dfrac{\partial F}{\partial r} & \dfrac{\partial F}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$



Example 5.5.4. $\mathbf{r} : (0,1) \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$. Let

$$F(t) = f\big(\mathbf{r}(t)\big) : (0,1) \to \mathbb{R}.$$

Then

$$\underbrace{F'(t)}_{\mathcal{B}(\mathbb{R};\mathbb{R})} = \underbrace{Df\big(\mathbf{r}(t)\big)}_{\mathcal{B}(\mathbb{R}^n;\mathbb{R})} \underbrace{\mathbf{r}'(t)}_{\mathcal{B}(\mathbb{R};\mathbb{R}^n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(\mathbf{r}(t)\big) r_i'(t)$$

where $\mathbf{r}(t) = \big(r_1(t), \cdots, r_n(t)\big)$.

Example 5.5.5. Let $f(u,v,w) = u^2v + wv^2$, $g(x,y) = (\underset{u}{xy}, \underset{v}{\sin x}, \underset{w}{e^x})$. Let $h(x,y) = f\big(g(x,y)\big) : \mathbb{R}^2 \to \mathbb{R}$.

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ &= 2uv \cdot y + (u^2 wv) \cdot \cos x + v^2 \cdot e^x \\ &= 2xy^2 \sin x + (x^2y^2 + 2xy \sin x)\cos x + e^x \sin^2 x \\ \frac{\partial h}{\partial y} &= \cdots \end{aligned}$$

Review:

$$\begin{aligned} D\mathbf{f}(\mathbf{a}) : \ & \mathbb{R}^n \to \mathbb{R}^m \\ & \mathbf{v} \to D\mathbf{f}(\mathbf{a})\mathbf{v} \end{aligned} \qquad \begin{aligned} D\mathbf{g}(\mathbf{b}) : \ & \mathbb{R}^m \to \mathbb{R}^\ell \\ & \mathbf{u} \to D\mathbf{g}(\mathbf{b})\mathbf{u} \end{aligned}$$

$$D\mathbf{F}(\mathbf{a}) : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$$
$$D\mathbf{F}(\mathbf{a})\mathbf{v} = D\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)\big(D\mathbf{f}(\mathbf{a})\mathbf{v}\big)$$

$$\begin{aligned}\mathbf{F} &= \mathbf{g} \circ \mathbf{f} \\ \mathbf{F}(x_1, \cdots, x_n) &= \mathbf{g}\big(\mathbf{f}(x_1, \cdots, x_n)\big).\end{aligned}$$

For $\mathbf{f} = (f_1, \cdots, f_m)$, $\mathbf{g} = (g_1, \cdots, g_\ell)$ and $\mathbf{F} = (F_1, \cdots, F_\ell)$, then $F_i = g_i \circ \mathbf{f}$ for $i = 1, \cdots, \ell$.

Let $g_i = g_i(y_1, \cdots, y_m)$ and $y_k = f_k(x_1, \cdots, x_n)$. We have

$$\frac{\partial F_i}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\frac{\partial f_k}{\partial x_j}$$

setting $y_i = f_i(x_1, \cdots, x_n)$.

$$\big[(D\mathbf{F}(\mathbf{a}))\big]_{\ell\times n} = \big[D\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)\big]_{\ell\times m}\,\big[D\mathbf{f}(\mathbf{a})\big]_{m\times n}$$

$$\begin{bmatrix} & \vdots & \\ \cdots & \dfrac{\partial F_i(\mathbf{a})}{\partial x_j} & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} & \cdots\ \cdots & \\ \dfrac{\partial g_i(\mathbf{b})}{\partial y_1} & \cdots\ \cdots & \dfrac{\partial g_i(\mathbf{b})}{\partial y_m} \\ & \cdots\ \cdots & \end{bmatrix} \begin{bmatrix} & \dfrac{\partial f_1(\mathbf{a})}{\partial x_j} & \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ & \dfrac{\partial f_m(\mathbf{a})}{\partial x_j} & \end{bmatrix}$$

5.6 Directional Derivative, Gradients, Tangent Plane and Linear Approximation

In this section, we will discuss multi-variable real-ralued function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition 5.6.1. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open $\mathbf{a} \in \mathcal{U}$, $f : \mathcal{U} \to \mathbb{R}$. Let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. We say that "*f has directional derivtive at* $\mathbf{a}$ *in the direction* $\mathbf{v}$" if the limilt

$$\lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} \qquad \left(\frac{d}{dt}\Big|_{t=0} f(\mathbf{a} + t\mathbf{v})\right)$$

exists. Denote by $D_{\mathbf{v}} f(\mathbf{a})$.

Remark. Let $\mathbf{e}_j = \langle 0, \cdots, 1, \cdots, 0\rangle$. $D_{\mathbf{e}_j} f(\mathbf{a}) = \dfrac{\partial f}{\partial x_j}(\mathbf{a})$ is the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{e}_j$.

Theorem 5.6.2. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$ be differentiable at* $\mathbf{a}$. *The directional derivative of f at* $\mathbf{a}$ *in the direction* $\mathbf{v}$ *is* $\big(Df\big)(\mathbf{a})\mathbf{v}$. *That is,* $D_{\mathbf{v}} f(\mathbf{a}) = Df(\mathbf{a})\mathbf{v}$.

Proof. Since f is differentiable at $\mathbf{a}$,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|f(\mathbf{x}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{x} - \mathbf{a})|}{\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n}} = 0.$$

Let $\mathbf{x} = \mathbf{a} + t\mathbf{v}$, $\|\mathbf{v}\|_{\mathbb{R}^n} = 1$. Then

$$\lim_{t\to 0} \frac{|f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a}) - Df(\mathbf{a})(t\mathbf{v})|}{\|t\mathbf{v}\|_{\mathbb{R}^n}} = \lim_{t\to 0} \left| \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t} - Df(\mathbf{a})\mathbf{v} \right| = 0.$$

Hence,

$$\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t} = Df(\mathbf{a})\mathbf{v}.$$

□

Remark. To compute the directional derivative $D_{\mathbf{v}}f(\mathbf{a})$, we have to check that $\mathbf{v}$ is a unit vector in advance.

Question: How about $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_{\mathbb{R}^n} \neq 1$?

Let $\mathbf{v} = \dfrac{\mathbf{u}}{\|\mathbf{u}\|_{\mathbb{R}^n}}$, then compute $Df(\mathbf{a})\mathbf{v} = D_{\mathbf{v}}f(\mathbf{a}) = \lim\limits_{t\to 0} \dfrac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t}$.

$$\begin{aligned}
\big(Df\big)(\mathbf{a})\mathbf{u} &= \big(Df\big)(\mathbf{a})\big(\|\mathbf{u}\|_{\mathbb{R}^n}\mathbf{v}\big) = \|\mathbf{u}\|_{\mathbb{R}^n} Df(\mathbf{a})\mathbf{v} \\
&= \|\mathbf{u}\|_{\mathbb{R}^n} \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t} \\
&= \lim_{t\to 0} \frac{f(\mathbf{a} + \frac{t\mathbf{u}}{\|\mathbf{u}\|_{\mathbb{R}^n}}) - f(\mathbf{a})}{\frac{t}{\|\mathbf{u}\|_{\mathbb{R}^n}}} \\
\left(s = \frac{t}{\|\mathbf{u}\|_{\mathbb{R}^n}}\right) \quad &= \lim_{s\to 0} \frac{f(\mathbf{a}+s\mathbf{u}) - f(\mathbf{a})}{s}
\end{aligned}$$

Remark. Let $\mathcal{U} \subseteq \mathbb{R}^n$, $f : \mathcal{U} \to \mathbb{R}$ and $\mathbf{a} \in \mathcal{U}$.

f is differentiable at $\mathbf{a}$ $\implies$ the directional derivatives of f at $\mathbf{a}$ in all directions exist. $D_{\mathbf{v}}f(\mathbf{a}) = Df(\mathbf{a})\mathbf{v}$

$\not\Longleftarrow$

Example 5.6.3. $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ Let $\mathbf{v} = \langle v_1, v_2 \rangle$. Then

$$D_{\mathbf{v}}f(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \frac{v_1^3}{v_1^2 + v_2^2}.$$

But f is not differentiable at $(0,0)$. Moreover,

$$\frac{v_1^3}{v_1^2+v_2^2} = \big(D_{\mathbf{v}}f\big)(0,0) \neq Jf(0,0)\mathbf{v} = v_1 \quad \text{where } Jf(0,0) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

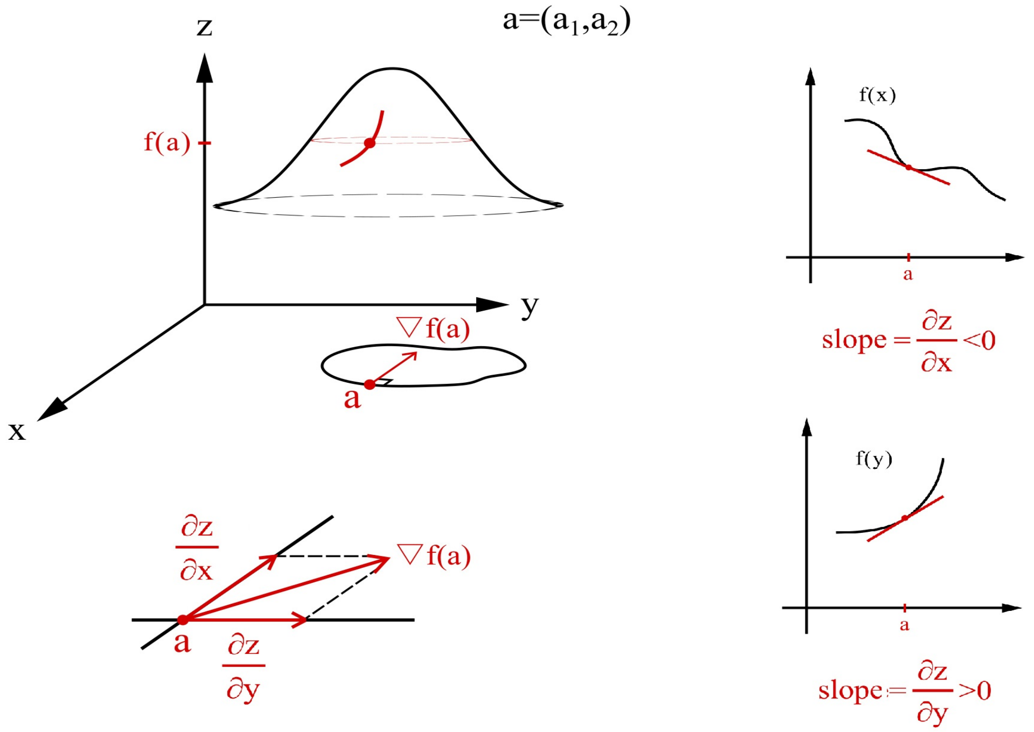
Remark. The existence of the directional derivative of f at $\mathbf{a}$ in all directions does NOT imply that f is continuous at $\mathbf{a}$. For example, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } x + y^2 \neq 0 \\ 0 & \text{if } x + y^2 = 0 \end{cases}$

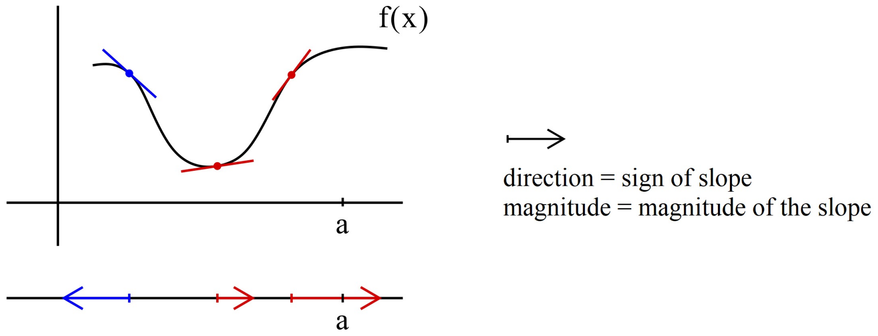
$$\big(D_{\mathbf{v}}f\big)(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^2 v_1 v_2}{t(tv_1 + t^2 v_2^2)} = \begin{cases} v_2 & \text{if } v_1 \neq 0 \\ 0 & \text{if } v_1 = 0 \end{cases}$$

But f is not continuous at $(0,0)$ along $x = y^2$.

■ The Gradients of Functions

Definition 5.6.4. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$ be differentiable at $\mathbf{a}$. The row vector of $Jf(\mathbf{a}) = \left[\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}\right]$ is also called "*the gradient of f at* $\mathbf{a}$" and denoted by $\nabla f(\mathbf{a})$. Therefore, if f is differentiable at $\mathbf{a}$, then $\left[Df(\mathbf{a})\right] = Jf(\mathbf{a}) = \left[\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}\right]$.





Remark. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$ be differentiable at $\mathbf{a}$. For $\mathbf{v} \in \mathbb{R}^n$, the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{v}$ is

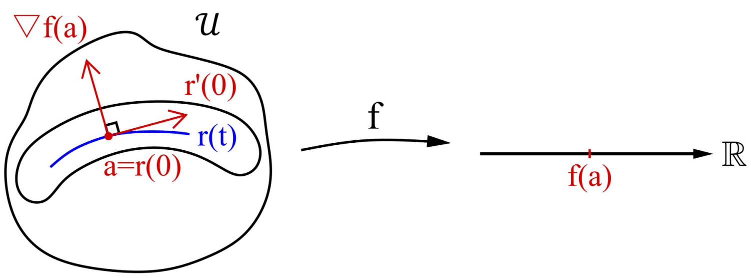
$$Df(\mathbf{a})\mathbf{v} = \nabla f(\mathbf{a}) \cdot \mathbf{v}.$$

Proposition 5.6.5. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \mathcal{U}$ and $f \in C^1(\mathcal{U}, \mathbb{R})$. Then if $\nabla f(\mathbf{a}) \neq \mathbf{0}$, the vector $\nabla f(\mathbf{a})$ is normal to the level set $\{x \in \mathcal{U} \mid f(\mathbf{x}) = f(\mathbf{a})\}$.*

Proof. Let $\mathbf{r} : (-\delta, \delta) \to \mathbb{R}^n$ be the curve such that $\mathbf{r}(t) \in \{\mathbf{x} \in \mathcal{U} \mid f(\mathbf{x}) = f(\mathbf{a})\}$, $\mathbf{r}(0) = \mathbf{a}$ and $\mathbf{r}'(t) \neq \mathbf{0}$. Then $f(\mathbf{r}(t)) \equiv f(\mathbf{a})$ for every $t \in (-\delta, \delta)$. By the chain rule,

$$\frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big) = \nabla f\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = 0 \quad \text{for every } t \in (-\delta, \delta).$$

Then $\nabla f\big(\mathbf{r}(0)\big) \cdot \mathbf{r}'(0) = 0$ and hence $\nabla f(\mathbf{a}) \perp \mathbf{r}'(0)$. Since $\mathbf{r}$ is an arbitrary curve on the level set passing $\mathbf{a}$, $\nabla f(\mathbf{a})$ is normal to the level set at $\mathbf{a}$.



□

Proposition 5.6.6. *Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathcal{U}$. Then $\dfrac{\nabla f}{\|\nabla f\|_{\mathbb{R}^n}}$ $\Big(-\dfrac{\nabla f}{\|\nabla f\|_{\mathbb{R}^n}}\Big)$ is the direction in which the function increases (decreases) most rapidly.*

Proof. Let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. The directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{v}$ is

$$\Big|Df(\mathbf{a})(\mathbf{v})\Big| = \Big|\nabla f(\mathbf{a}) \cdot \mathbf{v}\Big| \leq \|\nabla f(\mathbf{a})\|_{\mathbb{R}^n}\|\mathbf{v}\|_{\mathbb{R}^n} = \|\nabla f(\mathbf{a})\|_{\mathbb{R}^n}.$$

The equality holds if $\nabla f(\mathbf{a})$ is parallel to $\mathbf{v}$. (i.e. $\nabla f(\mathbf{a}) = c\mathbf{v}$ for some $c \in \mathbb{R}$). Hence, if $\mathbf{v} = \dfrac{\nabla f(\mathbf{a})}{\|\nabla f(\mathbf{a})\|_{\mathbb{R}^n}}$, then $Df(\mathbf{a})\mathbf{v}$ has maximum and if $\mathbf{v} = -\dfrac{\nabla f(\mathbf{a})}{\|\nabla f(\mathbf{a})\|_{\mathbb{R}^n}}$, then $Df(\mathbf{a})\mathbf{v}$ has minimum.

□

■ Tangent Planes (Spaces) to the Graph

The directional derivative of f at $\mathbf{a}$ in the direction (unit vector) $\mathbf{v}$ is the rate of change of f in the direction $\mathbf{v}$. Choose a (continuously) differentiable curve $\mathbf{r}(t) : (-\delta, \delta) \to \mathcal{U}$ such that $\mathbf{r}(0) = \mathbf{a}$ and $\mathbf{r}'(0) = \mathbf{v}$. Then $f\big(\mathbf{r}(t)\big)$ is a cruve on the graph of f and

$$\frac{d}{dt}\Big|_{t=0} f\big(\mathbf{r}(t)\big) = Df\big(\mathbf{r}(0)\big)\mathbf{r}'(0) = Df(\mathbf{a})\mathbf{v} = D_{\mathbf{v}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

which is the slope of the tangent line to the graph of f passing $(\mathbf{a}, f(\mathbf{a}))$. For $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{v} = \langle v_1, \cdots, v_n \rangle$, the equation of the tangent line is

$$\begin{cases} x_1 = a_1 + tv_1 \\ \quad\vdots \\ x_n = a_n + tv_n \\ x_{n+1} = f(\mathbf{a}) + tDf(\mathbf{a})\mathbf{v} \end{cases}$$

Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be (continuously) differentiable at $\mathbf{a}$. The tangent plane P to the graph of f passing $\big(\mathbf{a}, f(\mathbf{a})\big)$ is defined as the plane consisting of all tangent lines to the graph of f passing $\big(\mathbf{a}, f(\mathbf{a})\big)$. Hence, the equation of P is

$$x_{n+1} = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

or

$$x_{n+1} = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}).$$

Note: For $n = 2$, let $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(x_0, y_0) \in \mathcal{U}$. Then the tangent plane to the graph of f at (x_0, y_0, z_0) is

$$z = z_0 + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle = z_0 + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

■ **Linear Approximation**

Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathcal{U}$. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{|f(\mathbf{x}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{x} - \mathbf{a})|}{||\mathbf{x} - \mathbf{a}||_{\mathbb{R}^n}} = 0.$$

This implies that

$$f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + o(||\mathbf{x} - \mathbf{a}||_{\mathbb{R}^n}) \quad \text{as } \mathbf{x} \to \mathbf{a}$$

Define

$$\begin{aligned} L(\mathbf{x}) &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) \\ &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}). \end{aligned}$$

Then $f(\mathbf{x}) \approx L(\mathbf{x})$ as $\mathbf{x}$ is near $\mathbf{a}$.

Remark. Let $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in \mathcal{U}$. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})||_{\mathbb{R}^m}}{||\mathbf{x} - \mathbf{a}||_{\mathbb{R}^n}} = 0.$$

Let

$$L(\mathbf{x}) = f(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

We still have

$$\mathbf{f}(\mathbf{x}) = L(\mathbf{x}) + o(||\mathbf{x} - \mathbf{a}||_{\mathbb{R}^n}) \quad \text{as } \mathbf{x} \to \mathbf{a}.$$

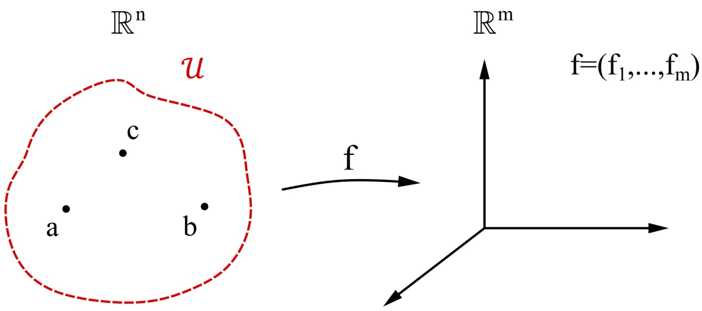
Heuristically, if $\mathbf{f}$ is differentiable at $\mathbf{a}$, the behavior of $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})$ is like the one of the linear map $D\mathbf{f}(a)(\mathbf{x} - \mathbf{a})$ when $\mathbf{x}$ is near $\mathbf{a}$.

5.7 The Mean Value Theorem

Recall: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and is differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Question: Is there similar result for $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$? That is, for $\mathbf{a}, \mathbf{b} \in \mathcal{U}$, is there $\mathbf{c} \in \mathcal{U}$ such that $f(\mathbf{b}) - f(\mathbf{a}) = Df(\mathbf{c})(\mathbf{b} - \mathbf{a})$?



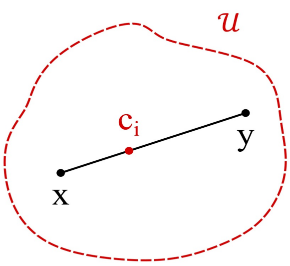
Answer: No! For example, $f : [0, 1] \to \mathbb{R}^2$ by $f(t) = (t^2, t^3)$. Then $f(1) - f(0) = (1, 1)$.

For any $s \in [0, 1]$, $Df(s)v = (2sv, 3s^2 v)$ for every $v \in \mathbb{R}$. But there exists no $s \in [0, 1]$ such that $(1, 1) = Df(x)(1 - 0) = (2s, 3s^2)$.

But we still have similar result for each component functions f_i.

Theorem 5.7.1. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ with $\mathbf{f} = (f_1, \cdots, f_m)$. Suppose that $\mathbf{f}$ is differentiable on $\mathcal{U}$ and the line segment joining $\mathbf{x}$ and $\mathbf{y}$ lies in $\mathcal{U}$. Then there exists $\mathbf{c}_1, \cdots, \mathbf{c}_n$ on the segment such that*

$$f_i(\mathbf{y}) - f_i(\mathbf{x}) = \big(Df_i\big)(\mathbf{c}_i)(\mathbf{y} - \mathbf{x}) \quad \textit{for } i = 1, \cdots, m.$$



Proof. Let $\mathbf{r} : [0, 1] \to \mathbb{R}^n$ such that $\mathbf{r}(t) = (1 - t)\mathbf{x} + t\mathbf{y}$. Then $f_i \circ \mathbf{r} : [0, 1] \to \mathbb{R}$ is differentiable on $[0, 1]$ and $f_i(\mathbf{x}) = f_i(\mathbf{r}(0))$ and $f_i(\mathbf{y}) = f_i(\mathbf{r}(1))$.

By the Mean value Theorem, there exists $t_0 \in [0, 1]$ such that

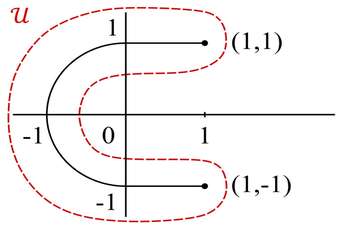
$$f_i(\mathbf{y}) - f_i(\mathbf{x}) = \frac{d}{dt}\Big[f_i\big(\mathbf{r}(t)\big)\Big]\Big|_{t=t_0}(1 - 0) = Df_i\big(\mathbf{r}(t_0)\big)\mathbf{r}'(t_0) = Df_i(\mathbf{c}_i)(\mathbf{y} - \mathbf{x}).$$

□

Corollary 5.7.2. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and convex and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ be differentiable on $\mathcal{U}$. Then for every $\mathbf{x}, \mathbf{y} \in \mathcal{U}$, there exists $\mathbf{c}_1, \cdots, \mathbf{c}_m$ on $\overline{\mathbf{xy}}$ such that*

$$f_i(\mathbf{y}) - f_i(\mathbf{x}) = (Df_i)(\mathbf{c}_i)(\mathbf{y} - \mathbf{x}).$$

Remark. The line segment joining $\mathbf{x}$ and $\mathbf{y}$ lies in $\mathcal{U}$ is necessary.



For example, let

$$C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2+y^2 = 1,\ x \leq 0\right\} \cup \left\{(x, \pm 1) \,\middle|\, 0 \leq x \leq 1\right\}$$

and U be a small neighborhood of C.

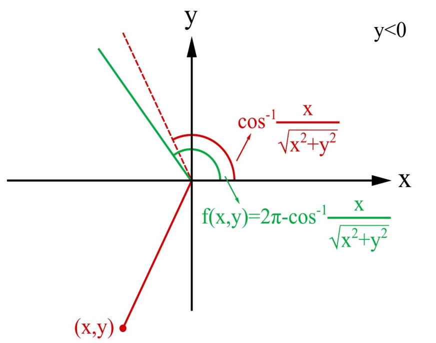
For $\mathbf{a} = (1, 1)$ and $\mathbf{b} = (1, -1)$, $\mathbf{b} - \mathbf{a} = (0, -2)$. Define

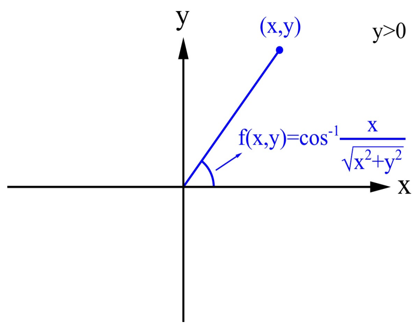
$$f(x, y) = \begin{cases} \cos^{-1} \dfrac{x}{\sqrt{x^2 + y^2}} & \text{if } y > 0 \\ \pi & \text{if } y = 0 \\ 2\pi - \cos^{-1} \dfrac{x}{\sqrt{x^2 + y^2}} & \text{if } y < 0 \end{cases}$$

Thus $f(1, -1) - f(1, 1) = \dfrac{3\pi}{2}$. But

$$\big(Df(x, y)\big) \underbrace{(0, -2)}_{(\mathbf{b}-\mathbf{a})} = \begin{bmatrix} \dfrac{-y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = -\frac{2x}{x^2 + y^2} \neq \frac{3\pi}{2}$$

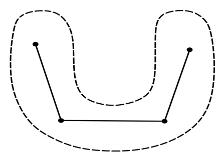
for any $(x, y) \in \mathcal{U}$ since $\left|\dfrac{2x}{x^2 + y^2}\right| \leq 3$.





Example 5.7.3. (1) Suppose that $\mathcal{U} \subseteq \mathbb{R}^n$ is an open and convex set, $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ is differentiable on $\mathcal{U}$ and $D\mathbf{f}(\mathbf{x}) = \mathbf{0}$ [$\in \mathscr{B}(\mathbb{R}^n; \mathbb{R}^m)$] for all $\mathbf{x} \in \mathcal{U}$. Then $\mathbf{f}$ is a constant function on $\mathcal{U}$.

(2) Moreover, if $\mathcal{U}$ is open and connected, $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ is differentiable and $D\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathcal{U}$, then $\mathbf{f}$ is constant on $\mathcal{U}$.



Proof. (Exercise) □

Theorem 5.7.4. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $K \subseteq \mathcal{U}$ be compact and $f : \mathcal{U} \to \mathbb{R}$ be of class C^1. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left| f(\mathbf{y}) - f(\mathbf{x}) - (Df)(\mathbf{x})(\mathbf{y}-\mathbf{x}) \right| \le \varepsilon \|\mathbf{y}-\mathbf{x}\|_{\mathbb{R}^n}$$

if $\|\mathbf{y}-\mathbf{x}\|_{\mathbb{R}^n} < \delta$ and $\mathbf{x}, \mathbf{y} \in K$.

Proof. Define $g : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ by

$$g(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{\left| f(\mathbf{y}) - f(\mathbf{x}) - (Df)(\mathbf{x})(\mathbf{y}-\mathbf{x}) \right|}{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^n}} & \text{if } \mathbf{y} \neq \mathbf{x} \\ 0 & \text{if } \mathbf{y} = \mathbf{x} \end{cases}$$

To check that g is continuous on $\mathcal{U} \times \mathcal{U}$.

Let $\mathbf{x} \neq \mathbf{y}$ and $(\mathbf{x},\mathbf{y}) \in \mathcal{U} \times \mathcal{U}$. Since f is of class C^1,

$$\lim_{(\mathbf{z},\mathbf{w})\to(\mathbf{x},\mathbf{y})} g(\mathbf{z},\mathbf{w}) = \lim_{(\mathbf{z},\mathbf{w})\to(\mathbf{x},\mathbf{y})} \frac{\left| f(\mathbf{w}) - f(\mathbf{z}) - (Df)(\mathbf{z})(\mathbf{w}-\mathbf{z}) \right|}{\|\mathbf{w}-\mathbf{z}\|_{\mathbb{R}^n}} = \frac{\left| f(\mathbf{y}) - f(\mathbf{x}) - (Df)(\mathbf{x})(\mathbf{y}-\mathbf{x}) \right|}{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^n}} = g(\mathbf{x},\mathbf{y}).$$

For $\mathbf{x} \in \mathcal{U}$ and $B(\mathbf{x}, r) \subseteq \mathcal{U}$, consider $\mathbf{w}, \mathbf{z} \in B(\mathbf{x}, r)$. Then the segment $\overline{\mathbf{wz}} \subseteq B(\mathbf{x}, r)$. By Mean Value Theorem, there exists $\xi \in \overline{\mathbf{wz}}$ such that

$$f(\mathbf{w}) - f(\mathbf{z}) = (Df)(\xi)(\mathbf{w}-\mathbf{z}).$$

Then

$$\begin{aligned} \lim_{\substack{(\mathbf{w},\mathbf{z})\to(\mathbf{x},\mathbf{x}) \\ \mathbf{z}\neq\mathbf{w}}} \frac{|f(\mathbf{w}) - f(\mathbf{z}) - Df(\mathbf{z})(\mathbf{w}-\mathbf{z})|}{\|\mathbf{w}-\mathbf{z}\|_{\mathbb{R}^n}} &= \lim_{\substack{(\mathbf{w},\mathbf{z})\to(\mathbf{x},\mathbf{x}) \\ \mathbf{z}\neq\mathbf{w}}} \frac{\left| \big(Df(\xi) - Df(\mathbf{z})\big)(\mathbf{w}-\mathbf{z}) \right|}{\|\mathbf{w}-\mathbf{z}\|_{\mathbb{R}^n}} \\ &\le \lim_{\substack{(\mathbf{w},\mathbf{z})\to(\mathbf{x},\mathbf{x}) \\ \mathbf{z}\neq\mathbf{w}}} \|Df(\xi) - Df(\mathbf{z})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R})} = 0 \end{aligned}$$

Hence, $\lim_{(\mathbf{w},\mathbf{z})\to(\mathbf{x},\mathbf{x})} g(\mathbf{z},\mathbf{w}) = 0 = g(\mathbf{x},\mathbf{x})$ and g is continuous at $(\mathbf{x},\mathbf{x})$. Thus, g is continuous on $\mathcal{U} \times \mathcal{U}$.

Since $K \times K \subseteq \mathcal{U} \times \mathcal{U}$ is compact, g is uniformly continuous on $K \times K$. Then for $\varepsilon > 0$, there exists $\delta > 0$ such that for every $(\mathbf{x},\mathbf{x}) \in K \times K$ and $\|(\mathbf{z},\mathbf{w}) - (\mathbf{x},\mathbf{x})\|_{\mathbb{R}^n \times \mathbb{R}^n} < \delta$,

$$\left| g(\mathbf{z},\mathbf{w}) - g(\mathbf{x},\mathbf{y}) \right| < \varepsilon.$$

Hence, if $\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^n} < \delta$, then $\|(\mathbf{x},\mathbf{y}) - (\mathbf{x},\mathbf{x})\|_{\mathbb{R}^n \times \mathbb{R}^n} < \delta$. We have

$$\Big| g(\mathbf{x},\mathbf{y}) - \underbrace{g(\mathbf{x},\mathbf{x})}_{=0} \Big| = \left| g(\mathbf{x},\mathbf{y}) \right| < \varepsilon.$$

□

Corollary 5.7.5. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $K \subseteq \mathcal{U}$ be compact and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ be of class C^1. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{y} \in K$ and $\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^n} < \delta$,*

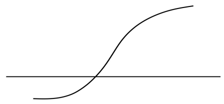
$$\|f(\mathbf{y}) - f(\mathbf{x}) - (D\mathbf{f})(\mathbf{x})(\mathbf{y}-\mathbf{x})\|_{\mathbb{R}^n} \le \varepsilon \|\mathbf{y}-\mathbf{x}\|_{\mathbb{R}^n}.$$

5.8 The Inverse Function Theorem

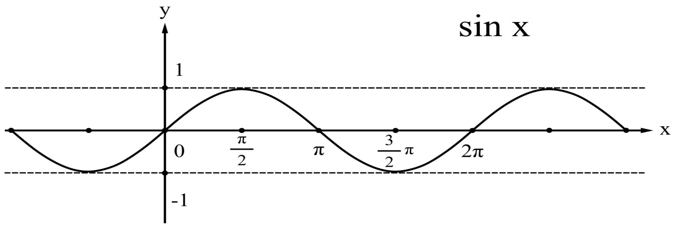
Recall: Let $f : (a,b) \to \mathbb{R}$. The function f is invertible from (a,b) to $f\big((a,b)\big)$ if and only if f is one-to-one.

Question: What is the sufficient condition such that f is one-to-one?

Observe that $f'(x) > 0$ ($f' < 0$) $\Longrightarrow f$ is increasing (decreasing) and then f is one-to-one.



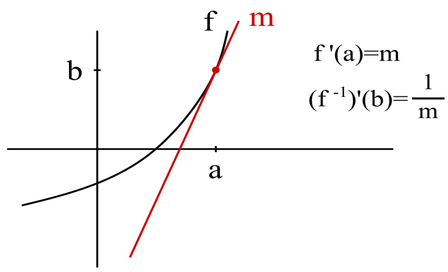
Question: In general, we cannot ask a function has this property everywhere (for exampe, $f(x) = \sin x$). Is there a sufficient condition for f such that f is invertible near a point?



Guess: $f'(a) \neq 0$.

Question: Is it enough? Consider $f(x) = \begin{cases} x^2 \sin \frac{1}{x} + \frac{1}{2}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Then $f'(0) = \frac{1}{2} > 0$ but $f'(x)$ is not continuous at 0.

Hence f is oscillatory near 0 and f is not one-to-one in any neighborhood of 0. We may guess $f \in C^1$ is necessary. Moreover, if $f : I \to \mathbb{R}$ is continuously differentiable near $a \in I$, $f'(a) \neq 0$ and $f(a) = b$, then $\big(f^{-1}\big)'(b) = \dfrac{1}{f'(a)}$.



Consider $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{a} \in D$.

Question: What is the sufficient condition of $\mathbf{f}$ at $\mathbf{a}$ such that $\mathbf{f}$ is invertible near $\mathbf{a}$?

Guess: (i) $D\mathbf{f}(\mathbf{a})$ is invertible (full rank) and (ii) $\mathbf{f}$ is of class C^1 near $\mathbf{a}$.

Heuristically, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + o\big(\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}\big)$ as $\mathbf{x} \to \mathbf{a}$. If $D\mathbf{f}(\mathbf{a})$ is invertible, then $\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})$ is one-to-one. Moreover, if $\mathbf{f}$ is of class C^1, then $\mathbf{f}$ is one-to-one near $\mathbf{a}$.

Theorem 5.8.1. *(Inverse Function Theorem) Let $D \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in D$, $\mathbf{f} : D \to \mathbb{R}^n$ be of class C^1 and $D\mathbf{f}(\mathbf{a})$ be invertible. Then there exists an open neighborhood $\mathcal{U}$ of $\mathbf{a}$ and an open neighborhood $\mathcal{V}$ of $\mathbf{f}(\mathbf{a})$ such that*

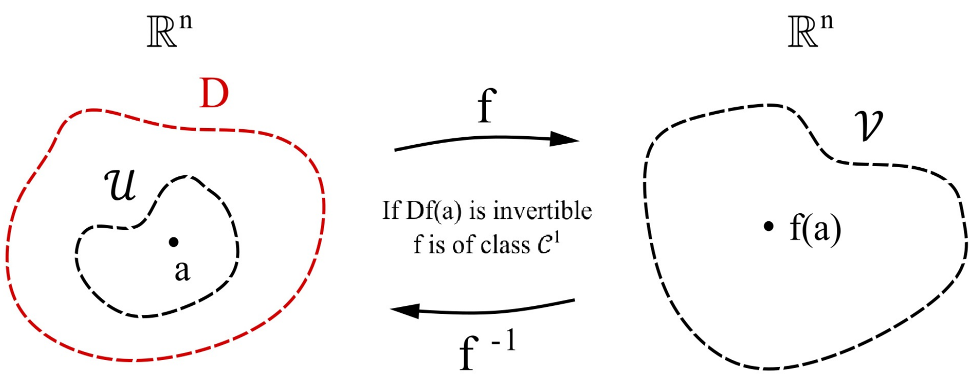
(1) $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ *is one-to-one and onto.*

(2) The inverse function $\mathbf{f}^{-1} : \mathcal{V} \to \mathcal{U}$ *is of class* C^1

(3) For $\mathbf{y} \in \mathcal{V}$ *and* $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{y})$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = \left(D\mathbf{f}(\mathbf{x})\right)^{-1}$$

(4) If $\mathbf{f}$ *is of class* C^r *for some* $r > 1$, *so is* $\mathbf{f}^{-1}$



f: 𝒰→𝒱 is 1-1 onto

Proof.

Recall:

(i) (*Contraction Mapping Theorem*) Let (M, d) be complete and $\phi : M \to M$ be a contraction mapping. That is,

$$d\big(f(x), f(y)\big) \le c d(x, y) \quad \text{for some } 0 < c < 1 \text{ and for every } x, y \in M$$

Then there exists a unique fixed point $x_0 \in M$. That is, $x_0 = f(x_0)$.

(ii) (*Secant Method*)

Let $\phi(x) = x - \dfrac{f(x) - y}{M}$ where $M = \sup |f'(x)| + 1$. Then

- $f(x_0) = y$ if and only if x_0 is a fixed point of ϕ.
- ϕ is a contraction mapping near x_0.

(iii) By Theorem 5.1.10, if $A \in GL(n)$ and $K \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\|A - K\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} < 1,$$

then $K \in GL(n)$.

Let $A = D\mathbf{f}(\mathbf{a}) \in GL(n)$. Then A^{-1} exists and $A^{-1} \in GL(n) \subseteq \mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^n\big)$. Choose $\lambda > 0$ such that

$$2\lambda\|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} = 1.$$

Since $\mathbf{f}$ is of class C^1 and D is open, there exists $\delta > 0$ such that $B(\mathbf{a},\delta) \subseteq D$ and if $\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n} < \delta$, then

$$\|D\mathbf{f}(\mathbf{x}) - A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} = \|D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} < \lambda.$$

Hence,

$$\|D\mathbf{f}(\mathbf{x}) - A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}\|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} < \frac{1}{2} \tag{5.10}$$

for every $x \in B(\mathbf{a},\delta)$.

Step 1: Let $\mathcal{U} = B(\mathbf{a},\delta)$. Then $\mathbf{f} : \mathcal{U} \to \mathbb{R}^n$ is one-to-one. (Hence, $\mathbf{f} : \mathcal{U} \to \mathbf{f}(\mathcal{U})$ is bijective.)
Proof of Step1: To prove that for every $\mathbf{y} \in \mathbb{R}^n$, at most one $\mathbf{x} \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Fix $\mathbf{y} \in \mathbb{R}^n$, define $\phi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} - A^{-1}\big(\mathbf{f}(\mathbf{x}) - \mathbf{y}\big)$. $\boxed{\mathbf{f}(\mathbf{x}) = \mathbf{y} \Leftrightarrow \phi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x}}$ Then

$$D\phi_{\mathbf{y}}(\mathbf{x}) = Id + A^{-1}\big(D\mathbf{f}\big)(\mathbf{x}) = A^{-1}\big(A - D\mathbf{f}(\mathbf{x})\big)$$

where Id is the identity map. By (5.10),

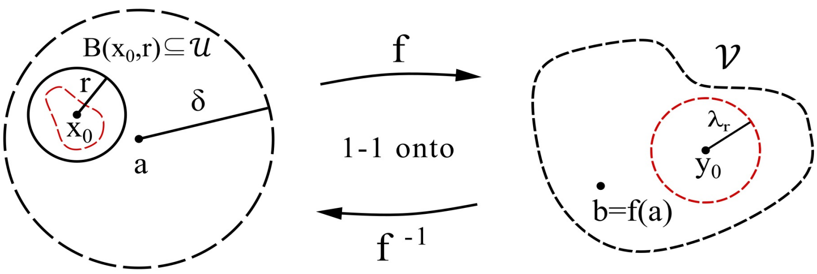
$$\|D\phi_{\mathbf{y}}(\mathbf{x})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \le \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}\|A - D\mathbf{f}(\mathbf{x})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} < \frac{1}{2}. \tag{5.11}$$

Thus, if $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{a},\delta)$, by the mean value theorem,

$$\|\phi_{\mathbf{y}}(\mathbf{x}_1) - \phi_{\mathbf{y}}(\mathbf{x}_2)\|_{\mathbb{R}^n} \le \Big[\sup_{\xi\in B(\mathbf{a},\delta)} \|D\phi_{\mathbf{y}}(\xi)\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}\Big]\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathbb{R}^n} < \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathbb{R}^n} \tag{5.12}$$

Hence, $\phi_{\mathbf{y}}$ has at most one fixed point in $\mathcal{U}$. That is, at most one $\mathbf{x} \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{y}$ is arbitrary in $\mathbb{R}^n$, $\mathbf{f}$ is one-to-one in $\mathbb{R}^n$.

Step 2: Let $\mathcal{V} = \mathbf{f}(\mathcal{U})$. Then $\mathcal{V}$ is open.



Proof of Step2: Let $\mathbf{y}_0 \in \mathcal{V}$. There exists $\mathbf{x}_0 \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$. Since $\mathcal{U}$ is open, there exists $r > 0$ such that $\overline{B(\mathbf{x}_0, r)} \subseteq \mathcal{U}$.

To prove that for every $\mathbf{z} \in B(\mathbf{y}_0, \lambda r)$, there exists $\mathbf{w} \in B(\mathbf{x}_0, r)$ such that $\mathbf{f}(\mathbf{w}) = \mathbf{z}$. Then $B(\mathbf{y}_0, \lambda r) \subseteq \mathcal{V}$.

Let $\mathbf{z} \in B(\mathbf{y}_0, \lambda r)$. To prove $\phi_{\mathbf{z}}(\mathbf{x}) = \mathbf{x} - A^{-1}\big(\mathbf{f}(\mathbf{x}) - \mathbf{z}\big)$ is a contraction mapping on $\overline{B(\mathbf{x}_0, r)}$.

$\boxed{\mathbf{f}(\mathbf{w}) = \mathbf{z} \Longleftrightarrow \phi_{\mathbf{z}}(\mathbf{w}) = \mathbf{w}}$

(i) To prove $\phi_{\mathbf{z}}$ maps from $\overline{B(\mathbf{x}_0, r)}$ into $\overline{B(\mathbf{x}_0, r)}$.
For $\mathbf{x} \in \overline{B(\mathbf{x}_0, r)}$,

$$\begin{aligned} \|\phi_{\mathbf{z}}(\mathbf{x}) - \mathbf{x}_0\|_{\mathbb{R}^n} &\le \|\phi_{\mathbf{z}}(\mathbf{x}) - \phi_{\mathbf{z}}(\mathbf{x}_0)\|_{\mathbb{R}^n} + \|\phi_{\mathbf{z}}(\mathbf{x}_0) - \mathbf{x}_0\|_{\mathbb{R}^n} \\ &\le \underbrace{\sup_{\xi \in \mathcal{U}} \|D\phi_{\mathbf{z}}(\xi)\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}}_{<\frac{1}{2}} \underbrace{\|\mathbf{x} - \mathbf{x}_0\|_{\mathbb{R}^n}}_{<r} + \|A^{-1}\big(\mathbf{f}(\mathbf{x}_0) - \mathbf{z}\big)\|_{\mathbb{R}^n} \\ &< \frac{r}{2} + \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \underbrace{\|\mathbf{f}(\mathbf{x}_0) - \mathbf{z}\|_{\mathbb{R}^n}}_{<\lambda r} \\ &< \frac{r}{2} + \underbrace{\lambda \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}}_{=\frac{1}{2}} r \\ &= r. \end{aligned}$$

Thus, $\phi_{\mathbf{z}}(\mathbf{x}) \in \overline{B(\mathbf{x}_0, r)}$ for every $\mathbf{x} \in \overline{B(\mathbf{x}_0, r)}$.

(ii) By (5.12),

$$\|\phi_{\mathbf{z}}(\mathbf{x}_1) - \phi_{\mathbf{z}}(\mathbf{x}_2)\|_{\mathbb{R}^n} < \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathbb{R}^n} \quad \text{for every } \mathbf{x}_1, \mathbf{x}_2 \in \overline{B(\mathbf{x}_0, r)}$$

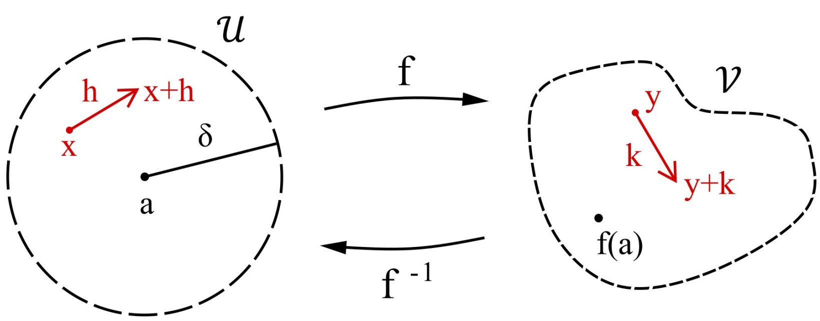
Hence, $\phi_{\mathbf{z}}$ is a contraction mapping on $\overline{B(\mathbf{x}_0, r)}$.

By the contraction mapping theorem, there exists $\mathbf{w} \in \overline{B(\mathbf{x}_0, r)}$ such that $\phi_{\mathbf{z}}(\mathbf{w}) = \mathbf{w}$. Thus $\mathbf{f}(\mathbf{w}) = \mathbf{z}$. We have $B(\mathbf{y}_0, \lambda r) \subseteq \mathcal{V}$ and therefore $\mathcal{V}$ is open. The statement (1) is proved.

Step 3: $\mathbf{f}^{-1} : \mathcal{V} \to \mathcal{U}$ is differentiable.
Proof of Step3: For $\mathbf{y} \in \mathcal{V}$, there exists $\mathbf{x} \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Since

$$\|D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} < \lambda \|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} = \frac{1}{2} < 1.$$

Then $D\mathbf{f}(\mathbf{x})$ is invertible and thus $\big[D\mathbf{f}(\mathbf{x})\big]^{-1}$ exists.



To prove that there exists a bounded linear map $L \in \mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\lim_{\mathbf{k}\to\mathbf{0}} \frac{\|\mathbf{f}^{-1}(\mathbf{y}+\mathbf{k}) - \mathbf{f}^{-1}(\mathbf{y}) - L\mathbf{k}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}} = 0.$$

We geuss that $L = \left[Df(\mathbf{x})\right]^{-1}$.

For every $\mathbf{k} \in \mathbb{R}^n$ such that $\mathbf{y}+\mathbf{k} \in \mathcal{V}$, there exists $\mathbf{h} = \mathbf{h}(\mathbf{k})$ such that $\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{y}+\mathbf{k}$. Then $\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \mathbf{k}$. By Mean Value Theorem, for $\mathbf{y} \in \mathcal{U}$,

$$\begin{aligned}
\|\mathbf{h} - A^{-1}\mathbf{k}\|_{\mathbb{R}^n} &= \|(\mathbf{x}+\mathbf{h}) - \mathbf{x} - A^{-1}\big(\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x})\big)\|_{\mathbb{R}^n} \\
&= \|\big[(\mathbf{x}+\mathbf{h}) - A^{-1}\big(\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{y}\big)\big] - \big[\mathbf{x} - A^{-1}\big(\mathbf{f}(\mathbf{x}) - \mathbf{y}\big)\big]\|_{\mathbb{R}^n} \\
&= \|\phi_{\mathbf{y}}(\mathbf{x}+\mathbf{h}) - \phi_{\mathbf{y}}(\mathbf{x})\|_{\mathbb{R}^n} \\
(M.V.T) \quad &\le \frac{1}{2}\|\mathbf{h}\|_{\mathbb{R}^n}.
\end{aligned}$$

Then

$$\|\mathbf{h}\|_{\mathbb{R}^n} \le \|A^{-1}\mathbf{k}\|_{\mathbb{R}^n} + \|\mathbf{h} - A^{-1}\mathbf{k}\|_{\mathbb{R}^n} \le \|A^{-1}\mathbf{k}\|_{\mathbb{R}^n} + \frac{1}{2}\|\mathbf{h}\|_{\mathbb{R}^n}.$$

We have

$$\|\mathbf{h}\|_{\mathbb{R}^n} \le 2\|A^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}\|\mathbf{k}\|_{\mathbb{R}^n} \le \frac{1}{\lambda}\|\mathbf{k}\|_{\mathbb{R}^n}. \tag{5.13}$$

Hence,

$$\begin{aligned}
\frac{\|\mathbf{f}^{-1}(\mathbf{y}+\mathbf{k}) - \mathbf{f}^{-1}(\mathbf{y}) - \big(D\mathbf{f}(\mathbf{x})\big)^{-1}\mathbf{k}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}} &= \frac{\|(\mathbf{x}+\mathbf{h}) - \mathbf{x} - \big(D\mathbf{f}(\mathbf{x})\big)^{-1}\mathbf{k}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}} = \frac{\|\big(D\mathbf{f}(\mathbf{x})\big)^{-1}\big[D\mathbf{f}(\mathbf{x})\mathbf{h} - \mathbf{k}\big]\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}} \\
&\le \|\big(D\mathbf{f}(\mathbf{x})\big)^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)} \frac{\|\mathbf{k} - D\mathbf{f}(\mathbf{x})\mathbf{h}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}} \\
&\le \underbrace{\|\big(D\mathbf{f}(\mathbf{x})\big)^{-1}\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^n)}}_{\text{bounded}} \underbrace{\frac{\|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})\mathbf{h}\|_{\mathbb{R}^n}}{\|\mathbf{h}\|_{\mathbb{R}^n}}}_{(\triangle)} \underbrace{\frac{\|\mathbf{h}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^n}}}_{\text{bounded}}.
\end{aligned}$$

Since $\mathbf{h} \to \mathbf{0}$ as $\mathbf{k} \to \mathbf{0}$, by (5.13) and $\mathbf{f}$ is differentiable at $\mathbf{x}$, $(\triangle) \to \mathbf{0}$ as $\mathbf{k} \to \mathbf{0}$. Then

$$\lim_{\mathbf{k}\to\mathbf{0}} \frac{\|\mathbf{f}^{-1}(\mathbf{y}+\mathbf{k}) - \mathbf{f}^{-1}(\mathbf{y}) - \big(D\mathbf{f}(\mathbf{x})\big)^{-1}\mathbf{k}\|_{\mathbb{R}^n}}{\|\mathbf{k}\|_{\mathbb{R}^\mathbf{n}}} = 0.$$

Therefore, $\mathbf{f}^{-1}$ is differentiable at $\mathbf{y}$ and $\big(D\mathbf{f}^{-1}\big)(\mathbf{y}) = \Big(D\mathbf{f}(\mathbf{x})\Big)^{-1}$. The statement (3) is proved.

Step 4: To prove the statements (2) and (4).
Proof of Step 4: Since the map $\mathbf{g} : GL(n) \to GL(n)$ by $\mathbf{g}(L) = L^{-1}$ is infinitely many times differentiable,

$$\big(D\mathbf{f}^{-1}\big)(\mathbf{y}) = \big(D\mathbf{f}(\mathbf{x})\big)^{-1} = \mathbf{g}\big(D\mathbf{f}(\mathbf{x})\big) = \mathbf{g}\Big(D\mathbf{f}\big(\mathbf{f}^{-1}(\mathbf{y})\big)\Big) = \Big(\mathbf{g} \circ \big(D\mathbf{f}\big) \circ \mathbf{f}^{-1}\Big)(\mathbf{y}). \tag{5.14}$$

By Chain rule, let $\mathbf{f} \in C^r$, then $D\mathbf{f} \in C^{r-1}$. For $k = 0, 1, \cdots, r-1$ and by (5.14), if $\mathbf{f}^{-1} \in C^k$ then $D\mathbf{f}^{-1} \in C^k$. This implies $\mathbf{f} \in C^{k+1}$. Continue this process until $\mathbf{f}^{-1} \in C^{r-1}$. We have $D\mathbf{f}^{-1} \in C^{r-1}$ and hence $\mathbf{f}^{-1} \in C^r$. The statements (2) and (4) are proved.

$$\mathbf{f}^{-1} \in \mathcal{C}^{k} \longrightarrow D\mathbf{f}^{-1} \in \mathcal{C}^{k}$$

$$\mathbf{f}^{-1} \in \mathcal{C}^{k+1} \longleftarrow D\mathbf{f}^{-1} \in \mathcal{C}^{k}$$

until

$$\mathbf{f}^{-1} \in \mathcal{C}^{r-1} \Rightarrow D\mathbf{f}^{-1} \in \mathcal{C}^{r-1} \Rightarrow \mathbf{f}^{-1} \in \mathcal{C}^{r}$$

□

Remark. If $\mathbf{f} \in C(\mathcal{U};\mathbb{R}^n)$ and $D\mathbf{f}(\mathbf{x})$ is invertible for every $\mathbf{x} \in \mathcal{U}$, then each $\mathbf{x} \in \mathcal{U}$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Hence, $\mathbf{f}$ is locally one-to-one in $\mathcal{U}$, but $\mathbf{f}$ need not be globally one-to-one in $\mathcal{U}$.

Example 5.8.2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Then

$$\left[Df(x,y)\right] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

$\det\left[Df(x,y)\right] = J_f(x,y) = e^{2x} \neq 0$. But f is not globally one-to-one.

Remark. Let $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and $D\mathbf{f}(\mathbf{a})$ is invertible and $\mathbf{f}$ is of class C^r near $\mathbf{a}$. By the Inverse Function Theorem, there exists open neighborhoods $\mathcal{U}$ of $\mathbf{a}$ and $\mathcal{V}$ of $\mathbf{f}(\mathbf{a})$ such that

$$\mathbf{f} : \mathcal{U} \to \mathcal{V} \quad \text{is one-to-one and onto and } \mathbf{f}^{-1} \text{ is of class } C^r.$$

Hence, for every $\mathbf{y} \in \mathcal{V}$, there exists a unique $\mathbf{x} \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. That is, we can solve $\mathbf{y}$ in terms of $\mathbf{x}$. Similarly, we can also solve $\mathbf{x}$ in terms of $\mathbf{y}$.

Remark. Let $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $D\mathbf{f}(\mathbf{x}_0) \in \mathcal{B}\left(\mathbb{R}^n;\mathbb{R}^n\right)$ is invertible if and only if $\det\left[D\mathbf{f}(\mathbf{x}_0)\right] \neq 0$. For $\mathbf{f} = (f_1, \cdots, f_n)$ and $\mathbf{x} = (x_1, \cdots, x_n)$, the determinant of the Jacobian matrix of f at $\mathbf{x}_0$ is called "*the Jacobian of* $\mathbf{f}$ *at* $\mathbf{x}_0$" and denoted by "$J_{\mathbf{f}}(\mathbf{x}_0)$" or "$\dfrac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}(\mathbf{x}_0)$". The value $|J_{\mathbf{f}}(\mathbf{x}_0)|$ is the volume of the parallel hexahedrom generated by the column vector of the Jacobian matrix.

Example 5.8.3. Let $\begin{cases} u(x,y) = \dfrac{x^4 + y^4}{x} \\ v(x,y) = \sin x + \cos y \end{cases}$ The equation says that u and v are expressed in terms of x and y. Find the points (x, y) where we can solve for x, y in terms of u, v.

Proof. Let $f(x,y) = (u, v) = \left(\dfrac{x^4 + y^4}{x}, \sin x + \cos y\right) : \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{3x^4 - y^4}{x^2} & \dfrac{4y^3}{x} \\ \cos x & -\sin y \end{vmatrix} = \frac{\sin y}{x^2}(y^4 - 3x^4) - \frac{4y^3}{x}\cos x.$$

Hence for those (x, y) such that $x \neq 0$ and $\dfrac{\sin y}{x^2}(y^4 - 3x^4) - \dfrac{4y^3}{x}\cos x \neq 0$, x, y can be solved in terms of u, v.

For example $(x_0, y_0) = (\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\partial(u,v)}{\partial(x,y)}\Big|_{(x_0,y_0)=(\frac{\pi}{2},\frac{\pi}{2})} \neq 0$. We can solve x, y in terms of u, v. That is, near $(\frac{\pi}{2}, \frac{\pi}{2})$, $x = x(u,v)$, $y = y(u,v)$ and $(x,y) = f^{-1}(u,v)$. Moreover, we can find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$ at $f(\frac{\pi}{2}, \frac{\pi}{2})$. Consider

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \left[Df^{-1}(u,v)\right]_{(u,v)=f(\frac{\pi}{2},\frac{\pi}{2})} = \left[Df(x,y)\right]^{-1}_{(x,y)=(\frac{\pi}{2},\frac{\pi}{2})}.$$

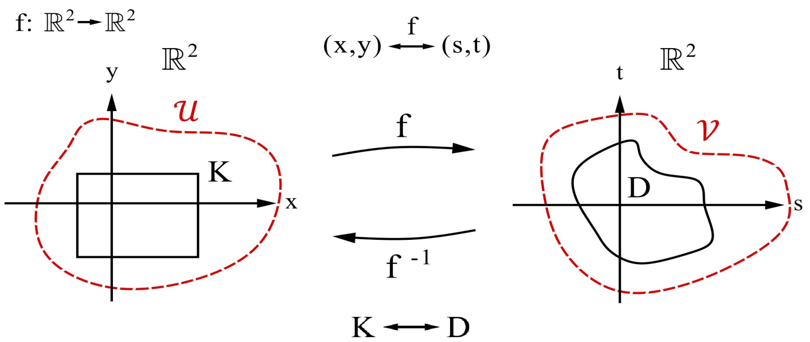
$$\left[Df(x,y)\right] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{3x^4 - y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{bmatrix}$$

$$\left[Df(x,y)\right]^{-1} = \frac{1}{|Jf|}\begin{bmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} = \frac{1}{\frac{\sin y}{x^2}(y^4 - 3x^4) - \frac{4y^3}{x}\cos x}\begin{bmatrix} -\sin y & -\frac{4y^3}{x} \\ -\cos x & \frac{3x^4 - y}{x} \end{bmatrix}$$

Taking $(x,y) = (\frac{\pi}{2}, \frac{\pi}{2})$, then $\frac{\partial x}{\partial u} = \frac{2}{\pi^2}, \cdots$. □

❏ Applications for Inverse Function Theorem

(I) (**Change of Variables**)



For example, let $g(s,t) : D \to \mathbb{R}$ Find the maximum of g on D. We define $h(x,y) = g\big(f(x,y)\big) : K \to \mathbb{R}$. Hence, we consider the extreme problem for h on K.

(II) (**Geometric Application**)
A surface $S \subseteq \mathbb{R}^3$ is locally a graph of a function define on an open subset $\mathcal{V} \subseteq \mathbb{R}^2$.

$$S = \left\{(x,y,z) \;\middle|\; x = x(u,v),\; y = y(u,v),\; z = z(u,v)\right\}.$$

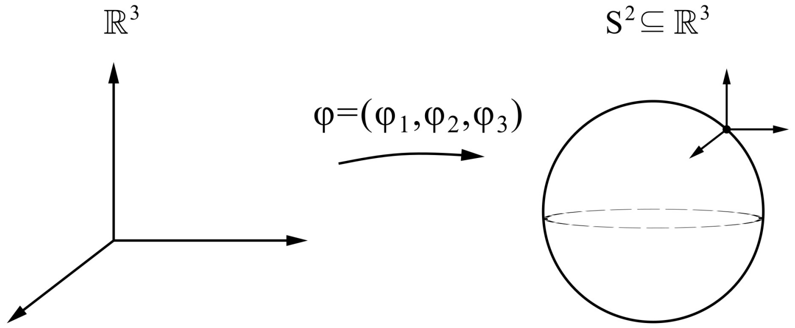
S is parametrized by two variables. It is reasonable to think z as a function of x, y (or y as a function of x, z; or x as a function of y, z) locally. If that $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$, by Inverse Function Theorem, there exists a one-to-one correspondence between the variables (u, v) and (x, y) locally. Hence, $u = u(x, y)$ and $v = v(x, y)$. We have

$$S = \Big\{ \Big(x, y, z\big(u(x,y), v(x,y)\big) \Big) \Big\}.$$

(III) (**PDE Applications**)
Consider the wave equation $u(t, x_1, x_2, x_3)$ where $\mathbf{x} = (x_1, x_2, x_3)$ satisfies

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x}) + \sum_{ij=1}^{3} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}(\mathbf{x}) = 0 \tag{5.15}$$

which is defined on $S^2 \subseteq \mathbb{R}^3$.



Define $v(t, x_1, x_2, x_3) = u(t, \phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \phi_3(\mathbf{x}))$. Then we can convert the wave equation (5.15) into

$$\frac{\partial^2 v}{\partial t^2}(\mathbf{x}) = \sum_{ij=1}^{3} b_{ij}(\mathbf{x}) \frac{\partial^2 v}{\partial x_i \partial x_j}(\mathbf{x}) = 0$$

(IV) (**Others**) Lagrange Multipliers, etc

❏ Open Mappings

Definition 5.8.4. Let X and Y be two metric spaces and $f : X \to Y$, $x_0 \in X$.

(1) We say that f is an "*open mapping*" if for every open set $\mathcal{U} \subseteq X$, $f(\mathcal{U})$ is open in Y.

(2) We say that f is a "*local open mapping at* x_0" if there exists an open neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U})$ is open in Y.

Remark. If f^{-1} is continuous, then f is an open mapping.

Corollary 5.8.5. *(1) If* $\mathbf{f} \in C^1(\mathcal{U}; \mathbb{R}^n)$ *and* $D\mathbf{f}(\mathbf{x})$ *is invertible for every* $\mathbf{x} \in \mathcal{U}$*, then* $\mathbf{f}(\mathcal{W})$ *is an open subset of* $\mathbb{R}^n$ *for every open set* $\mathcal{W} \subseteq \mathcal{U}$*. That is,* $\mathbf{f}$ *is an open mapping of* $\mathcal{U}$ *into* $\mathbb{R}^n$*.*

(2) If $\mathbf{f} \in C^1(\mathcal{U}; \mathbb{R}^n)$ *and* $D\mathbf{f}(\mathbf{x}_0)$ *is invertible, then* $\mathbf{f}$ *is a local open mapping at* $\mathbf{x}_0$*.*

5.9 The Implicit Function Theorem

Recall: (Implicit Differentiation) Consider

$$x^2y + xy^5 = 2$$

Find $\dfrac{dy}{dx}$ at $(1, 1)$. Differentiating the both sides with respect to x,

$$2xy + x^2\frac{dy}{dx} + y^5 + 5xy^4\frac{dy}{dx} = 0$$

Then

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \left.\frac{-(2xy + y^5)}{x^2 + 5xy^4}\right|_{(x,y)=(1,1)} - \frac{1}{2}.$$

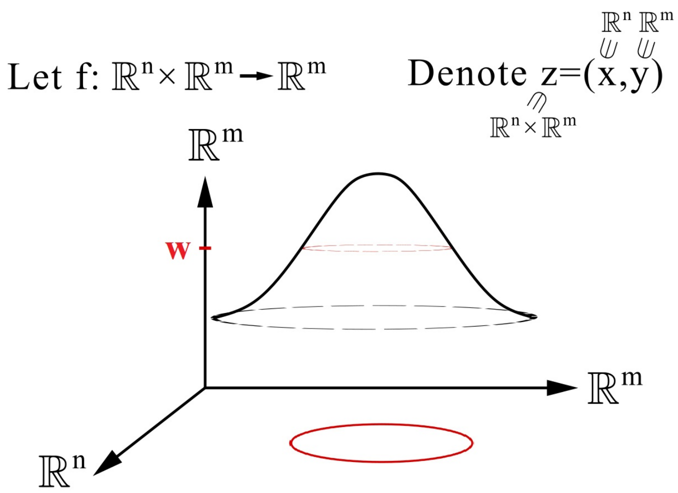
Similarly, we can compute $\left.\dfrac{dx}{dy}\right|_{(x,y)=(1,1)} = -2.$

As x and y satisfy the equation $x^2y + xy^5 = 2$, we can regard y as a function of x, or x as a function of y.

Question: For a function $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\ell$, suppose that $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfies $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Can we express $\mathbf{y}$ as a function in $\mathbf{x}$? That is, $\mathbf{y} = \mathbf{y}(\mathbf{x})$ such that

$$\mathbf{F}\big(\mathbf{x}, \mathbf{y}(\mathbf{x})\big) = \mathbf{0}.$$

Let $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Denote $\overset{\in \mathbb{R}^{n+m}}{\mathbf{z}} = (\overset{\in \mathbb{R}^n}{\mathbf{x}}, \overset{\in \mathbb{R}^m}{\mathbf{y}})$.



There must be some $\mathbf{w} \in Range(\mathbf{f})$ such that the level set $\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m} \mid \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{w}\}$ (the preimage of $\mathbf{w}$ under $\mathbf{f}$) contains infinitely many points. Heuristically, it is a n-dimensional (geometric) object.

Question: (Geometry) Is the level set a geometric surface? How smooth is it?
(Analysis) Can we express $\mathbf{y}$ as a function of $\mathbf{x}$ such that preimage of $\mathbf{w}$ under $\mathbf{f}$ is the graph of $\mathbf{y}(\mathbf{x})$?

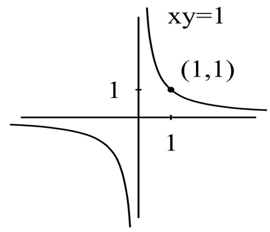
Example 5.9.1. Consider the equation

$$x^2y + xy^5 = 2.$$

Let $f(x,y) = x^2y + xy^5$. Then f maps from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ and the point $(1,1)$ is on the level set with $f(x,y) = 2$. We may think whether the level set is locally a graph of function $y = y(x)$ or $x = x(y)$.
Heuristically, the level set is a 1 dimension curve. It is supposed to be expressed by a single variable.

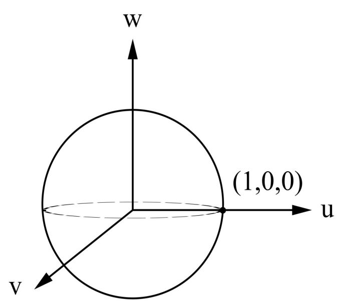
Example 5.9.2. For example, let $f(x,y) = xy : \mathbb{R}^{1+1} \to \mathbb{R}$. Then $f(1,1) = 1$ and the preimagle

$$f^{-1}(1) = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, xy = 1\right\} = \left\{\big(x, y(x)\big) \in \mathbb{R}^2 \,\middle|\, y(x) = \frac{1}{x}\right\}.$$



Hence, the preimage of 1 under f containing $(1,1)$ is the graph of $y(x) = \dfrac{1}{x}$ and $f\big(x, y(x)\big) = 1$.

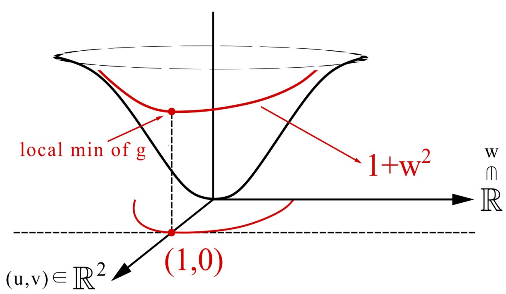
Example 5.9.3. Let $f(\underbrace{u,v}_{\mathbb{R}^2},\underbrace{w}_{\mathbb{R}}) = u^2 + v^2 + w^2$ with $f(1,0,0) = 1$. Then



the preimage (level set) of 1 under f is the sphere

$$\begin{aligned} S &= \left\{(u,v,w) \in \mathbb{R}^3 \,\middle|\, u^2 + v^2 + w^2 = 1\right\} \\ &= \left\{(u,v,w) \in \mathbb{R}^3 \,\middle|\, f(u,v,w) = 1\right\} \end{aligned}$$

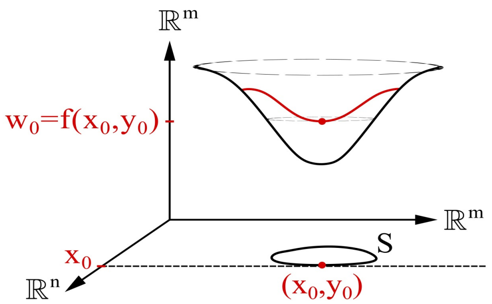
Question: Is there a function $w = w(u,v)$ (locally) such that $\big(u, v, w(u,v)\big) \subseteq S$?
Answer: No!, Clearly, by vertical line test, the surface is not a graph of a single function $w = w(u,v)$ near $(1,0,0)$.
Question: What's happen at $(1,0,0)$?



Let $g(w) = f(1,0,w) = 1 + w^2 : \mathbb{R} \to \mathbb{R}$. $g'(0) = 0$ and g has a local extreme value at 1. Hence, the graph of g will go forward and backward. Also, $g'(0) = f_w(1,0,0) = 0$.

In general, $\mathbf{f}(\mathbf{x},\mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. For $(\mathbf{x}_0, \mathbf{y}_0) \in \mathbb{R}^n \times \mathbb{R}^m$. Consider the level set $S = \left\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \,\middle|\, \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right\}$.

Suppose that there exists no function $\mathbf{y} = \mathbf{y}(\mathbf{x})$ such that S = the graph of $\mathbf{y}(\mathbf{x})$ near $(\mathbf{x}_0, \mathbf{y}_0)$. What's happen?



Let $\mathbf{g}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0, \mathbf{y}) : \mathbb{R}^m \to \mathbb{R}^m$. If $m = 1$, there exists a local minimum of $\mathbf{g}$ at $\mathbf{y}_0$. If $m > 1$, the space $\{(\mathbf{x}_0, \mathbf{y})\}$ will be tangent to the level set S at $(\mathbf{x}_0, \mathbf{y_0})$. Since any curve $\mathbf{r}(t)$ on S $\mathbf{g}$ is not invertible near $\mathbf{y} = \mathbf{y}_0$, then $D_\mathbf{y}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = D\mathbf{g}(\mathbf{y}_0)$ is not invertible.

■ Linear Maps

Let $L \in \mathcal{B}(\mathbb{R}^{n+m}; \mathbb{R}^m)$. We can split L into two linear maps $L_\mathbf{x} \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^m)$ and $L_\mathbf{y} \in \mathcal{B}(\mathbb{R}^m; \mathbb{R}^m)$ by

$$L_\mathbf{x}\mathbf{h} = L(\underbrace{\mathbf{h}}_{\in\mathbb{R}^n}, \underbrace{\mathbf{0}_m}_{\in\mathbb{R}^m}) \text{ and } L_\mathbf{y}\mathbf{k} = L(\underbrace{\mathbf{0}_n}_{\in\mathbb{R}^n}, \underbrace{\mathbf{k}}_{\in\mathbb{R}^m})$$

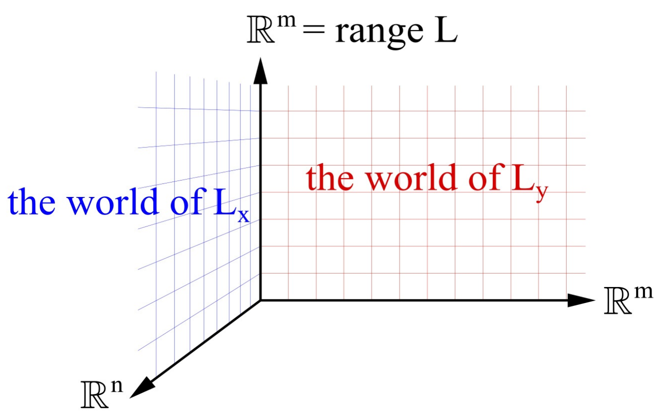
where $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{k} \in \mathbb{R}^m$ and $(\mathbf{h}, \mathbf{k}) \in \mathbb{R}^{n+m}$. Hence,

$$L(\mathbf{h}, \mathbf{k}) = L(\mathbf{h}, \mathbf{0}_m) + L(\mathbf{0}_n, \mathbf{k}) = L_\mathbf{x}\mathbf{h} + L_\mathbf{y}\mathbf{k}$$

Write

$$[L]_{m\times(n+m)} = [\underbrace{L_\mathbf{x}}_{m\times n} | \underbrace{L_\mathbf{y}}_{m\times m}] = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mm} \end{array}\right]$$

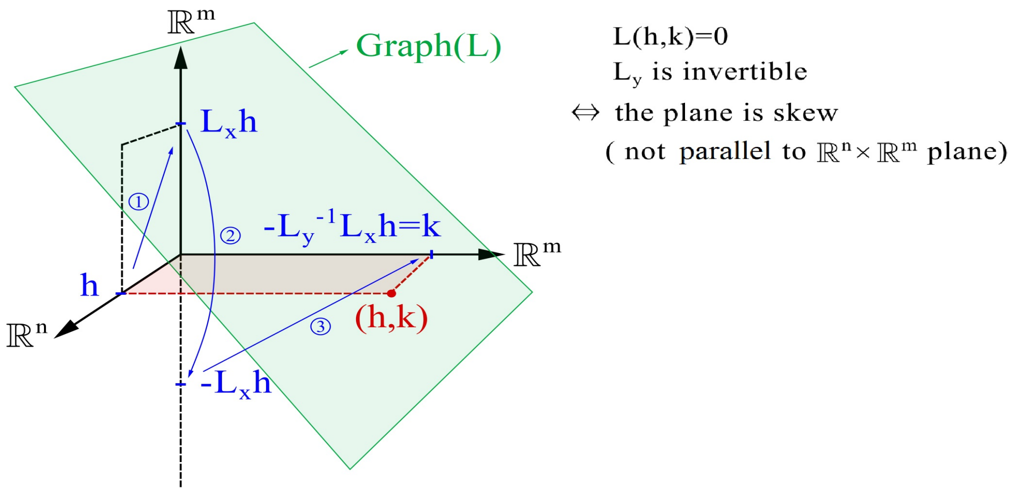
$$\qquad\qquad\qquad\qquad L_\mathbf{x} \qquad\qquad\qquad L_\mathbf{y}$$



If $rank(L) = m$, then the dimensions of $Ker(L) = n$. That is, the level set of $\mathbf{0}_m$ under L has dimension n. Also, $Ker(L)$ is the graph of a function of variable $\mathbf{x}$. In other words, there exists a function $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that $Ker(L) = \{(\mathbf{h}, \mathbf{k}(\mathbf{h})) \mid \mathbf{h} \in \mathbb{R}^n\}$.

Theorem 5.9.4. *If $L \in \mathcal{B}(\mathbb{R}^{n+m}; \mathbb{R}^m)$ and $L_\mathbf{y}$ is invertible, then there corresponds to every $\mathbf{h} \in \mathbb{R}^n$ a unique $\mathbf{k} \in \mathbb{R}^m$ such that $L(\mathbf{h}, \mathbf{k}) = \mathbf{0}_m$. (That is, $\mathbf{k} = \mathbf{k}(\mathbf{h})$ is a function of $\mathbf{h}$). Moreover, $\mathbf{k}$ can be computed from $\mathbf{h}$ by*

$$\mathbf{k} = -(L_\mathbf{y})^{-1} L_\mathbf{x}\mathbf{h}.$$



Proof. Since $L(\mathbf{h},\mathbf{k}) = L_{\mathbf{x}}\mathbf{h} + L_{\mathbf{y}}\mathbf{k}$, we have

$$L(\mathbf{h},\mathbf{k}) = \mathbf{0}_m \quad \text{if and only if} \quad L_{\mathbf{x}}\mathbf{h} + L_{\mathbf{y}}\mathbf{k} = \mathbf{0}_m.$$

Thus, if $L(\mathbf{h},\mathbf{k}) = \mathbf{0}_m$ and $L_{\mathbf{y}}$ is invertible, then $\mathbf{k} = -L_{\mathbf{y}}^{-1}L_{\mathbf{x}}\mathbf{h}$. □

Moreover, if $L(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{w}_0$, then the preimage of $\mathbf{w}_0$ is $L^{-1}(\mathbf{w}_0) = (\mathbf{x}_0,\mathbf{y}_0) + Ker(L)$ is also the graph of a function of variable $\mathbf{x}$. In other words, there exists a function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ such that the preimage of $\mathbf{w}_0$ under L is $L^{-1}(\mathbf{w}_0) = \{(\mathbf{x},\mathbf{g}(\mathbf{x})) \mid \mathbf{x} \in \mathbb{R}^n\}$.

On the other hands, if $rank(L) < m$, then $L^{-1}(\mathbf{w}_0)$ has dimension greater than n and it must not be a graph of a function of variable $\mathbf{x}$. That is, for $L(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{w}_0$, there exists $\mathbf{y}_1 \in \mathbb{R}^m$ and $\mathbf{y}_0 \neq \mathbf{y}_1$ such that $L(\mathbf{x}_0,\mathbf{y}_0) = L(\mathbf{x}_0,\mathbf{y}_1)$. Thus,

$$L_{\mathbf{x}}\mathbf{x}_0 + L_{\mathbf{y}}\mathbf{y}_0 = L(\mathbf{x}_0,\mathbf{y}_0) = L(\mathbf{x}_0,\mathbf{y}_1) = L_{\mathbf{x}}\mathbf{x}_0 + L_{\mathbf{y}}\mathbf{y}_1.$$

Therefore,

$$L_{\mathbf{y}}\mathbf{y}_0 = L_{\mathbf{y}}\mathbf{y}_1.$$

We have $L_{\mathbf{y}} : \mathbb{R}^m \to \mathbb{R}^m$ is not invertible. This implies the $m \times m$ matrix

$$\begin{bmatrix} b_{11} & \cdots & b_1 m \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix}$$

is not invertible.

<u>*Notation:*</u> Let $\mathbf{F} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ where $\mathbf{F} = (F_1, \cdots, F_m)$, $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, , y_m)$. Denote

$$[D\mathbf{F}] = \left[\begin{array}{ccc|ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array}\right]$$

$$D_{\mathbf{x}}\mathbf{F} \qquad\qquad D_{\mathbf{y}}\mathbf{F}$$

Theorem 5.9.5. *(Implicit Function Theorem) Let $D \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{F} : D \to \mathbb{R}^m$ be a function of class C^r, $r \in \mathbb{N}$. Suppose that $\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}_m$ for some $(\mathbf{x}_0, \mathbf{y}_0) \in D$ and*

$$\left[D_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)\right] = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{bmatrix} (\mathbf{x}_0, \mathbf{y}_0)$$

is invertible. Then there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$, an open neighborhood $\mathcal{V} \subseteq \mathbb{R}^m$ of $\mathbf{y}_0$ and $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ such that

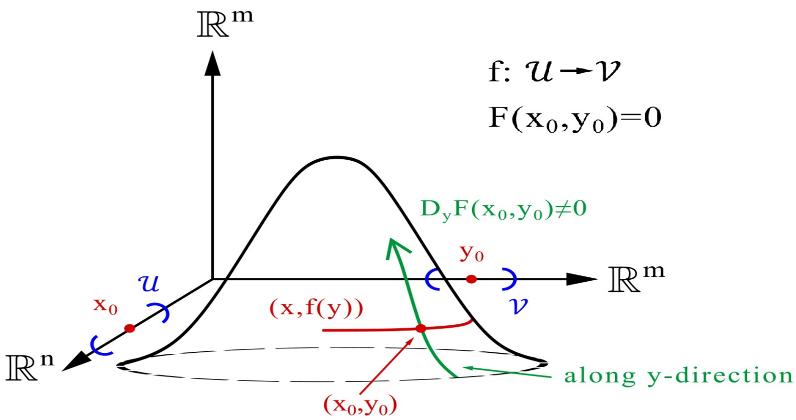
(1) $\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{0}_m$ *for every* $\mathbf{x} \in \mathcal{U}$.

(2) $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$.

(3) $D\mathbf{f}(\mathbf{x}) = -\left[D_{\mathbf{y}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\right]^{-1}\left[D_{\mathbf{x}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\right]$ *for every* $\mathbf{x} \in \mathcal{U}$ *where*

$$\left[\big(D_{\mathbf{x}}\mathbf{F}\big)(\mathbf{x}, \mathbf{y})\right] = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} (\mathbf{x}, \mathbf{y})$$

(4) $\mathbf{f}$ *is of class* C^r



Proof. Denote $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ and $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$ and $\mathbf{y}, \mathbf{v} \in \mathbb{R}^m$. Define $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \big(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})\big)$. Then $\mathbf{G} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ and

$$\left[D\mathbf{G}(\mathbf{x}, \mathbf{y})\right] = \begin{bmatrix} I_n & \mathbf{0}_{n\times m} \\ D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y}) & D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y}) \end{bmatrix}$$

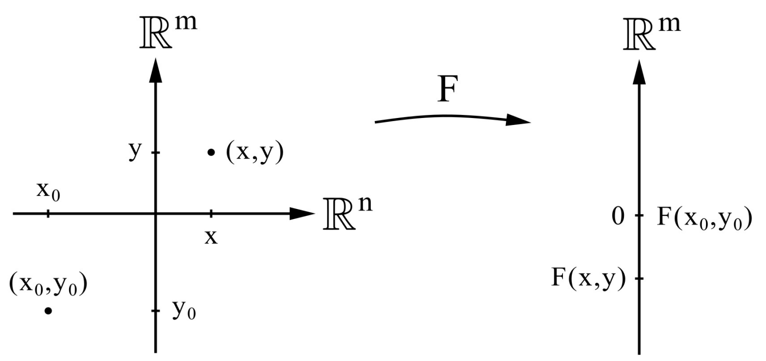
where I_n is the $n \times n$ identity matrix. Since the matrix $\left[D_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)\right]$ is invertible and $\mathbf{F} \in C^r$, the matrix $\left[D\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)\right]$ is invertible, $\mathbf{G} \in C^r$.

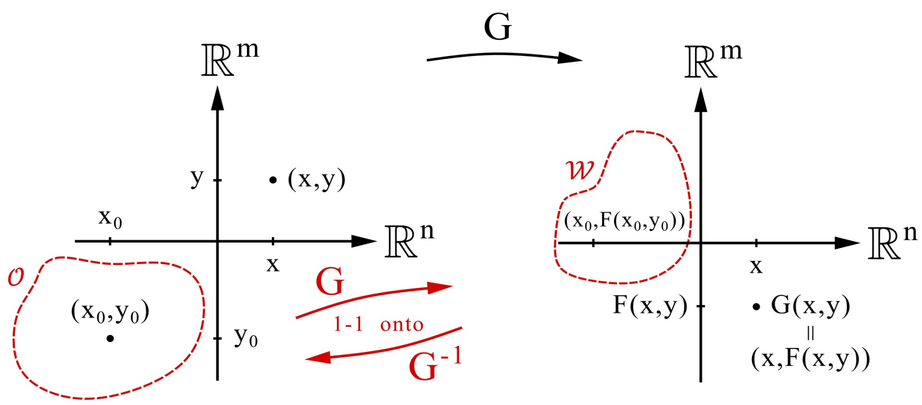
By the Inverse Function Theorem, there are an open neighborhood O of $(\mathbf{x}_0, \mathbf{y}_0)$ and an open neighborhood $\mathcal{W}$ of $\big(\mathbf{x}_0, \mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)\big) = (\mathbf{x}_0, \mathbf{0}_m)$ such that

(i) $\mathbf{G} : \mathcal{O} \to \mathcal{W}$ is one-to-one and onto.

(ii) the inverse function $\mathbf{G}^{-1} : \mathcal{W} \to \mathcal{O}$ is of class C^r.

(iii) $D\mathbf{G}^{-1}\big(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})\big) = \big(D\mathbf{G}(\mathbf{x}, \mathbf{y})\big)^{-1}$ for every $(\mathbf{x}, \mathbf{y}) \in \mathcal{O}$.



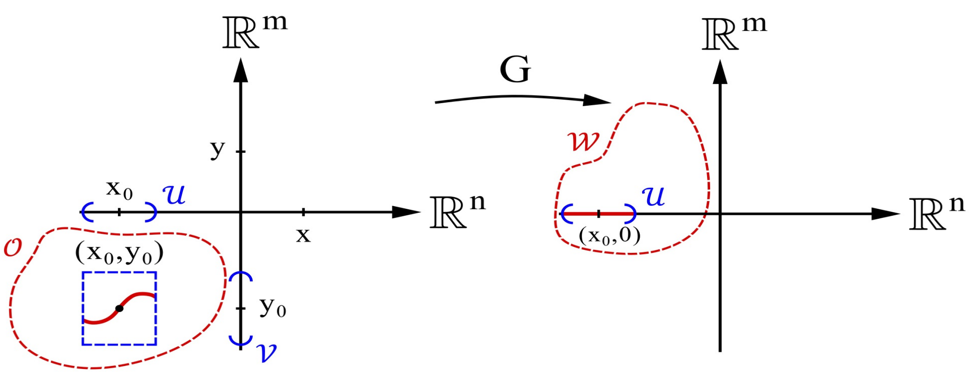


Choose an open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ and an open neighborhood $\mathcal{V}$ of $\mathbf{y}_0$ such that

(a) $(\mathbf{x}, \mathbf{0}_m) \in \mathcal{W}$ for every $\mathbf{x} \in \mathcal{U}$;

(b) $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{O}$;

(c) $\mathbf{G}^{-1}(\mathbf{x}, \mathbf{0}_m) \in \mathcal{U} \times \mathcal{V}$ for every $\mathbf{x} \in \mathcal{U}$.



Hence, if $\mathbf{x} \in \mathcal{U}$, then $(\mathbf{x}, \mathbf{0}_m) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \big(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})\big)$ for some $\mathbf{y} \in \mathcal{V}$ since $O \xrightarrow{\mathbf{G}} \mathcal{W}$ is bijective. Then $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}_m$ for this $\mathbf{y}$.

So far, we have shown that for every $\mathbf{x} \in \mathcal{U}$, there exists $\mathbf{y} \in \mathcal{V}$ such that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}_m$. Now, we will show that $\mathbf{y}$ is the unique point in $\mathcal{V}$ such that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}_m$ and hence $\mathbf{x} \to \mathbf{y}$ is a function.

With the same $\mathbf{x}$, suppose that there exists $\mathbf{y}' \in \mathcal{V}$ such that $\mathbf{F}(\mathbf{x}, \mathbf{y}') = \mathbf{0}_m$. Then

$$\mathbf{G}(\mathbf{x}, \mathbf{y}') = \big(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y}')\big) = (\mathbf{x}, \mathbf{0}_m) = \big(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})\big) = \mathbf{G}(\mathbf{x}, \mathbf{y})$$

Since $\mathbf{G}$ is one-to-one, $\mathbf{y} = \mathbf{y}'$ and hence we can define $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ such that $\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{0}_m$. Moreover, due to $\mathbf{G}$ is one-to-one, $\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{x}_0, \mathbf{0}_m) = \mathbf{G}\big(\mathbf{x}_0, \mathbf{f}(\mathbf{x}_0)\big)$. Then $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$.

Thanks to $\underbrace{\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{G}^{-1}(\mathbf{x}, \mathbf{0}_m)}_{(*)}$ and $\mathbf{G}^{-1}$ is of class C^r, $\mathbf{f}$ is of class C^r. The statement (4) is proved.

For $(\mathbf{u}, \mathbf{v}) = \mathbf{G}(\mathbf{x}, \mathbf{y})$, since $D\mathbf{G}^{-1}(\mathbf{u}, \mathbf{v}) = \big(D\mathbf{G}(\mathbf{x}, \mathbf{y})\big)^{-1}$,

$$\big[D\mathbf{G}(\mathbf{x}, \mathbf{y})\big]^{-1} = \begin{bmatrix} I_n & \mathbf{0}_{n\times m} \\ D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y}) & D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y}) \end{bmatrix}^{-1} = \begin{bmatrix} I_n & \mathbf{0}_{n\times m} \\ -\big(D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y})\big)^{-1}\big(D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y})\big) & \big(D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y})\big)^{-1} \end{bmatrix}$$

By $(*)$,

Let $\mathbf{H}(\mathbf{x}) = \big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) : \mathbb{R}^n \to \mathbb{R}^{n+m}$. Then

$$\big[D\mathbf{H}(\mathbf{x})\big] = \begin{bmatrix} I_n \\ D\mathbf{f}(\mathbf{x}) \end{bmatrix} = \big[D_{\mathbf{x}}\mathbf{G}^{-1}(\mathbf{x}, \mathbf{0}_m)\big]$$

Note that $\mathbf{G}^{-1}(\mathbf{x}, \mathbf{0}_m) = (\mathbf{x}, \mathbf{f}(\mathbf{x}))$. Thus,

$$D\mathbf{f}(\mathbf{x}) = \big[D_{\mathbf{y}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\big]^{-1}\big[D_{\mathbf{x}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\big]. \tag{5.16}$$

Check (5.16).

Consider $\mathbf{F} : D \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m$, $D\mathbf{F}(\mathbf{x}, \mathbf{y}) \in \mathcal{B}\big(\mathbb{R}^{n+m}; \mathbb{R}^n\big)$. Let $(\mathbf{h}, \mathbf{k}) \in \mathbb{R}^{n+m}$ be a vector.

$$\big(D\mathbf{F}(\mathbf{x}, \mathbf{y})\big)(\mathbf{h}, \mathbf{k}) = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y})\mathbf{h} + D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y})\mathbf{k} \tag{5.17}$$

Define $\Phi(\mathbf{x}) = \big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$. Then

$$D\Phi(\mathbf{x}) = \big(Id, D\mathbf{f}(\mathbf{x})\big) \quad \in \mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^{n+m}\big).$$

For $\mathbf{h} \in \mathbb{R}^n$,

$$D\Phi(\mathbf{x})\mathbf{h} = (\mathbf{h}, D\mathbf{f}(\mathbf{x})\mathbf{h}) \quad \in \mathbb{R}^{n+m}. \tag{5.18}$$

Since $\mathbf{F}\big(\Phi(\mathbf{x})\big) \equiv \mathbf{0}_m$ ($\mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$), by the chain rule,

$$\mathcal{B}\big(\mathbb{R}^n;\mathbb{R}^m\big) \ni \mathbf{0} = D\Big(\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big) = \underbrace{D\mathbf{F}\big(\Phi(\mathbf{x})\big)}_{\mathcal{B}(\mathbb{R}^{n+m};\mathbb{R}^m)} \underbrace{D\Phi(\mathbf{x})}_{\mathcal{B}(\mathbb{R}^n;\mathbb{R}^{n+m})}$$

For every $\mathbf{h} \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbf{0}_m = D\Big(\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big)\mathbf{h} &= D\mathbf{F}\big(\Phi(\mathbf{x})\big)D\Phi(\mathbf{x})\mathbf{h} \\
&\overset{(5.18)}{=} D\mathbf{F}\big(\Phi(\mathbf{x})\big)\big(\mathbf{h}, D\mathbf{f}(\mathbf{x})\mathbf{h}\big) \\
&\overset{(5.17)}{=} D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\mathbf{h} + D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)D\mathbf{f}(\mathbf{x})\mathbf{h}
\end{aligned}$$

Then

$$-D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\mathbf{h} = D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)D\mathbf{f}(\mathbf{x})\mathbf{h}.$$

Thus,

$$D\mathbf{f}(\mathbf{x})\mathbf{h} = -\Big(D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big)^{-1} D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\mathbf{h}.$$

We have

$$D\mathbf{f}(\mathbf{x}) = -\Big(D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big)^{-1} D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big).$$

Moreover, consider $\Phi(\mathbf{x}) = \big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)$ and $\mathbf{F} : D \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m$. Then

$$\big[D\Phi(\mathbf{x})\big] = \begin{bmatrix} I_n \\ D\mathbf{f}(\mathbf{x}) \end{bmatrix}_{(n+m)\times n} \quad \text{and} \quad \big[D\mathbf{F}(\mathbf{x})\big] = \Big[\underbrace{D_{\mathbf{x}}\mathbf{F}}_{m\times n} \;\; \underbrace{D_{\mathbf{y}}\mathbf{F}}_{m\times m}\Big]_{m\times(n+m)}$$

$$\begin{aligned}
\Big[D\Big(\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big)\Big] &= \Big[D\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big]\,\big[D\Phi(\mathbf{x})\big] = \Big[D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big) \quad D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)\Big] \begin{bmatrix} I_n \\ D\mathbf{f}(\mathbf{x}) \end{bmatrix} \\
&= \Big[D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big) + D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)D\mathbf{f}(\mathbf{x})\Big]_{m\times n} = \big[0\big]_{m\times n}
\end{aligned}$$

Therefore,

$$D_{\mathbf{x}}\mathbf{F}\big(\Phi(\mathbf{x})\big) + D_{\mathbf{y}}\mathbf{F}\big(\Phi(\mathbf{x})\big)D\mathbf{f}(\mathbf{x}) = \mathbf{0}_m$$

□

Remark. (1) In the Implicit Function Theorem, we can generally write the value $\mathbf{0}_m$ as $\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$. Then the statement (1) is changed by $\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$ for every $\mathbf{x} \in \mathcal{U}$.

(2) In the Implicit Function Theorem, $\mathbf{F} : D \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m$. The variables $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ are only notations. We only concern the hypothesis $D_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$ is invertible.

For example, if $\mathbf{F}(x_1, x_2, x_3, x_4, x_5) : \mathbb{R}^5 \to \mathbb{R}^2$ is a C^1 mapping where $\mathbf{F} = (F_1, F_2)$. Suppose that

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_5} \end{bmatrix} (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0)$$

is invertible. Then $\mathbf{x} = (x_1, x_3, x_4)$ and $\mathbf{y} = (x_2, x_5)$ as well as $\mathbf{x}_0 = (x_1^0, x_3^0, x_4^0)$ and $\mathbf{y}_0 = (x_2^0, x_5^0)$.

By the Implicit Function Theorem, there exist an open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$, an open neighborhood $\mathcal{V}$ of $\mathbf{y}_0$ and a C^1 mapping $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ such that $\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$.

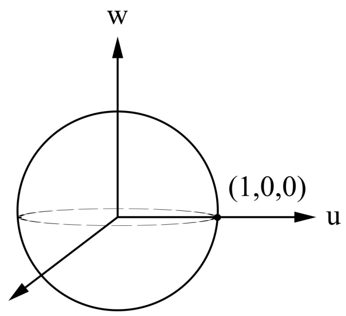
For example, $f(u, v, w) = u^2 + v^2 + w^2$ and $\big[Df\big] = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} = \begin{bmatrix} 2u & 2v & 2w \end{bmatrix}$.

At $(0, 1, 0)$, $\frac{\partial f}{\partial v} = 2 \neq 0$. Then $\mathbf{y} = v$ and $\mathbf{x} = (u, w)$ as well as $\mathbf{x}_0 = (0, 0)$ and $\mathbf{y}_0 = 1$. By the Implicit Function Theorem, there exist an open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$, an open neighborhood $\mathcal{V}$ of $\mathbf{y}_0$ and a function $g : \mathcal{U} \to \mathcal{V}$ such that $f\big(u, g(u, w), w\big) = 1$.

Geometrically, the sphere

$$S = \big\{(u, v, w) \mid u^2 + v^2 + w^2 = 1\big\}$$

can be expressed as the graph of the function $g(u, w)$ near the point $(0, 1, 0)$.

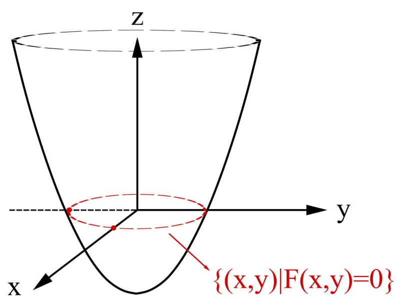


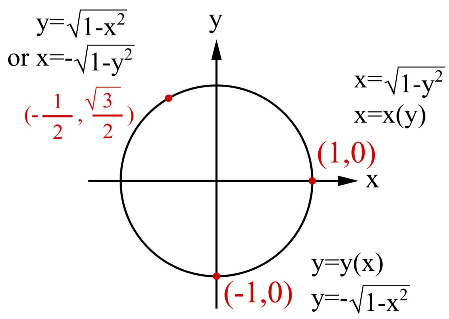
Example 5.9.6. Let $F(x, y) = x^2 + y^2 - 1$.

(i) At $(1, 0)$, $D_x F(1, 0) = 2 \neq 0$. By the implicit function theorem, near $(1, 0)$, $x = x(y)$ such that $F\big(x(y), y\big) = 0$.

(ii) At $(0, -1)$, $D_y F(0, -1) = -2 \neq 0$. By the implicit function theorem, near $(0, -1)$, $y = y(x)$ such that $F\big(x, y(x)\big) = 0$.

(iii) At $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $D_x F(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = -1 \neq 0$ and $D_y F(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \sqrt{3} \neq 0$. By the implicit function theorem, x can be expressed as a function of y, say $x = x(y)$ such that $F\big(x(y), y\big) = 0$. Similary, y can be expressed as a function of x, say $y = y(x)$ such that $F\big(x, y(x)\big) = 0$.





To find $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\big(D_y F(x,y)\big)^{-1}\big(D_x F(x,y)\big) = -\frac{2y}{2x}\Big|_{(x,y)=(x_0,y_0)}$$

(i) (differentiation of single variable function) $F(x,y)=0 \Leftrightarrow x^2+y^2=1$. Then

$$\frac{d}{dx}(x^2+y^2) = 2x + 2y\frac{dy}{dx} = 0.$$

We have $\frac{dy}{dx} = -\frac{2y}{2x}$.

(ii) (partial derivative of two variables function) $F\big(x, y(x)\big) = 0$. Then

$$\frac{d}{dx}\Big(F\big(x, y(x)\big)\Big) = F_x + F_y \cdot \frac{dy}{dx} = 0.$$

Then $\frac{dy}{dx} = -\frac{F_y}{F_x} = -\frac{2y}{2x}$.

Example 5.9.7. Consider the equation

$$\begin{cases} xu + yv^2 = 0 \\ xv^3 + y^2u^6 = 0 \end{cases} \quad \text{near } (x_0, y_0, u_0, v_0) = (1, -1, 1, -1). \tag{5.19}$$

Let $\mathbf{F}(x,y,u,v) = (\underbrace{xu + yv^2}_{F_1}, \underbrace{xv^3 + y^2u^6}_{F_2})$. Then

$$\big[D_{x,y}\mathbf{F}\big]_{(1,-1,1,-1)} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}_{(1,-1,1,-1)} = \begin{bmatrix} u & v^2 \\ v^3 & 2yu^6 \end{bmatrix}_{(1,-1,1,-1)} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{is invertible.}$$

By the implicit function theorem, to satisfy the equation (5.19), (x,y) can be expressed as a function of (u,v), say $x = g_1(u,v)$, $y = g_2(u,v)$ near $(1,-1)$ such that

$$\mathbf{F}\big(x(u,v), y(u,v), u, v\big) = \mathbf{F}(1,-1,1,-1) = (0,0)$$

Let $(x,y) = \mathbf{g}(u,v) = \big(g_1(u,v), g_2(u,v)\big)$. Then

$$D\mathbf{g}(u,v) = -\big[D_{x,y}\mathbf{F}(x,y,u,v)\big]^{-1}\big[D_{u,v}\mathbf{F}(x,y,u,v)\big].$$

Example 5.9.8. Consider the equation $\mathbf{f}(x,y,z) = (\underbrace{xe^y + ye^z}_{f_1(x,y,z)}, \underbrace{xe^z + ze^y}_{f_2(x,y,z)}) : \mathbb{R}^3 \to \mathbb{R}^2$ near $(-1,1,1)$.

$$\left[D_{y,z}\mathbf{f}\right]_{(-1,1,1)} = \begin{bmatrix} \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{bmatrix}_{(-1,1,1)} = \begin{bmatrix} xe^y + e^z & ye^z \\ ze^y & xe^z + e^y \end{bmatrix}_{(-1,1,1)} = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix} \quad \text{is invertible.}$$

By the implicit function theorem, to satisfy $f(x,y,z) = f(-1,1,1)$, y,z can be expressed as a function of x, say $y = g_1(x)$, $z = g_2(x)$ such that $f\big(x, y(x), z(x)\big) = f(-1,1,1)$ near -1.

Let $\mathbf{g}(x) = (y,z) = \big(g_1(x), g_2(x)\big)$. Then

$$\left[D\mathbf{g}(x)\right] = -\left[D_{y,z}\mathbf{f}(x,y,z)\right]^{-1}\left[D_x\mathbf{f}(x,y,z)\right] = -\begin{bmatrix} xe^y + e^z & ye^z \\ ze^y & xe^z + e^y \end{bmatrix}^{-1}\begin{bmatrix} e^y \\ e^z \end{bmatrix}$$

Example 5.9.9. Let $\mathbf{f} : \mathbb{R}^{2+3} \to \mathbb{R}^2$ where $\mathbf{f} = (f_1, f_2)$ is given by

$$\begin{aligned} f_1(x_1,x_2,y_1,y_2,y_3) &= 2e^{x_1} + x_2y_1 - 4y_2 + 3 \\ f_2(x_1,x_2,y_1,y_2,y_3) &= x_2\cos x_1 - 6x_1 + 2y_1 - y_3 \end{aligned}$$

Let $\mathbf{x}_0 = (0,1)$ and $\mathbf{y}_0 = (3,2,7)$. Then $\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}$. Consider

$$\left[D\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right] = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix}$$

We have

$$\left[D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} \quad \text{and} \quad \left[D_{\mathbf{y}}\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

Then, $\left[D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right]$ is invertible. By the Implicit Function Theorem, there exist an open neighborhood $\mathcal{V}$ of $(0,1)$, an open neighborhood $\mathcal{U}$ of $(3,2,7)$ and a C^1-mapping $\mathbf{g} = (g_1,g_2) : \mathcal{U} \to \mathcal{V}$ such that

$$\mathbf{f}\big(\mathbf{g}(\mathbf{y}),\mathbf{y}\big) = \mathbf{0}.$$

Moreover,

$$\begin{aligned} \left[D\mathbf{g}(3,2,7)\right] &= \begin{bmatrix} \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_2} & \dfrac{\partial g_1}{\partial y_3} \\ \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_2} & \dfrac{\partial g_2}{\partial y_3} \end{bmatrix}(3,2,7) = \left[D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right]^{-1}\left[D_{\mathbf{y}}\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)\right] \\ &= \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -5 & -4 & 3 \\ 10 & -24 & -2 \end{bmatrix} \end{aligned}$$

❑ Prove the Inverse Function Theorem by using the Implicit Function Theorem

■ **Implicit Function Theorem**

Let $\mathbf{F} : D \to\subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m$ is of class C^1. Denote $\mathbf{z} = (\underset{\in\mathbb{R}^n}{\mathbf{x}}, \underset{\in\mathbb{R}^m}{\mathbf{y}})$. If $D_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$ is invertible and $\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}_m$, then there exist open neighborhoods $\mathcal{U} \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $\mathcal{V} \subseteq \mathbb{R}^m$ of $\mathbf{y}_0$ and $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ such that

(1) $\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = \mathbf{0}_m$ for every $\mathbf{x} \in \mathcal{U}$.

(2) $\mathbf{f}(x_0) = \mathbf{y}_0$

(3) $D\mathbf{f}(\mathbf{x}) = \Big(D_{\mathbf{y}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\Big)^{-1}\Big(D_{\mathbf{x}}\mathbf{F}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big)\Big)$

(4) if $\mathbf{F}$ is of class C^r, then so is $\mathbf{f}$.

■ **Inverse Function Theorem**

Let $\mathbf{f} : D \to\subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class C^1, $\mathbf{f}(\mathbf{a}) = \mathbf{b}$, $D\mathbf{f}(\mathbf{a})$ is invertible, then there exist open neighborhoods $\mathcal{U} \subseteq D$ of $\mathbf{a}$ and $\mathcal{V} \subseteq \mathbf{f}(D)$ of $\mathbf{b}$ such that

(a) $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is ono-to-one and onto.

(b) $\mathbf{f}^{-1} : \mathcal{V} \to \mathcal{U}$ is of class C^1.

(c) $D\mathbf{f}^{-1}(\mathbf{y}) = \big(D\mathbf{f}(\mathbf{x})\big)^{-1}$ for every $\mathbf{y} \in \mathcal{V}$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$

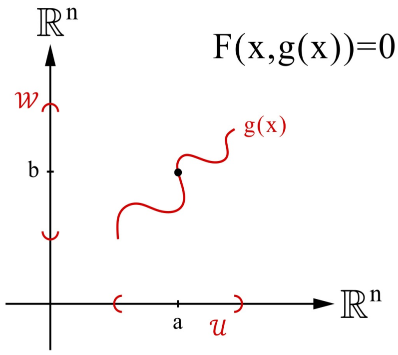
(d) if $\mathbf{f}$ is of class C^r, so is $\mathbf{f}^{-1}$.

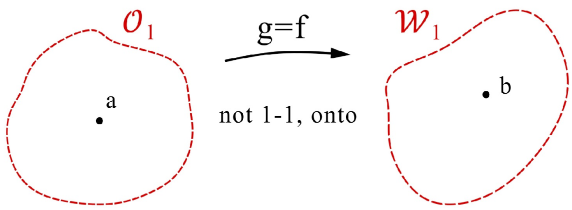
■ **Sketch the proof of the Inverse Function Theorem by using Implicit Function Theorem**

(1) Let $\mathbf{F}(\mathbf{x}, \mathbf{y}) : D \times \mathbb{R}^n \to \mathbb{R}^n$ be given by $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}) - \mathbf{y}$. Then $\mathbf{F} \in C^1$, $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}_n$ and $D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})$ is invertible.

(2) By the Implicit Function Theorem, ther exist open nbighborhoods $O_1 \subseteq D$ of $\mathbf{a}$ and $\mathcal{W}_1 \in \mathbb{R}^n$ of $\mathbf{b}$ and $\mathbf{g} : \mathcal{U} \to \mathcal{W}$ such that $\mathbf{g} : \mathcal{U} \to \mathcal{V}$ such that (1)-(4) hold. [Note that $\mathcal{U}$ and $\mathcal{V}$ is not given. Also, $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is not one-to-one and onto.]
Hence

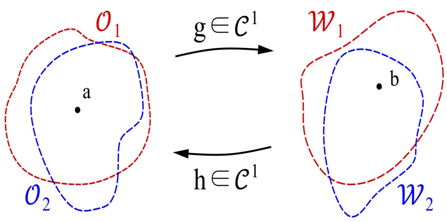
$$\begin{aligned} & \mathbf{F}\big(\mathbf{x}, \mathbf{g}(\mathbf{x})\big) = \mathbf{0}_n \quad \text{for every } \mathbf{x} \in O_1 \\ \iff & \mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x}) = \mathbf{0}_n \quad \text{for every } \mathbf{x} \in O_1 \\ \iff & \mathbf{f} = \mathbf{g} \text{ on } O_1. \end{aligned}$$





(3) Since $D\mathbf{f}(\mathbf{a})$ is invertible, $D_{\mathbf{x}}\mathbf{F}(\mathbf{a},\mathbf{b})$ is invertible. By the Implicit Function Theorem, there exists open neighborhoods $\mathcal{W}_2$ of $\mathbf{b}$ and O_2 of $\mathbf{a}$ and $\mathbf{h} : \mathcal{W}_2 \to O_2$ such that (1)-(4) hold.

$$\mathbf{F}\big(\mathbf{h}(\mathbf{y}),\mathbf{y}\big) = \mathbf{0}_n \quad \text{for every } \mathbf{y} \in \mathcal{W}_2 \quad \Longleftrightarrow \quad \mathbf{f}\big(\mathbf{h}(\mathbf{y})\big) = \mathbf{y} \quad \text{for every } \mathbf{y} \in \mathcal{W}_2.$$



(i) There exists O_1 of $\mathbf{a}$, $\mathcal{W}_1$ of $\mathbf{b}$, $\mathbf{g} : O_1 \to \mathcal{W}_1$ such that $\mathbf{F}\big(\mathbf{x},\mathbf{g}(\mathbf{x})\big) = \mathbf{0}_n$ for every $\mathbf{x} \in O_1$. Thus, $\mathbf{f} = \mathbf{g}$.

(ii) There exists $\mathcal{W}_2$ of $\mathbf{b}$, O_2 of $\mathbf{a}$, $\mathbf{h} : \mathcal{W}_2 \to O_2$ such that $\mathbf{F}\big(\mathbf{h}(\mathbf{y}),\mathbf{y}\big) = \mathbf{0}_n$ for every $\mathbf{y} \in \mathcal{W}_2$. Thus, $\mathbf{f}\big(\mathbf{h}(\mathbf{y})\big) = \mathbf{y}$ for every $\mathbf{y} \in \mathcal{W}_2$.

Let $\mathcal{U} = O_1 \cap O_2$ and $\mathcal{V} = \mathcal{W}_1 \cap \mathcal{W}_2$. To prove

(i) $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is well-defined, one-to-one and onto.

(ii) $\mathbf{h} = \mathbf{f}^{-1}$ on $\mathcal{V}$.

(iii) $D\mathbf{h}(\mathbf{y}) = \big(D\mathbf{f}(\mathbf{x})\big)^{-1}$ for every $\mathbf{y} \in \mathcal{V}$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$.

(iv) if $\mathbf{f} \in C^r$, then $\mathbf{h}$ is C^r.

Proof of (i): For $\mathbf{x} \in \mathcal{U}$, there exists $\mathbf{y} \in \mathcal{W}_2$ such that $\mathbf{x} = \mathbf{h}(\mathbf{y})$. Then

$$\mathbf{y} = \mathbf{f}\big(\mathbf{h}(\mathbf{y})\big) = \mathbf{f}(\mathbf{x}) \in \mathcal{W}_1.$$

Hence,

$$\mathbf{f}(\mathbf{x}) = \mathbf{y} \in \mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{V}.$$

We have $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is well-defined.

If $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{U}$ such that $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$, then there exists $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{W}_2$ such that $\mathbf{y}(\mathbf{y}_1) = \mathbf{x}_1$ and $\mathbf{h}(\mathbf{y}_2) = \mathbf{x}_2$. Then

$$\mathbf{y}_1 = \mathbf{f}\big(\mathbf{h}(\mathbf{y}_1)\big) = \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2) = \mathbf{f}\big(\mathbf{h}(\mathbf{y}_2)\big) = \mathbf{y}_2.$$

Hence,

$$\mathbf{x}_1 = \mathbf{h}(\mathbf{y}_1) = \mathbf{h}(\mathbf{y}_2) = \mathbf{x}_2.$$

We have $\mathbf{f}$ is ono-to-one.

If $\mathbf{y} \in \mathcal{V} = \mathcal{W}_1 \cap \mathcal{W}_2$, then $\mathbf{h}(\mathbf{y}) \in O_2$ and $\mathbf{f}\big(\mathbf{h}(\mathbf{y})\big) = \mathbf{y}$.

5.10 Higher Derivatives

Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ be differentiable on $\mathcal{U}$. Suppose that $D\mathbf{f}(\mathbf{x})$ exists for every $\mathbf{x} \in \mathcal{U}$. Then $D\mathbf{f}$ is a map from $\mathcal{U}$ into $\mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^m\big)$. We may ask whether this map is differentiable.

Definition 5.10.1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed space, $\mathcal{U} \subseteq X$ be open and $\mathbf{a} \in \mathcal{U}$. A function $f : \mathcal{U} \to Y$ is said to be "*twice differentiable at* $\mathbf{a}$" if

(1) f is differentiable in a neighborhood at $\mathbf{a}$.

(2) there exists $L_2 \in \mathcal{B}\big(X; \mathcal{B}(X; Y)\big)$ such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|Df(\mathbf{x}) - Df(\mathbf{a}) - L_2(\mathbf{x} - \mathbf{a})\|_{\mathcal{B}(X;Y)}}{\|\mathbf{x} - \mathbf{a}\|_X} = 0$$

or

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|Df(\mathbf{a} + \mathbf{h}) - Df(\mathbf{a}) - L_2\mathbf{h}\|_{\mathcal{B}(X;Y)}}{\|\mathbf{h}\|_X} = 0.$$

The linear map, L_2, is denoted by "$D^2 f(\mathbf{a})$" and is called "*the second derivative of f at* $\mathbf{a}$".

Remark. For every $\mathbf{u}, \mathbf{v} \in X$, $\underbrace{\big(D^2 f(\mathbf{a})\big)}_{\in\mathcal{B}(X;\mathcal{B}(X;Y))}(\mathbf{v}) \in \mathcal{B}(X; Y)$ and $\underbrace{\big[\big(D^2 f(\mathbf{a})\big)(\mathbf{v})\big]}_{\in\mathcal{B}(X;Y)}(\mathbf{u}) \in Y$.

$D^2 f(\mathbf{a})(\mathbf{v})(\mathbf{u})$ is usually denoted by $D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v})$.

Definition 5.10.2. In general, a function is said to be "*k-times differentiable at* $\mathbf{a} \in \mathcal{U}$" if

(1) f is $(k - 1)$ times differentiable in a neighborhood of $\mathbf{a}$.

(2) there exists $L_k \in \mathcal{B}(\underbrace{X; \mathcal{B}(X; \mathcal{B}(X, \cdots \mathcal{B}(X; Y))) \cdots}_{k \text{ times}})$ such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|D^{k-1} f(\mathbf{x}) - D^{k-1} f(\mathbf{a}) - L_k(\mathbf{x} - \mathbf{a})\|_{\mathcal{B}(X;\mathcal{B}(X;\mathcal{B}(X,\cdots\mathcal{B}(X;Y)))\cdots)}}{\|\mathbf{x} - \mathbf{a}\|_X} = 0$$

or

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|D^{k-1} f(\mathbf{a} + \mathbf{h}) - D^{k-1} f(\mathbf{a}) - L_k\mathbf{h}\|_{\mathcal{B}(X;\mathcal{B}(X;\mathcal{B}(X,\cdots\mathcal{B}(X;Y)))\cdots)}}{\|\mathbf{h}\|_X} = 0.$$

The linear map L_k is denoted by $D^k f(\mathbf{a})$ and is called "*the k-th derivative of f at* $\mathbf{a}$".

Remark. For any k vectors $\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(k)} \in X$,

$$\begin{aligned}
D^k f(\mathbf{a})(\mathbf{u}^{(k)}) &\in \mathcal{B}(\overbrace{X; \mathcal{B}(X; \mathcal{B}(X; \cdots \mathcal{B}(X; \mathcal{B}(X; Y)) \cdots))}^{k-1}) \\
D^k f(\mathbf{a})(\mathbf{u}^{(k)})(\mathbf{u}^{(k-1)}) &\in \mathcal{B}(\overbrace{X; \mathcal{B}(X; \mathcal{B}(X; \cdots \mathcal{B}(X; \mathcal{B}(X; Y)) \cdots)}^{k-2}) \\
&\vdots \\
D^k f(\mathbf{a})(\mathbf{u}^{(k)})(\mathbf{u}^{(k-1)}) \cdots (\mathbf{u}^{(1)}) &\in Y
\end{aligned}$$

$D^k f(\mathbf{a})(\mathbf{u}^{(k)})(\mathbf{u}^{(k-1)})\cdots(\mathbf{u}^{(1)})$ is usually denoted by $D^k f(\mathbf{a})(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{(k)})$.

Example 5.10.3. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be normed spaces and $L\in\mathcal{B}(X;Y)$. Then for any $\mathbf{a}\in X$, $DL(\mathbf{a})=L$. Hence,

$$\lim_{\substack{\mathbf{h}\to\mathbf{0}\\ \mathbf{h}\in X}}\frac{\|DL(\mathbf{a}+\mathbf{h})-DL(\mathbf{a})-\mathbf{0}\mathbf{h}\|_{\mathcal{B}(X;Y)}}{\|\mathbf{h}\|_X}=\lim_{\substack{\mathbf{h}\to\mathbf{0}\\ \mathbf{h}\in X}}\frac{\|L-L\|_{\mathcal{B}(X;Y)}}{\|\mathbf{h}\|_X}=0$$

Hence, $D^2L(\mathbf{a})=\mathbf{0}$.

Note. In order to find a representation of $D^2 f(\mathbf{a})$, let us look at the following two observations

Remark. Let $f:\mathcal{U}\subseteq X\to Y$ be twice differentiable at $\mathbf{a}\in\mathcal{U}$. Consider the "directional derivative" of Df at $\mathbf{a}$ in the direction $\mathbf{v}\in X$. Let $\mathbf{x}=\mathbf{a}+t\mathbf{v}$ with $\|\mathbf{v}\|_X=1$.

$$\lim_{t\to 0}\frac{\|Df(\mathbf{a}+t\mathbf{v})-Df(\mathbf{a})-tD^2f(\mathbf{a})(\mathbf{v})\|_{\mathcal{B}(X;Y)}}{\|t\mathbf{v}\|_X}=0.$$

Hence, for $\mathbf{u}\in X$ with $\|\mathbf{u}\|_X=1$,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\|Df(\mathbf{a}+t\mathbf{v})(\mathbf{u})-Df(\mathbf{a})(\mathbf{u})-tD^2f(\mathbf{a})(\mathbf{v})(\mathbf{u})\|_Y}{\|t\mathbf{v}\|_X}\\
=&\lim_{t\to 0}\frac{\|\big(Df(\mathbf{a}+t\mathbf{v})-Df(\mathbf{a})-tD^2f(\mathbf{a})(\mathbf{v})\big)(\mathbf{u})\|_Y}{|t|}\\
\le&\lim_{t\to 0}\frac{\|Df(\mathbf{a}+t\mathbf{v})-Df(\mathbf{a})-tD^2f(\mathbf{a})(\mathbf{v})\|_{\mathcal{B}(X;Y)}}{|t|}\\
=&\ 0
\end{aligned}$$

Since

$$Df(\mathbf{a}+t\mathbf{v})(\mathbf{u})-Df(\mathbf{a})(\mathbf{u})=\lim_{s\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v}+s\mathbf{u})-f(\mathbf{a}+t\mathbf{v})}{s}-\frac{f(\mathbf{a}+s\mathbf{u})-f(\mathbf{a})}{s}\right],$$

we have

$$\begin{aligned}
D^2f(\mathbf{a})(\mathbf{v})(\mathbf{u}) &= \lim_{t\to 0}\lim_{s\to 0}\frac{f(\mathbf{a}+t\mathbf{v}+s\mathbf{u})-f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a}+s\mathbf{u})+f(\mathbf{a})}{st}\\
&= \lim_{t\to 0}\lim_{s\to 0}\frac{\frac{f(\mathbf{a}+t\mathbf{v}+s\mathbf{u})-f(\mathbf{a}+t\mathbf{v})}{s}-\frac{f(\mathbf{a}+s\mathbf{u})-f(\mathbf{a})}{s}}{t}\\
&= \lim_{t\to 0}\frac{1}{t}\Big(\lim_{s\to 0}\frac{f(\mathbf{a}+t\mathbf{v}+s\mathbf{u})-f(\mathbf{a}+t\mathbf{v})}{s}-\frac{f(\mathbf{a}+s\mathbf{u})-f(\mathbf{a})}{s}\Big)\\
&= \lim_{t\to 0}\frac{D_{\mathbf{u}}f(\mathbf{a}+t\mathbf{v})-D_{\mathbf{u}}f(\mathbf{a})}{t}\\
&= D_{\mathbf{v}}\big(D_{\mathbf{u}}f\big)(\mathbf{a}).
\end{aligned}$$

Proposition 5.10.4. *Let $\mathcal{U}\subseteq\mathbb{R}^n$ be open, $\mathbf{a}\in\mathcal{U}$ and $\mathbf{f}=(f_1,\cdots,f_m):\mathcal{U}\to\mathbb{R}^m$. Then $\mathbf{f}$ is k-times differentiable at $\mathbf{a}$ if and only if f_i is k-times differentiable at $\mathbf{a}$ for all $i=1,\cdots,m$.*

Proof. (Exercise) by induction. □

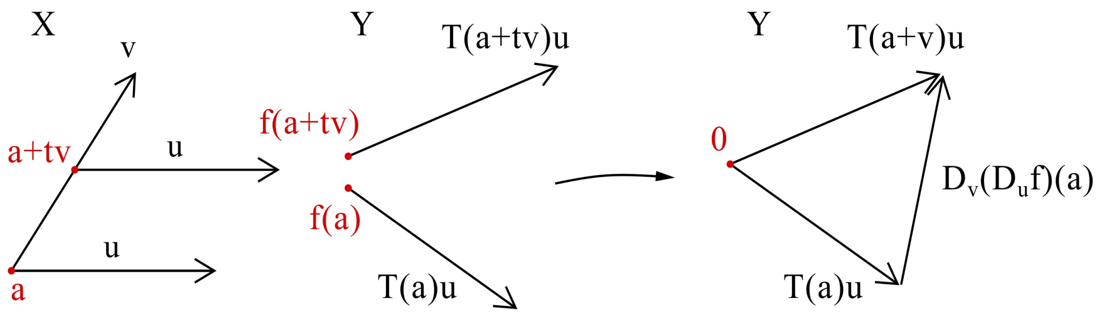
Note. The proposition suggests that, in order to study the differentiation of $\mathbf{f} = (f_1, \cdots, f_m) : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, it sufficies to study the differentiation of $f_i : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ since

$$D^k\mathbf{f}(\mathbf{a})(\mathbf{u}_1, \cdots, \mathbf{u}_k) = \Big(D^k f_1(\mathbf{a})(\mathbf{u}_1, \cdots, \mathbf{u}_k), \cdots, D^k f_m(\mathbf{a})(\mathbf{u}_1, \cdots, \mathbf{u}_k)\Big)$$

Another viewpoint: By Theorem 5.6.2, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in X with $\|\mathbf{u}\|_X = \|\mathbf{v}\|_X = 1$.

$$\begin{aligned} D^2\mathbf{f}(\mathbf{a})(\mathbf{v}) &= D_{\mathbf{v}}(D\mathbf{f})(\mathbf{a}) \\ D^2\mathbf{f}(\mathbf{a})(\mathbf{v})(\mathbf{u}) &= \Big(D^2\mathbf{f}(\mathbf{a})(\mathbf{v})\Big)(\mathbf{u}) = \Big(D_{\mathbf{v}}\big(D\mathbf{f}\big)(\mathbf{a})\Big)(\mathbf{u}) \end{aligned}$$

For $\mathbf{f} : X \to Y$, $D\mathbf{f}(\mathbf{a})$ 反映 $\mathbf{f}$ 各方向的瞬間改變率，$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{a})\mathbf{u}$ 反映 $\mathbf{f}$ 在 $\mathbf{a}$ 點沿 $\mathbf{u}$ 方向的瞬間改變率 (量)(since $\|\mathbf{u}\|_X = 1$). $D_{\mathbf{v}}\big(D\mathbf{f}\big)(\mathbf{a})$ 反映 $D\mathbf{f}$ 在 $\mathbf{a}$ 點沿 $\mathbf{v}$ 方向上的瞬間改變率，亦在考量 $\mathbf{f}$ 在 $\mathbf{a}$ 點沿各方向上改變量的比率。因此考慮 $D\mathbf{f}(\mathbf{a})\mathbf{u}$ 為 $\mathbf{f}$ 在 $\mathbf{a}$ 點沿 $\mathbf{u}$ 方向上 $\mathbf{f}$ 的改變量。此改變量隨 $\mathbf{a}$ 點變化而改變。
$D_{\mathbf{v}}\big(D_{\mathbf{u}}\mathbf{f}\big)(\mathbf{a})$ 為上述改變中 $\mathbf{v}$ 方向上的改變量



Therefore, $D^2\mathbf{f}(\mathbf{a})(\mathbf{v})(\mathbf{u})$ is obtained by first differentiating $\mathbf{f}$ in the $\mathbf{u}$-direction and then differentiating $D_{\mathbf{u}}\mathbf{f}$ at $\mathbf{a}$ in the $\mathbf{v}$-direction.
Similarly, $\big(D^k\mathbf{f}\big)(\mathbf{a})(\mathbf{u}_k)\cdots(\mathbf{u}_1)$ is obtained by first differentiating $\mathbf{f}$ at $\mathbf{a}$ in the $\mathbf{u}_1$-direction. Then continuing similar procedure, $D_{\mathbf{u}_{k-1}}(D_{\mathbf{u}_{k-2}}(D_{\mathbf{u}_{k-3}}(\cdots D_{\mathbf{u}_1}\mathbf{f}))(\mathbf{a})$ in the $\mathbf{u}_k$-direction.

Remark. (1) The second derivative $D^2\mathbf{f}(\mathbf{a}) \in \mathcal{B}(X; \mathcal{B}(X; Y))$ is a linear map. Then, for $\mathbf{v}_1, \mathbf{v}_2 \in X$ and $c \in \mathbb{R}$,

$$D^2\mathbf{f}(c\mathbf{v}_1 + \mathbf{v}_2) = cD^2\mathbf{f}(\mathbf{a})(\mathbf{v}_1) + D^2\mathbf{f}(\mathbf{a})(\mathbf{v}_2) \quad \in \mathcal{B}(X; Y).$$

For $\mathbf{u} \in X$,

$$D^2\mathbf{f}(\mathbf{a})\big(c\mathbf{v}_1 + \mathbf{v}_2\big)(\mathbf{u}) = cD^2\mathbf{f}(\mathbf{a})\big(\mathbf{v}_1\big)(\mathbf{u}) + D^2\mathbf{f}\big(\mathbf{v}_2\big)(\mathbf{u}) \quad \in Y.$$

Also, for every $\mathbf{v} \in X$, since $D^2\mathbf{f}(\mathbf{a})(\mathbf{v}) \in \mathcal{B}(X; Y)$ is a linear map, for $\mathbf{u}_1, \mathbf{u}_2 \in X$,

$$D^2\mathbf{f}(\mathbf{a})(\mathbf{v})(c\mathbf{u}_1 + \mathbf{u}_2) = cD^2\mathbf{f}(\mathbf{a})(\mathbf{v})(\mathbf{u}_1) + D^2\mathbf{f}(\mathbf{a})(\mathbf{v})(\mathbf{u}_2).$$

Hence, $D^2\mathbf{f}(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D^2\mathbf{f}(\mathbf{a})(\mathbf{v})(\mathbf{u})$ is linear in both $\mathbf{u}$ and $\mathbf{v}$. We call a map with this property a "*bilinear map*".

(2) Similarly, $D^k\mathbf{f}(\mathbf{a})(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ is linear in $\mathbf{u}_1, \cdots, \mathbf{u}_k$. A map with this property is called "*k-linear map*".

■ **Matrix Representatioin of $D^2\mathbf{f}(\mathbf{a})$**

This remark suggests that, in order to define $D^2\mathbf{f}(\mathbf{a})$ clearly, it suffices to define $D^2\mathbf{f}(\mathbf{a})$ on the basis pair $(\mathbf{e}_i, \mathbf{e}_j)$ where $1 \le i, j \le n$.

Let X be a finite dimension with $\dim X = n$ and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a basis of X and $Y = \mathbb{R}$. Then $D^2f(\mathbf{a}) : X \times X \to Y$ is a bilinear form.

Let $\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$ and $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$. Then

$$
\begin{aligned}
D^2f(\mathbf{a})(\mathbf{u},\mathbf{v}) &= D^2f(\mathbf{a})\Big(\sum_{i=1}^n u_i\mathbf{e}_i, \sum_{j=1}^n v_j\mathbf{e}_j\Big) = \sum_{i=1}^n\sum_{j=1}^n u_iv_jD^2f(\mathbf{a})(\mathbf{e}_i,\mathbf{e}_j) \\
&= \begin{bmatrix} u_1 & \cdots & \cdots & u_n \end{bmatrix} \begin{bmatrix} D^2f(\mathbf{a})(\mathbf{e}_1,\mathbf{e}_1) & \cdots & \cdots & D^2f(\mathbf{a})(\mathbf{e}_1,\mathbf{e}_n) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ D^2f(\mathbf{a})(\mathbf{e}_n,\mathbf{e}_1) & \cdots & \cdots & D^2f(\mathbf{a})(\mathbf{e}_n,\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} \\
&= \begin{bmatrix} v_1 & \cdots & \cdots & v_n \end{bmatrix} \begin{bmatrix} D^2f(\mathbf{a})(\mathbf{e}_1,\mathbf{e}_1) & \cdots & \cdots & D^2f(\mathbf{a})(\mathbf{e}_n,\mathbf{e}_1) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ D^2f(\mathbf{a})(\mathbf{e}_1,\mathbf{e}_n) & \cdots & \cdots & D^2f(\mathbf{a})(\mathbf{e}_n,\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{bmatrix} \\
&= \begin{bmatrix} v_1 & \cdots & \cdots & v_n \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \cdots & \cdots & \dfrac{\partial^2 f}{\partial x_1\partial x_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\partial x_1} & \cdots & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{bmatrix}
\end{aligned}
$$

Example 5.10.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable at (a, b).

$$
\begin{bmatrix} Df(x,y) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x}(x,y) & \dfrac{\partial f}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \end{bmatrix}
$$

Denote $L_2 = D^2f(a,b) \in \mathcal{B}(\mathbb{R}^2; \mathcal{B}(\mathbb{R}^2;\mathbb{R}))$. Then

$$
\begin{aligned}
0 &= \lim_{(x,y)\to(a,b)} \frac{\big\|Df(x,y) - Df(a,b) - L_2\big(x-a, y-b\big)\big\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R})}}{\|(x-a, y-b)\|_{\mathbb{R}^2}} \\
&= \lim_{(x,y)\to(a,b)} \frac{\big\| \begin{bmatrix} f_x(x,y) & f_y(x,y) \end{bmatrix} - \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} - L_2\big(x-a, y-b\big)\big\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R})}}{\sqrt{(x-a)^2 + (y-b)^2}} \\
&= \lim_{(x,y)\to(a,b)} \frac{\big\| \begin{bmatrix} f_x(x,y) - f_x(a,b) & f_y(x,y) - f_y(a,b) \end{bmatrix} - L_2\big(x-a, y-b\big)\big\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R})}}{\sqrt{(x-a)^2 + (y-b)^2}}.
\end{aligned}
$$

Let $(x,b) \to (a,b)$ and $\begin{bmatrix} L_2\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \in \mathcal{B}(\mathbb{R}^2;\mathbb{R})$. Then

$$\begin{aligned}
& 0 = \lim_{x\to a} \frac{\left\| \begin{bmatrix} f_x(x,b) - f_x(a,b) & f_y(x,b) - f_y(a,b) \end{bmatrix} - (x-a)\begin{bmatrix} L_2\mathbf{e}_1 \end{bmatrix} \right\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R})}}{|x-a|} \\
\iff \quad & 0 = \lim_{x\to a} \frac{|f_x(x,b) - f_x(a,b) - (x-a)a_{11}|}{|x-a|} \quad \text{and} \quad 0 = \lim_{x\to a} \frac{|f_y(x,b) - f_y(a,b) - (x-a)a_{12}|}{|x-a|} \\
\iff \quad & a_{11} = f_{xx}(a,b) \quad \text{and} \quad a_{12} = f_{yx}(a,b) \\
\iff \quad & \begin{bmatrix} L_2\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \end{bmatrix}.
\end{aligned}$$

Similarly, let $(a,y) \to (a,b)$, then

$$\begin{bmatrix} L_2\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}.$$

Hence, for $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$,

$$\begin{aligned}
\begin{bmatrix} L_2\mathbf{v} \end{bmatrix} &= v_1 \begin{bmatrix} L_2\mathbf{e}_1 \end{bmatrix} + v_2 \begin{bmatrix} L_2\mathbf{e}_2 \end{bmatrix} \\
&= v_1 \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \end{bmatrix} + v_2 \begin{bmatrix} f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \\
(\textit{symbolically}) \quad &= \begin{bmatrix} \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \end{bmatrix} & \begin{bmatrix} f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= \begin{bmatrix} L_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\end{aligned}$$

Let $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$. Then

$$\begin{aligned}
L_2(\mathbf{u},\mathbf{v}) &= L_2(\mathbf{v})(\mathbf{u}) = \begin{bmatrix} L_2\mathbf{v} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= v_1 \begin{bmatrix} L_2\mathbf{e}_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + v_2 \begin{bmatrix} L_2\mathbf{e}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= v_1 \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + v_2 \begin{bmatrix} f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \left(\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \left(\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{bmatrix} D^2f(a,b)(\mathbf{e}_1,\mathbf{e}_1) & D^2f(a,b)(\mathbf{e}_1,\mathbf{e}_2) \\ D^2f(a,b)(\mathbf{e}_2,\mathbf{e}_1) & D^2f(a,b)(\mathbf{e}_2,\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Note: The above matrix is an informal expression as

$$\begin{bmatrix} D^2f(a,b)(\mathbf{e}_1,\mathbf{e}_1) & D^2f(a,b)(\mathbf{e}_1,\mathbf{e}_2) \\ D^2f(a,b)(\mathbf{e}_2,\mathbf{e}_1) & D^2f(a,b)(\mathbf{e}_2,\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} D^2f(a,b)\mathbf{e}_1 & D^2f(a,b)\mathbf{e}_2 \end{bmatrix}.$$

It applies $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ by

$$\begin{bmatrix} D^2 f(a,b)\mathbf{e}_1 & D^2 f(a,b)\mathbf{e}_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \in \mathcal{B}(\mathbb{R}^2; \mathbb{R}).$$

However, we usually express it as

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} D^2 f(a,b)\mathbf{e}_1 \\ D^2 f(a,b)\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} D^2 f(a,b)(\mathbf{e}_1, \mathbf{e}_1) & D^2 f(a,b)(\mathbf{e}_2, \mathbf{e}_1) \\ D^2 f(a,b)(\mathbf{e}_1, \mathbf{e}_2) & D^2 f(a,b)(\mathbf{e}_2, \mathbf{e}_2) \end{bmatrix}$$

Viewpoint of Identification :

We identify $\mathcal{B}(\mathbb{R}^2; \mathbb{R})$ as $\mathbb{R}^2$ (That is, $\mathcal{B}(\mathbb{R}^2; \mathbb{R}) \backsimeq \mathbb{R}^2$). Then

$$\begin{aligned} Df \quad &: \quad U \subseteq \mathbb{R}^2 \to \mathcal{B}(\mathbb{R}^2; \mathbb{R}) \\ &\backsimeq \quad U \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \end{aligned}$$

Define

$$g(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} Df \end{bmatrix}^T : U \subseteq \mathbb{R}^2 \to \mathbb{R}.$$

$$Dg(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

Hence,

$$D^2 f(a,b) \in \mathcal{B}(\mathbb{R}^2; \mathcal{B}(\mathbb{R}^2; \mathbb{R}))$$

$$\Big\updownarrow \text{ 1-1 correspondence}$$

$$\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

such that for $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

$$\begin{aligned} D^2 f(a,b)(\mathbf{u}, \mathbf{v}) &= D^2 f(a,b)(\mathbf{v})(\mathbf{u}) \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \frac{\partial^2 f}{\partial x^2} v_1 u_1 + \frac{\partial^2 f}{\partial y \partial x} v_2 u_1 + \frac{\partial^2 f}{\partial x \partial y} v_1 u_2 + \frac{\partial^2 f}{\partial y^2} v_2 u_2 \\ &= \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j} v_i u_j \end{aligned}$$

Note. 在這裡 $\left[Dg(x,y)\right]$ 為 Hessian matrix or $\left[Df(x,y)\right]$ 的轉置矩陣，原因是上述的分析是以右乘的方式 $\left[D^2 f(x,y)\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ 作用上 **v** 向量。而未來實際操作上，我們以左乘方式 $\begin{bmatrix} v_1 & v_2 \end{bmatrix}\left[D^2 f(x,y)\right]^T$ 表達作用上 **v** 之後的 linear map。

Question: How about k-times derivative on $\mathbb{R}^n$?

Proposition 5.10.6. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$. Suppose that f is k-times differentiable at $\mathbf{a} \in \mathcal{U}$. Then for k vector $\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(k)} \in \mathbb{R}^n$,*

$$\begin{aligned} D^k f(\mathbf{a})\big(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(k)}\big) &= \sum_{j_1,\cdots j_k=1}^{n} \frac{\partial^k f(\mathbf{a})}{\partial x_{j_k}\partial x_{j_{k-1}}, \cdots, \partial x_{j_1}} u_{j_1}^{(1)} \cdots u_{j_k}^{(k)} \\ &= \sum_{j_1,\cdots j_k=1}^{n} \frac{\partial}{\partial x_{j_k}}\Big(\frac{\partial}{\partial x_{j_{k-1}}}\big(\cdots \frac{\partial}{\partial x_{j_2}}(\frac{\partial f}{\partial x_{j_1}})\big)\Big)(\mathbf{a}) u_{j_1}^{(1)} \cdots u_{j_k}^{(k)} \end{aligned}$$

where $\mathbf{u}^{(i)} = (u_1^{(i)}, \cdots, u_n^{(i)})$ for all $i = 1, \cdots, k$.

Proof. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Since $D^k f(\mathbf{a})$ is a k-linear map, it suffices to show that

$$D^k f(\mathbf{a})(\mathbf{e}_{j_k})(\mathbf{e}_{j_{k-1}}) \cdots (\mathbf{e}_{j_2})(\mathbf{e}_{j_1}) = D^k f(\mathbf{a})(\mathbf{e}_{j_1} \cdots, \mathbf{e}_{j_k}) = \frac{\partial^k f}{\partial x_{j_k} \cdots \partial x_{j_1}}(\mathbf{a}). \tag{5.20}$$

If so,

$$\begin{aligned} D^k f(\mathbf{a})\big(\mathbf{u}^{(1)}, \cdots \mathbf{u}^{(k)}\big) &= D^k f(\mathbf{a})\Big(\sum_{j_1=1}^{n} u_{j_1}^{(1)}\mathbf{e}_{j_1}, \cdots, \sum_{j_k=1}^{n} u_{j_k}^{(k)}\mathbf{e}_{j_k}\Big) \\ &= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} D^k f(\mathbf{a})\big(\mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_k}\big) u_{j_1}^{(1)} \cdots u_{j_k}^{(k)} \end{aligned}$$

When $k = 1$,

$$Df(\mathbf{a})\mathbf{e}_j = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{a}) \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \frac{\partial f}{\partial x_j}(\mathbf{a}) \quad \text{for } j = 1, \cdots, n.$$

Therefore, the proposition holds when $k = 1$. Assume that (5.20) holds when $k = \ell$. That is, f is $(\ell - 1)$-times differentiable in a neighborhood of $\mathbf{a}$ and f is ℓ-times differentiable at $\mathbf{a}$.

Suppose that f is $(\ell + 1)$-times differentiable at $\mathbf{a}$ and f is ℓ-times differentiable in a neighborhood of $\mathbf{a}$. We will prove that (5.20) holds when $k = \ell + 1$. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|D^\ell f(\mathbf{x}) - D^\ell f(\mathbf{a}) - D^{\ell+1} f(\mathbf{a})(\mathbf{x} - \mathbf{a})\|_{\mathcal{B}(\mathbb{R}^n;\cdots\mathcal{B}(\mathbb{R}^n;\mathbb{R})\cdots)}}{\|\mathbf{x} - \mathbf{a}\|_{\mathbb{R}^n}} = 0 \tag{5.21}$$

and

$$D^\ell f(\mathbf{x})\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell}\big) = \frac{\partial^\ell f}{\partial x_{j_\ell}\cdots\partial x_{j_1}}(\mathbf{x}) \tag{5.22}$$

for every $\mathbf{x}$ in a neighborhood of $\mathbf{a}$. Hence,

$$\begin{aligned}
& \lim_{\mathbf{x}\to\mathbf{a}} \frac{\left|\frac{\partial^\ell f}{\partial x_{j_1}\cdots\partial x_{j_\ell}}(\mathbf{x}) - \frac{\partial^\ell f}{\partial x_{j_1}\cdots\partial x_{j_\ell}}(\mathbf{a}) - D^{\ell+1}f(\mathbf{a})\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell},\mathbf{x}-\mathbf{a}\big)\right|}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \\
\overset{(5.22)}{=} & \lim_{\mathbf{x}\to\mathbf{a}} \frac{\left|D^\ell f(\mathbf{x})\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell}\big) - D^\ell f(\mathbf{a})\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell}\big) - D^{\ell+1}f(\mathbf{a})(\mathbf{x}-\mathbf{a})\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell}\big)\right|}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \\
= & \lim_{\mathbf{x}\to\mathbf{a}} \frac{\left|\left[D^\ell f(\mathbf{x}) - D^\ell f(\mathbf{a}) - D^{\ell+1}f(\mathbf{a})(\mathbf{x}-\mathbf{a})\right]\big(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_\ell}\big)\right|}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}}} \\
\leq & \lim_{\mathbf{x}\to\mathbf{a}} \frac{\left\|D^\ell f(\mathbf{x}) - D^\ell f(\mathbf{a}) - D^{\ell+1}f(\mathbf{a})(\mathbf{x}-\mathbf{a})\right\|_{\mathcal{B}(\mathbb{R}^n;\cdots\mathcal{B}(\mathbb{R}^n;\mathbb{R})\cdots)}\|\mathbf{e}_{j_1}\|_{\mathbb{R}^n}\cdots\|\mathbf{e}_{j_\ell}\|_{\mathbb{R}^n}}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}} \\
= & \ 0.
\end{aligned}$$

Let $\mathbf{x} = \mathbf{a} + t\mathbf{e}_{j_{\ell+1}}$. Then

$$\lim_{t\to 0} \frac{\left|\frac{\partial^\ell f}{\partial x_{j_\ell}\cdots\partial x_{j_1}}(\mathbf{a}+t\mathbf{e}_{j_{\ell+1}}) - \frac{\partial^\ell f}{\partial x_{j_\ell}\cdots\partial x_{j_1}}(\mathbf{a}) - tD^{\ell+1}f(\mathbf{a})(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_{\ell+1}})\right|}{|t|} = 0.$$

Thus,

$$\lim_{t\to 0} \left| \frac{\frac{\partial^\ell f}{\partial x_{j_\ell}\cdots\partial x_{j_1}}(\mathbf{a}+t\mathbf{e}_{j_{\ell+1}}) - \frac{\partial^\ell f}{\partial x_{j_\ell}\cdots\partial x_{j_1}}(\mathbf{a})}{t} - D^{\ell+1}f(\mathbf{a})(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_{\ell+1}}) \right| = 0.$$

We have

$$D^{\ell+1}f(\mathbf{a})(\mathbf{e}_{j_1},\cdots,\mathbf{e}_{j_{\ell+1}}) = \frac{\partial^{\ell+1}f}{\partial x_{j_{\ell+1}}\cdots\partial x_{j_1}}(\mathbf{a}).$$

□

Example 5.10.7. Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$, then $D^3 f(\mathbf{a}) \in \mathcal{B}\Big(\mathbb{R}^2;\mathcal{B}\big(\mathbb{R}^2;\mathcal{B}(\mathbb{R}^2;\mathbb{R})\big)\Big)$. Let $\mathbf{u} = (u_1,u_2)$, $\mathbf{v} = (v_1,v_2)$ and $\mathbf{w} = (w_1,w_2)$ be vectors in $\mathbb{R}^2$. Then

$$\begin{aligned}
D^3 f(\mathbf{a})(\mathbf{u},\mathbf{v},\mathbf{w}) &= \frac{\partial^3 f(\mathbf{a})}{\partial x\partial x\partial x}u_1v_1w_1 + \frac{\partial^3 f(\mathbf{a})}{\partial y\partial x\partial x}u_1v_1w_2 + \frac{\partial^3 f(\mathbf{a})}{\partial x\partial y\partial x}u_1v_2w_1 \\
&+ \frac{\partial^3 f(\mathbf{a})}{\partial y\partial y\partial x}u_2v_2w_1 + \frac{\partial^3 f(\mathbf{a})}{\partial x\partial x\partial y}u_1v_1w_2 + \frac{\partial^3 f(\mathbf{a})}{\partial y\partial x\partial y}u_2v_1w_2 \\
&+ \frac{\partial^3 f(\mathbf{a})}{\partial x\partial y\partial y}u_1v_2w_2 + \frac{\partial^3 f(\mathbf{a})}{\partial y\partial y\partial y}u_2v_2w_2.
\end{aligned}$$

Example 5.10.8. Let $f(x,y,z):\mathbb{R}^3\to\mathbb{R}$. Then $D^2f(\mathbf{a})\in\mathcal{B}\big(\mathbb{R}^3;\mathcal{B}(\mathbb{R}^3;\mathbb{R})\big)$. Let $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ be two vectors in $\mathbb{R}^2$. Then

$$\begin{aligned}
& D^2f(\mathbf{a})(\mathbf{u},\mathbf{v}) = D^2f(\mathbf{a})(\mathbf{v})(\mathbf{u}) \\
= \;& \frac{\partial^2 f(\mathbf{a})}{\partial x\partial x}u_1v_1+\frac{\partial^2 f(\mathbf{a})}{\partial x\partial y}u_1v_2+\frac{\partial^2 f(\mathbf{a})}{\partial x\partial z}u_1v_3 \\
+ \;& \frac{\partial^2 f(\mathbf{a})}{\partial y\partial x}u_2v_1+\frac{\partial^2 f(\mathbf{a})}{\partial y\partial y}u_2v_2+\frac{\partial^2 f(\mathbf{a})}{\partial y\partial z}u_2v_3 \\
+ \;& \frac{\partial^2 f(\mathbf{a})}{\partial z\partial x}u_3v_1+\frac{\partial^2 f(\mathbf{a})}{\partial z\partial y}u_3v_2+\frac{\partial^2 f(\mathbf{a})}{\partial z\partial z}u_3v_3.
\end{aligned}$$

Example 5.10.9. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x_1,x_2)=x_1\cos x_2$ and $\mathbf{u}^{(1)}=\langle 2,0\rangle$, $\mathbf{u}^{(2)}=\langle 1,1\rangle$, $\mathbf{u}^{(3)}=\langle 0,-1\rangle$..

$$\begin{aligned}
D^3f(0,0)\big(\mathbf{u}^{(1)},\mathbf{u}^{(2)},\mathbf{u}^{(3)}\big) &= \frac{\partial^3 f(0,0)}{\partial x_1\partial x_1\partial x_1}\cancelto{0}{u_1^{(1)}u_1^{(2)}u_1^{(3)}}+\frac{\partial^3 f(0,0)}{\partial x_1\partial x_1\partial x_2}u_1^{(1)}u_1^{(2)}u_2^{(3)} \\
&+ \frac{\partial^3 f(0,0)}{\partial x_1\partial x_2\partial x_1}\cancelto{0}{u_1^{(1)}u_2^{(2)}u_1^{(3)}}+\frac{\partial^3 f(0,0)}{\partial x_1\partial x_2\partial x_2}u_1^{(1)}u_2^{(2)}u_2^{(3)} \\
&+ \frac{\partial^3 f(0,0)}{\partial x_2\partial x_1\partial x_1}\cancelto{0}{u_2^{(1)}u_1^{(2)}u_1^{(3)}}+\frac{\partial^3 f(0,0)}{\partial x_2\partial x_1\partial x_2}\cancelto{0}{u_2^{(1)}u_1^{(2)}u_2^{(3)}} \\
&+ \frac{\partial^3 f(0,0)}{\partial x_2\partial x_2\partial x_1}\cancelto{0}{u_2^{(1)}u_2^{(2)}u_1^{(3)}}+\frac{\partial^3 f(0,0)}{\partial x_2\partial x_2\partial x_2}\cancelto{0}{u_2^{(1)}u_2^{(2)}u_2^{(3)}} \\
&= [-2\sin x_2]_{(0,0)}\cdot 2\cdot 1\cdot(-1)-[2x_1\cos x_2]_{(0,0)}\cdot 1\cdot 1\cdot(-1) \\
&= 0
\end{aligned}$$

Corollary 5.10.10. *Let $\mathcal{U}\subseteq\mathbb{R}^n$ be open and $f:\mathcal{U}\to\mathbb{R}$ be $(k+1)$-times differentiable at $\mathbf{a}\in\mathcal{U}$. Then for $\mathbf{u}^{(1)},\cdots,\mathbf{u}^{(k)},\mathbf{u}^{(k+1)}\in\mathbb{R}^n$,*

$$\big(D^{k+1}f\big)(\mathbf{a})\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{(k+1)}\big)=\sum_{j=1}^n u_j^{(k+1)}\frac{\partial}{\partial x_j}\Big|_{x=a}\big(D^kf\big)(\mathbf{x})\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{(k)}\big).$$

That is, $\big(D^{k+1}f\big)(\mathbf{a})\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{k+1}\big)$ is the directional derivative of $D^kf(\cdot)\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{k+1}\big)$ at $\mathbf{a}$ in the direction $\mathbf{u}^{(k+1)}$ by multiplying $\|\mathbf{u}^{(k+1)}\|_{\mathbb{R}^n}$.

Proof.

$$\begin{aligned}
\big(D^{k+1}f\big)(\mathbf{a})\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{k+1}\big) &= \sum_{j_1,\cdots,j_{k+1}=1}^n \frac{\partial^{k+1}f}{\partial x_{j_{k+1}}\cdots\partial x_{j_1}}(\mathbf{a})u_{j_1}^{(1)}\cdots u_{j_{k+1}}^{(k+1)} \\
&= \sum_{j_{k+1}=1}^n u_{j_{k+1}}^{(k+1)}\Big(\sum_{j_1,\cdots,j_k=1}^n\frac{\partial^{k+1}f}{\partial x_{j_{k+1}}\cdots\partial x_{j_1}}(\mathbf{a})u_{j_1}^{(1)}\cdots u_{j_k}^{(k)}\Big) \\
&= \sum_{j_{k+1}=1}^n u_{j_{k+1}}^{(k+1)}\frac{\partial}{\partial x_{j_{k+1}}}\Big|_{x=a}\underbrace{\Big(\sum_{j_1,\cdots,j_k=1}^n\frac{\partial^k f}{\partial x_{j_k}\cdots\partial x_{j_1}}(\mathbf{a})u_{j_1}^{(1)}\cdots u_{j_k}^{(k)}\Big)}_{D^kf(\mathbf{a})\big(\mathbf{u}^{(1)},\cdots,\mathbf{u}^{(k)}\big)}.
\end{aligned}$$

□

Example 5.10.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable at $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$. Then for $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$,

$$\begin{aligned} D^2 f(\mathbf{a})(\mathbf{v})(\mathbf{u}) = D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v}) &= \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) u_1 v_1 + \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) u_1 v_2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) u_2 v_1 + \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) u_2 v_2 \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

Definition 5.10.12. In general, if $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at $\mathbf{a}$ and $\mathbf{v} = \langle v_1, \cdots, v_n \rangle$, $\mathbf{u} = \langle u_1, \cdots, u_n \rangle$ be vectors in $\mathbb{R}^n$, then

$$D^2 f(\mathbf{a})(\mathbf{v})(\mathbf{u}) = D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{bmatrix}$$

We call this $n \times n$ matrix $\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix}$ "*Hessian matrix of f*" and denote $H(f)(\mathbf{a})$ or $H_f(\mathbf{a})$.

The bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$B(\mathbf{u}, \mathbf{v}) = \left(D^2 f\right)(\mathbf{a})(\mathbf{v})(\mathbf{u}) \quad \text{for every } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

is called the "*Hessian of f*".

Remark. (1) If all second partial derivatives of f at $\mathbf{a}$ exist, then the Hessian matrix of f is defined even if f is not twice differentiable at $\mathbf{a}$.

(2) The Hessian matrix may not be symmetric $\left(D^2 f(\mathbf{u}, \mathbf{v}) \neq D^2 f(\mathbf{v}, \mathbf{u})\right)$.

(3) If all second partial derivatives of f are continuous at $\mathbf{a}$, then f is twice differentiable at $\mathbf{a}$ and the Hessian matrix is symmetric.

(4)

$$D^2 f(\mathbf{a}) \text{ exists} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \text{ exists for every } i, j = 1, \cdots, n \implies H_f(\mathbf{a}) \text{ exists}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ are continuous} \quad &\Longleftrightarrow \quad D^2 f(\mathbf{a}) \text{ exists and are continuous} \\ &\Longrightarrow \quad H_f(\mathbf{a}) \text{ is symmetric.} \end{aligned}$$

For example, $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ then $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

Definition 5.10.13. (1) A function is said to be "*of class* C^r if the first r derivatives exist and are continuous.

(2) A function is said to be "*smooth*" or "*of class* C^∞" if it is of class C^r for all positive integer r.

Theorem 5.10.14. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$. Suppose that all k-times partial derivatives $\dfrac{\partial^k f}{\partial x_{j_k} \cdots \partial x_{j_1}}$ exist in a neighborhood of $\mathbf{a} \in \mathcal{U}$ and are continuous at $\mathbf{a}$. Then f is k-times differentiable at $\mathbf{a}$. Moreover, if $\dfrac{\partial^k f}{\partial x_{j_k} \cdots \partial x_{j_1}}$ is continuous on $\mathcal{U}$, then f is of class C^k.*

Theorem 5.10.15. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$. Suppose that the mixed partial derivatives $\dfrac{\partial f}{\partial x_i}, \dfrac{\partial f}{\partial x_j}, \dfrac{\partial^2 f}{\partial x_i \partial x_j}, \dfrac{\partial^2 f}{\partial x_j \partial x_i}$ exist in a neighborhood of $\mathbf{a}$ and are continuous at $\mathbf{a}$. Then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}).$$

Proof. W.L.O.G, it suffices to show the case $n = 2$ and

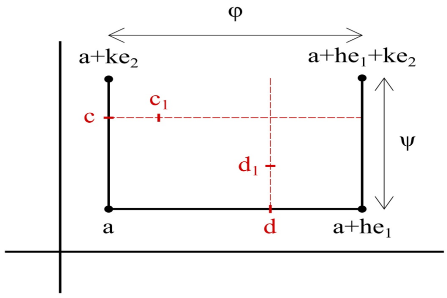
$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}).$$

Let $S(\mathbf{a}, h, k) = f(\mathbf{a} + h\mathbf{e}_1 + k\mathbf{e}_2) - f(\mathbf{a} + h\mathbf{e}_1) - f(a + k\mathbf{e}_2) + f(\mathbf{a})$.

Define $\phi(\mathbf{x}) = f(\mathbf{x} + h\mathbf{e}_1) - f(\mathbf{x})$ and $\psi(\mathbf{x}) = f(\mathbf{x} + k\mathbf{e}_2) - f(\mathbf{x})$. Then

$$S(\mathbf{a}, h, k) = \phi(\mathbf{a} + k\mathbf{e}_2) - \phi(\mathbf{a}) = \psi(\mathbf{a} + h\mathbf{e}_1) - \psi(\mathbf{a}).$$

By the Mean Value Theorem, there exist $\mathbf{c} = \mathbf{a} + \theta_2 k\mathbf{e}_2$ and $\mathbf{d} = \mathbf{a} + \theta_1 h\mathbf{e}_1$ such that



$$\begin{aligned} S(\mathbf{a}, h, k) &= \phi(\mathbf{a} + k\mathbf{e}_2) - \phi(\mathbf{a}) = k\frac{\partial \phi}{\partial x_2}(\mathbf{c}) = k\Big(\frac{\partial f}{\partial x_2}(\mathbf{c} + h\mathbf{e}_1) - \frac{\partial f}{\partial x_2}(\mathbf{c})\Big) \\ &= \psi(\mathbf{a} + h\mathbf{e}_1) - \psi(\mathbf{a}) = h\frac{\partial \psi}{\partial x_1}(\mathbf{d}) = h\Big(\frac{\partial f}{\partial x_1}(\mathbf{d} + k\mathbf{e}_2) - \frac{\partial f}{\partial x_1}(\mathbf{d})\Big). \end{aligned}$$

Hence, if $h, k \neq 0$,

$$\frac{1}{k}\Big(\frac{\partial f}{\partial x_1}(\mathbf{d} + k\mathbf{e}_2) - \frac{\partial f}{\partial x_1}(\mathbf{d})\Big) = \frac{S(\mathbf{a}, h, k)}{hk} = \frac{1}{h}\Big(\frac{\partial f}{\partial x_2}(\mathbf{c} + h\mathbf{e}_1) - \frac{\partial f}{\partial x_1}(\mathbf{c})\Big)$$

By the Mean Value Theorem, there exists $\mathbf{c}_1 \in \overline{\mathbf{c}(\mathbf{c} + h\mathbf{e}_1)}$ and $\mathbf{d}_1 \in \overline{\mathbf{d}(\mathbf{d} + k\mathbf{e}_2)}$ such that

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{d}_1) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{c}_1)$$

Since $\dfrac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x})$ and $\dfrac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x})$ are continuous at $\mathbf{a}$, then $\mathbf{d}_1 \to \mathbf{a}$ and $\mathbf{c}_1 \to \mathbf{a}$ as $h, k \to 0$ and thus

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}).$$

□

Corollary 5.10.16. *Let $\mathcal{U} \subseteq \mathbb{R}$ be open and f is of class C^2. Then*

$$D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D^2 f(\mathbf{a})(\mathbf{v}, \mathbf{u})$$

for $\mathbf{a} \in \mathcal{U}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Remark. If $f : \mathcal{U} \to \mathbb{R}$ is of class C^2 and $\mathbf{a} \in \mathcal{U}$, the Hessian of f at $\mathbf{a}$ is the bilinear form $H_f(\mathbf{a}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$H_f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v}) \quad \text{for every } \mathbf{u}, \mathbf{v} \in \mathbb{R}$$

Since $f \in C^2$, $H_f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D^2 f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = D^2 f(\mathbf{a})(\mathbf{v}, \mathbf{u}) = H_f(\mathbf{a})(\mathbf{v}, \mathbf{u})$. The Hessian matrix

$$\left[H_f(\mathbf{a})\right] = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}(\mathbf{a}) \quad \text{is a symmetric matrix}$$

and

$$\left[\mathbf{u}\right]^T \left[H_f(\mathbf{a})\right] \left[\mathbf{v}\right] = H_f(\mathbf{a})(\mathbf{v}, \mathbf{u}) = H_f(\mathbf{a})(\mathbf{u}, \mathbf{v}) = \left[\mathbf{v}\right]^T \left[H_f(\mathbf{a})\right] \left[\mathbf{u}\right]$$

5.11 Taylor Theorem

Review: Let $f : (a, b) \to \mathbb{R}$, $\in C^{k+1}$ and $c \in (a, b)$. For $x \in (a, b)$, there exists $\xi \in (a, b)$ and ξ is between c and x such that

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(c)}{j!}(x - c)^j + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x - c)^{k+1}.$$

Question: Is there a similar result for higher dimensional cases?

Question: For $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, $f \in C^{k+1}$, can we apply 1-dimensional result to higher dimensional cases?

Theorem 5.11.1. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$ be of class C^{k+1}. Let $\mathbf{x}, \mathbf{a} \in \mathcal{U}$ and the line segment $\overline{\mathbf{xa}} \subseteq \mathcal{U}$. Then there exists a point $\mathbf{c}$ on $\overline{\mathbf{xa}}$ such that*

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{j=1}^{k} \frac{1}{j!} D^j f(\mathbf{a})\big(\underbrace{\mathbf{x} - \mathbf{a}, \cdots, \mathbf{x} - \mathbf{a}}_{j\text{–}copies}\big) + \frac{1}{(k+1)!} D^{k+1} f(\mathbf{c})\big(\underbrace{\mathbf{x} - \mathbf{a}, \cdots, \mathbf{x} - \mathbf{a}}_{(k+1)\text{–}copies}\big).$$

Proof. Let $\mathbf{r}(t) : [0,1] \to \mathcal{U}$ be given by $\mathbf{r}(t) = (1-t)\mathbf{a} + t\mathbf{x}$. Hence, $\mathbf{r}(0) = \mathbf{a}$ and $\mathbf{r}(1) = \mathbf{x}$ and $\mathbf{r} \in C^\infty$. Define $g(t) = f\big(\mathbf{r}(t)\big)$. Then $g : [0,1] \to \mathbb{R}$ be of class C^{k+1}. By the Taylor theorem (for single variable fucnions), there exists $t_0 \in (0,1)$ such that

$$g(1) = g(0) + \sum_{j=1}^{k} \frac{g^{(j)}(0)}{j!}(1-0)^j + \frac{g^{(k+1)}(t_0)}{(k+1)!}(1-0)^{k+1}. \tag{5.23}$$

By the chain rule,

$$\begin{aligned} g'(t) &= Df\big(\mathbf{r}(t)\big)\mathbf{r}'(t) = \big[Df\big(\mathbf{r}(t)\big)\big](\mathbf{x}-\mathbf{a}) = \sum_{i=1}^{n} \frac{\partial f\big(\mathbf{r}(t)\big)}{\partial x_i}(x_i - a_i) \\ g''(t) &= \sum_{i\,j=1}^{n} \frac{\partial^2 f\big(\mathbf{r}(t)\big)}{\partial x_j \partial x_i}(x_i - a_i)(x_j - a_j) = D^2 f\big(\mathbf{r}(t)\big)\big(\mathbf{x}-\mathbf{a}, \mathbf{x}-\mathbf{a}\big). \end{aligned}$$

By the induction,

$$g^{(i)}(t) = D^{(i)} f\big(\mathbf{r}(t)\big)\big(\mathbf{x}-\mathbf{a}, \cdots, \mathbf{x}-\mathbf{a}\big).$$

By (5.23), let $\mathbf{c} = \mathbf{r}(t_0)$,

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{j=1}^{k} \frac{1}{j!} D^j f(\mathbf{a})\big(\underbrace{\mathbf{x}-\mathbf{a}, \cdots, \mathbf{x}-\mathbf{a}}_{j-copies}\big) + \frac{1}{(k+1)!} D^{k+1} f(\mathbf{c})\big(\underbrace{\mathbf{x}-\mathbf{a}, \cdots, \mathbf{x}-\mathbf{a}}_{(k+1)-copies}\big).$$

□

Definition 5.11.2. Let $\mathcal{U} \subseteq \mathbb{R}$ be open and $f : \mathcal{U} \to \mathbb{R}$ be of class C^{k+1}. We call

$$\sum_{j=0}^{k} \frac{1}{j!} D^j f(\mathbf{a})\big(\mathbf{x}-\mathbf{a}, \cdots, \mathbf{x}-\mathbf{a}\big)$$

"*the kth degree Taylor polynomial for f centered at* **a**.

Corollary 5.11.3. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$ be of class C^{k+1}, and define*

$$R_{k,\mathbf{a}}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=0}^{k} \frac{1}{j!} D^j f(\mathbf{a})\big(\mathbf{x}-\mathbf{a}, \cdots, \mathbf{x}-\mathbf{a}\big).$$

Then $\lim\limits_{\mathbf{x}\to\mathbf{a}} \dfrac{R_{k,\mathbf{a}}(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}^k} = 0$. *We usually write* $R_{k,\mathbf{a}}(\mathbf{x}) = o\big(\|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n}^k\big)$ *as* $\mathbf{x} \to \mathbf{a}$.

Example 5.11.4. Let $f(x,y) = \sin(x+y^2)$. Find the third degree Taylor polynomial for f centered at $(0,0)$.

Proof.

$$\begin{aligned} f(x,y) &= f(0,0) + \sum_{j=1}^{3} \frac{1}{j!} D^j f(0,0)\Big(\langle (x,y)-(0,0)\rangle, \langle (x,y)-(0,0)\rangle\Big) \\ &= f(0,0) + \Big(f_x(0,0)x + f_y(0,0)y\Big) + \frac{1}{2!}\Big[f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + f_{yx}(0,0)yx + f_{yy}(0,0)y^2\Big] \\ &\quad + \frac{1}{3!}\Big[f_{xxx}(0,0)x^3 + f_{xxy}(0,0)x^2y + f_{xyx}(0,0)x^2y + f_{xyy}(0,0)xy^2 \\ &\qquad + f_{yxx}(0,0)x^2y + f_{yxy}(0,0)xy^2 + f_{yyx}(0,0)xy^2 + f_{yyy}(0,0)y^3\Big] \end{aligned}$$

Then

$$
\begin{aligned}
f(x,y) &= 0+\big(\cos 0\cdot x+0\cdot\cos 0\cdot y\big)\\
&\quad+\frac{1}{2!}\Big[-\sin 0\cdot x^2-2\cdot 0\cdot\sin 0\cdot xy-2\cdot 0\cdot\sin 0\cdot yx+(2\cos 0-4\cdot\sin 0)y^2\Big]\\
&\quad+\frac{1}{3!}\Big[-\cos 0\cdot x^3+2\cdot 0\cos 0\cdot x^2y-2\cdot 0\cdot\cos 0\cdot x^2y+(-2\sin 0-4\cdot 0\cos 0)xy^2\\
&\qquad-2\cdot 0\cdot\cos 0\cdot x^2y+(-2\sin 0-4\cdot 0\cdot\cos 0)xy^2+(-2\sin 0-4\cdot 0\cdot\cos 0)xy^2\\
&\qquad+(-4\cdot 0\cdot\sin 0-8\cdot 0\cdot\sin 0-8\cdot 0\cdot\cos 0)y^3\Big]\\
&= x+y^2-\frac{1}{6}x^3.
\end{aligned}
$$

Note. We can check whether the above Taylor polynomial is reasonable. Let $t = x + y^2$. The third degree Taylor polynomial for $f(t) = \sin t$ at $t = 0$ is

$$
\begin{aligned}
\sum_{j=0}^{3}\frac{f^{(j)}(0)}{j!}t^j &= t-\frac{t^3}{3!}=(x+y^2)-\frac{(x+y^2)^3}{3!}\\
&= x+y^2-\frac{1}{6}(x^3+3x^2y^2+3xy^4+y^6)\\
&= \underbrace{x+y^2-\frac{1}{6}x^3}_{\substack{\textit{third degree}\\ \textit{Taylor polynomial}}}-\frac{1}{2}x^2y^2-\frac{1}{2}xy^4-\frac{1}{6}y^6.
\end{aligned}
$$

□

Remark. The second degree Taylor polynonial for f centered at $\mathbf{a}$ is

$$P_{\mathbf{a},f}(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}\big[\mathbf{x}-\mathbf{a}\big]^T \underbrace{\big[H_f(\mathbf{a})\big]}_{\substack{\textit{Hessian}\\ \textit{matrix}}} \big[\mathbf{x}-\mathbf{a}\big]$$

Remark. Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ and $f \in C^3$. If $\mathbf{a} \in \mathcal{U}$ is a critical point of f and $H_f(\mathbf{a})$ is positive definite, then f has a minimum value at $\mathbf{a}$.

Proof. By the Taylor theorem,

$$f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}\big[\mathbf{x}-\mathbf{a}\big]^T \big[H_f(\mathbf{a})\big]\big[\mathbf{x}-\mathbf{a}\big] + R_{2,\mathbf{a}}(\mathbf{x}).$$

where $\lim_{\mathbf{x}\to\mathbf{a}} \dfrac{R_{2,\mathbf{a}}(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|^2_{\mathbb{R}^n}} = 0$.

Since $\mathbf{a}$ is a critical point of f, $Df(\mathbf{a}) = 0$. Since $H_f(\mathbf{a})$ is positive definite, there exists $c > 0$ such that for every $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v}^T\big[H_f(\mathbf{a})\big]\mathbf{v} \geq c\|\mathbf{v}\|^2_{\mathbb{R}^n}.$$

Then

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{a}) &= \frac{1}{2}\left[\mathbf{x}-\mathbf{a}\right]^T \left[H_f(\mathbf{a})\right] \left[\mathbf{x}-\mathbf{a}\right] + R_{2,\mathbf{a}}(\mathbf{x}) \\ &\geq \frac{1}{2}c\|\mathbf{x}-\mathbf{a}\|^2_{\mathbb{R}^n} + R_{2,\mathbf{a}}(\mathbf{x}) \\ &\geq \frac{1}{4}c\|\mathbf{x}-\mathbf{a}\|^2_{\mathbb{R}^n} \quad \text{(as } \|\mathbf{x}-\mathbf{a}\|_{\mathbb{R}^n} \text{ is sufficiently small.)} \\ &\geq 0. \end{aligned}$$

Hence, $f(\mathbf{a})$ is a local minimum. Note that the number c is the smallest eigenvalue of $H_f(\mathbf{a})$. □

5.12 Maximum and Minimum

Review: Let $f : (a, b) \to \mathbb{R}$ be twice differentiable. Find the maxima (or minima) of f on (a, b).

(i) find all critical points ($f'(x) = 0$ or $f'(x)$ DNE)

(ii) Using the first derivative test or the second derivative test

Question: How about the two or more variables functions? Is there similar results for higher dimensional cases?

Definition 5.12.1. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$. We say that

(1) a point $\mathbf{x}_0$ is a "*global (absolute) minimum (maximum) point of f*" if

$$f(\mathbf{x}_0) \underset{(\geq)}{\leq} f(\mathbf{x}) \quad \text{for every } \mathbf{x} \in \mathcal{U}.$$

(2) a point $\mathbf{x}_0 \in \mathcal{U}$ is a "*local minimum (maximum) point of f*" if there exists a neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of $\mathbf{x}_0$ such that

$$f(\mathbf{x}_0) \underset{(\geq)}{\leq} f(\mathbf{x}) \quad \text{for every } \mathbf{x} \in \mathcal{V}.$$

(3) a point $\mathbf{x}_0 \in \mathcal{U}$ is a "*local (global) extreme point of f*" if $\mathbf{x}_0$ is either a lcoal (global) minimum point or a local (global) maximum point of f.

(4) a point $\mathbf{x}_0 \in \mathcal{U}$ is a "*critical point of f*" if either

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0$$

or at least one of $\dfrac{\partial f}{\partial x_i}(\mathbf{x}_0)$ does not exist.
Note that if f is differenitable at $\mathbf{x}_0$ and $\mathbf{x}_0$ is a critical point of f, then $Df(\mathbf{x}_0) = \mathbf{0}$ (or $\nabla f(\mathbf{x}_0) = \mathbf{0}$).

(5) a point $\mathbf{x}_0$ is a "*saddle point of* f" if f is differentiable at $\mathbf{x}_0$ and $\mathbf{x}_0$ is a critical point, but not an extreme point of f.

Theorem 5.12.2. *Let $\mathcal{U} \to \mathbb{R}^n$ be open, $f : \mathcal{U} \to \mathbb{R}$ be differentiable and $\mathbf{x}_0 \in \mathcal{U}$ be an extreme point of f. Then $\mathbf{x}_0$ is a critical point of f.*

Proof. W.L.O.G, let $\mathbf{x}_0$ be a local minimum point of f. Suppose that $Df(\mathbf{x}_0) \neq \mathbf{0}$. Then there exists a unit vector $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$ such that $Df(\mathbf{x}_0)\mathbf{u} = c \neq 0$. We may assume that $c < 0$ (otherwise replacing $\mathbf{u}$ by $-\mathbf{u}$).

Since f is differentiable at $\mathbf{x}_0$, there exists $\delta > 0$ such that if $\|\mathbf{h}\|_{\mathbb{R}^n} < \delta$,

$$\left|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)\mathbf{h}\right| < \frac{|c|}{2}\|\mathbf{h}\|_{\mathbb{R}^n}.$$

Taking $0 < \lambda < \delta$, then

$$\lambda|c| < \underbrace{\underbrace{f(\mathbf{x}_0 + \lambda\mathbf{u}) - f(\mathbf{x}_0)}_{>0} - \underbrace{Df(\mathbf{x}_0)(\lambda\mathbf{u})}_{=\lambda c<0}}_{>\lambda|c|} < \frac{|c|}{2}\|\lambda\mathbf{u}\|_{\mathbb{R}^n} = \frac{|\lambda c|}{2}.$$

Then we obtain a contradiction and hence $Df(\mathbf{x}_0) = \mathbf{0}$. ☐

Definition 5.12.3. Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form. B is called

(1) "*positive definite*" ("*negative definite*") if $B(\mathbf{u}, \mathbf{u}) > 0$ (< 0) for every $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$.

(2) "*positive semi-definite*" ("*negative semi-definite*") if $B(\mathbf{u}, \mathbf{u}) \geq 0$ (≤ 0") for every $\mathbf{u} \in \mathbb{R}^n$.

Remark. From the second degree Taylor polynomial for f centered at $\mathbf{x}_0$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}\left[\mathbf{x} - \mathbf{x}_0\right]^T \left[H_f(\mathbf{x}_0)\right] \left[\mathbf{x} - \mathbf{x}_0\right].$$

Let $\mathbf{x}_0$ be an extreme point of f. Then

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2}\left[\mathbf{x} - \mathbf{x}_0\right]^T \left[H_f(\mathbf{x}_0)\right] \left[\mathbf{x} - \mathbf{x}_0\right].$$

Hence,

(i) if $\left[H_f(\mathbf{x}_0)\right]$ is positive definite, then $\mathbf{x}_0$ is a local minimum.

(ii) if $\left[H_f(\mathbf{x}_0)\right]$ is negative definite, then $\mathbf{x}_0$ is a local maximum.

Theorem 5.12.4. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$ be a function of class C^2.*

(1) If $\mathbf{x}_0$ is a critical point of f such that the Hessian $H_f(\mathbf{x}_0)$ is negative (positive) definite, then f has a local maximum (minimum) point at $\mathbf{x}_0$. (sufficient condition)

(2) If f has a local maximum (minimum) point at $\mathbf{x}_0$, then $H_f(\mathbf{x}_0)$ is negative (positive) semi-definite. (necessary condition)

Idea: *Since $H_f(\mathbf{x}_0)$ is negative definite and $f \in C^2$, $H_f(\mathbf{x})$ is negative definite as $\mathbf{x} \approx \mathbf{x}_0$. By the Taylor theorem,*

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \underbrace{Df(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)}_{=0} + \underbrace{\frac{1}{2}D^2 f(\mathbf{c})(\mathbf{x}-\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)}_{\approx \frac{1}{2}\left[\mathbf{x}-\mathbf{x}_0\right]^T\left[H_f(\mathbf{x}_0)\right]\left[\mathbf{x}-\mathbf{x}_0\right]<0}.$$

Proof. (1) Let $S = \left\{\mathbf{u} \in \mathbb{R}^n \;\middle|\; \|\mathbf{u}\|_{\mathbb{R}^n} = 1\right\}$ be a compact subset in $\mathbb{R}^n$. Define $g : S \to \mathbb{R}$ by $g(\mathbf{u}) = H_f(\mathbf{x}_0)(\mathbf{u},\mathbf{u}) \left(= \mathbf{u}^T \left[H_f(\mathbf{x}_0)\right] \mathbf{u}\right)$. Then g is continuous on S and hence g attains its maximum. That is, there exists $\mathbf{u}_0 \in S$ such that

$$0 \underset{\substack{negative\\definite}}{>} H_f(\mathbf{x}_0)(\mathbf{u}_0,\mathbf{u}_0) = \underbrace{g(\mathbf{u}_0)}_{=\lambda} = \max_{\mathbf{u}\in S} g(\mathbf{u}) = \max_{\|\mathbf{u}\|_{\mathbb{R}^n}=1} H_f(\mathbf{x}_0)(\mathbf{u},\mathbf{u}).$$

Hence, for $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} \neq \mathbf{0}$,

$$H_f(\mathbf{x}_0)(\mathbf{u},\mathbf{u}) \underset{bilinear}{=} \|\mathbf{u}\|^2_{\mathbb{R}^n} H_f(\mathbf{x}_0)\Big(\frac{\mathbf{u}}{\|\mathbf{u}\|_{\mathbb{R}^n}}, \frac{\mathbf{u}}{\|\mathbf{u}\|_{\mathbb{R}^n}}\Big) < \lambda\|\mathbf{u}\|^2_{\mathbb{R}^n} < 0 \tag{5.24}$$

Since f is of class C^2, there exists $\delta > 0$ such that if $\|\mathbf{x}-\mathbf{x}_0\|_{\mathbb{R}^n} < \delta$,

$$\left\|H_f(\mathbf{x}) - H_f(\mathbf{x}_0)\right\|_{\mathcal{B}(\mathbb{R}^n;\mathcal{B}(\mathbb{R}^n;\mathbb{R}))} < \frac{|\lambda|}{2}.$$

Thus,

$$\Big|H_f(\mathbf{x})(\mathbf{u},\mathbf{u}) - H_f(\mathbf{x}_0)(\mathbf{u},\mathbf{u})\Big| \le \left\|H_f(\mathbf{x}) - H_f(\mathbf{x}_0)\right\|_{\mathcal{B}(\mathbb{R}^n;\mathcal{B}(\mathbb{R}^n;\mathbb{R}))}\|\mathbf{u}\|^2_{\mathbb{R}^n} < \frac{|\lambda|}{2}\|\mathbf{u}\|^2_{\mathbb{R}^n} \tag{5.25}$$

for every $\|\mathbf{x}-\mathbf{x}_0\|_{\mathbb{R}^n} < \delta$ and every $\mathbf{u} \neq \mathbf{0}$.

By Taylor Theorem, for $\|\mathbf{x}-\mathbf{x}_0\|_{\mathbb{R}^n} < \delta$,

$$\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{x}_0) + \underbrace{Df(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)}_{=0 \; critical\;\; point} + \frac{1}{2}D^2 f(\mathbf{c})(\mathbf{x}-\mathbf{x}_0,\mathbf{x}-\mathbf{x}_0) \quad \text{for some } \mathbf{c} \in \overline{\mathbf{x}\mathbf{x}_0} \\
&= f(\mathbf{x}_0) + \frac{1}{2}D^2 f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0,\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}\Big(D^2 f(\mathbf{x}) - D^2 f(\mathbf{x}_0)\Big)(\mathbf{x}-\mathbf{x}_0,\mathbf{x}-\mathbf{x}_0) \\
&= f(\mathbf{x}_0) + \frac{1}{2}\left[\mathbf{x}-\mathbf{x}_0\right]^T\left[H_f(\mathbf{x}_0)\right]\left[\mathbf{x}-\mathbf{x}_0\right] + \frac{1}{2}\left[\mathbf{x}-\mathbf{x}_0\right]^T\left[H_f(\mathbf{c}) - H_f(\mathbf{x}_0)\right]\left[\mathbf{x}-\mathbf{x}_0\right] \\
&\overset{(5.24)(5.25)}{<} f(\mathbf{x}_0) + \frac{1}{2}\lambda\|\mathbf{x}-\mathbf{x}_0\|^2_{\mathbb{R}^n} - \frac{\lambda}{2}\|\mathbf{x}-\mathbf{x}_0\|^2_{\mathbb{R}^n} \\
&\le f(\mathbf{x}_0)
\end{aligned}$$

Hence, $\mathbf{x}_0$ is a local maximum point of f.

(2) Assume that $H_f(\mathbf{x}_0)$ is not negative semi-definite. Then there exists $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\|_{\mathbb{R}^n} = 1$ such that $H_f(\mathbf{x}_0)(\mathbf{u}, \mathbf{u}) > 0$. $\boxed{\text{To prove that } f \text{ is not local maximum along the direction } \mathbf{u}.}$

Since $\mathbf{x}_0$ is a local maximum point of f, $Df(\mathbf{x}_0) = 0$ and there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{x}_0\|_{\mathbb{R}^n} < \delta$, $f(\mathbf{x}) \le f(\mathbf{x}_0)$. By the Taylor theorem,

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \underbrace{Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}_{=0} + \frac{1}{2}\left[\mathbf{x} - \mathbf{x}_0\right]^T \left[H_f(\mathbf{c}_\mathbf{x})\right] \left[\mathbf{x} - \mathbf{x}_0\right]$$

for some $\mathbf{c}_\mathbf{x} \in \overline{\mathbf{x}\mathbf{x}_0}$. Hence,

$$\left[\mathbf{x} - \mathbf{x}_0\right]^T \left[H_f(\mathbf{c}_\mathbf{x})\right] \left[\mathbf{x} - \mathbf{x}_0\right] = 2\big(f(\mathbf{x}) - f(\mathbf{x}_0)\big) \le 0. \tag{5.26}$$

Let $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$. Then $\mathbf{x} \to \mathbf{x}_0$ as $t \to 0$. Therefore, $\mathbf{c}_\mathbf{x} \to \mathbf{x}_0$ as $t \to 0$. By (5.26), $H_f(\mathbf{c}_\mathbf{x})(\mathbf{u}, \mathbf{u}) \le 0$ for $t \in (0, \delta)$. Since $f \in C^2$ and $\mathbf{c}_\mathbf{x} \to \mathbf{x}_0$ as $t \to 0$,

$$H_f(\mathbf{x}_0)(\mathbf{u}, \mathbf{u}) = \lim_{t \to 0} H_f(\mathbf{c}_\mathbf{x})(\mathbf{u}, \mathbf{u}) \le 0$$

We obtain a contradition and hence $H_f(\mathbf{x}_0)$ is negative semi-definite.

□

Remark.
Question: How to determine whether a matrix $A \in M_{n\times n}(\mathbb{R})$ is positive (negative) (semi)-definite?

Method 1: If A is symmetric, diagonalizing A.

$$A \quad \longrightarrow \quad \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{where } \lambda_i : \text{eigenvalue.}$$

(i) If $\lambda_1, \cdots, \lambda_n > 0$, then A is positive definite.

(ii) If $\lambda_1, \cdots, \lambda_n \ge 0$, then A is positive semi-definite.

(iii) If $\lambda_1, \cdots, \lambda_n < 0$, then A is negative definite.

(iv) If $\lambda_1, \cdots, \lambda_n \le 0$, then A is positive semi-definite.

Method 2: (Sylvester's criterion) For the matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$, we define

$$\triangle_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \quad \text{for } k = 1, \cdots, n.$$

(i) if $\det(\triangle_k) > 0$ for every $k = 1, \cdots, n$, then A is positive definite

(ii) if $\det(\triangle_k) < 0$ for $k = 1, 3, 5 \cdots$ and $\det(\triangle_k) > 0$ for $k = 2, 4, 6, \cdots$ (or write $(-1)^k \det(\triangle_k) > 0$), then A is negative definite.

(iii) if $\det(\triangle_k) > 0$ for $k = 1, 2, \cdots, n-1$, $\det A = 0$, then A is positive semi-definite.

(iv) if $(-1)^k \det(\triangle_k) > 0$ for $k = 1, 2, \cdots, n-1$ and $\det A = 0$, then A is negative semi-definite.

In particular, let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

(i) if $a > 0$ and $\det A > 0$, then A is positive definite.

(ii) if $a < 0$ and $\det A > 0$, then A is negative definite.

(iii) if $\det A \le 0$, then A is indefinite.

Theorem 5.12.5. *Let $f \in C^2(\mathbb{R}^2; \mathbb{R})$, $\nabla f(x_0, y_0) = 0$, $D = f_{xx} f_{yy} - (f_{xy})^2$.*

(1) If $f_{xx} > 0$ and $D > 0$, then f has a local minimum at (x_0, y_0).

(2) If $f_{xx} < 0$ and $D > 0$, then f has a local maximum at (x_0, y_0).

(3) If $D < 0$, then f has a saddle point at (x_0, y_0).

(4) If $D = 0$, no conclusion can be drawn.

Example 5.12.6. Let $f(x, y, z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2$. Then

$$\nabla f(x, y, z) = \langle e^{x-y} - e^{y-x} + 2xe^{x^2}, -e^{x-y} + e^{y-x}, 2z \rangle.$$

The point $(0, 0, 0)$ is the only critical point. The Hessian of f is

$$Hf(x, y, z) = \begin{bmatrix} e^{x-y} + e^{y-x} + 4x^2 e^{x^2} + 2e^{x^2} & -e^{x-y} - e^{y-x} & 0 \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

At $(0, 0, 0)$, $H_f(0, 0, 0) = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. We compute that $\det(\triangle_1) = 3$, $\det(\triangle_2) = 2$ and $\det(\triangle_3) = 4$. Hence, $H_f(0, 0, 0)$ is positive definite. We have $(0, 0, 0)$ is a local (global) minimum point of f.

❑ Lagrange Multipliers

In this section, we will study the "*Lagrange multipliers*" which gives a method to find the maximum or minimum of a function $\mathbf{h}(\mathbf{x})$ subject to a constraint (or side condition) $\mathbf{f}(\mathbf{x}) = \mathbf{C}$.

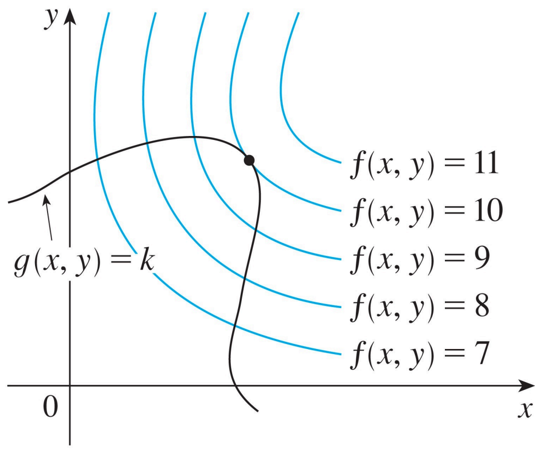
In the course of Elementary Calculus, we have learned some special cases. For example, to find the maximum (or minimum) of $f(x, y)$ subject to the constraint $g(x, y) = k$.

■ One Constraint

We want to find a point(s) (x_0, y_0) on the level curve $C = \{(x,y) \mid g(x,y) = k\}$ such that

$$f(x_0, y_0) \geq f(x,y) \quad \text{for all } (x,y) \in C. \tag{5.27}$$

Suppose that $(x_0, y_0) \in C$ satisfying (5.27) and $f(x_0, y_0) = M$. Then (x_0, y_0) is also on the level curve $C_1 = \{(x,y) \mid f(x,y) = M\}$. Moreover, since (x_0, y_0) is the maximum point, *the two level curve C and C_1 must be tangent each other at (x_0, y_0)*.



Since C and C_1 are level curves of g and f respectively, the gradient vectors $\nabla g \perp C$ and $\nabla f \perp C_1$. Then $\nabla g(x_0, y_0)$ is parallel to $\nabla f(x_0, y_0)$. Therefore, there exists a number λ ("*Lagrange multiplier*") such that

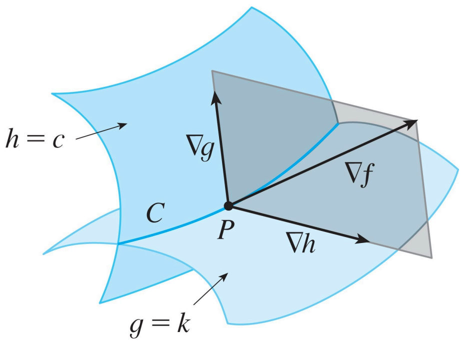
$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

■ **Two Constraints**

Furthermore, we also discuss the Lagrange multipliers with two constraints.

Find the maximum and minimum values of $f(x,y,z)$ subject to two constraints $g(x,y,z) = k$ and $h(x,y,z) = c$.

Let C be the intersection of the two level surfaces $g(x,y,z) = k$ and $h(x,y,z) = c$. Find $P(x_0, y_0, z_0) \in C$ such that $f(x_0, y_0, z_0)$ ahs extreme value along C.



To find the level surface $S = \{(x,y,z) \mid f(x,y,z) = M\}$ which tangnet to C. Then , at the intersection of C and S, $\nabla f \perp C$. We have

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

■ **General Cases**

Theorem 5.12.7. *Let $m < n$, V be open in $\mathbb{R}^n$, and $f, g_j : V \to \mathbb{R}$ be C^1 function on V for $j = 1, 2, \cdots, m$. Suppose that there is an $\mathbf{a} \in V$ such that*

$$\frac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_m)}(\mathbf{a}) \neq 0.$$

If $f(\mathbf{a})$ is a local extremum of f subject to the constraints $g_k(\mathbf{a}) = 0$ for $k = 1, \cdots m$, then there exist scalars $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that

$$\nabla f(\mathbf{a}) = \sum_{k=1}^{m} \lambda_k \nabla g_k(\mathbf{a}) = \mathbf{0}_m. \tag{5.28}$$

(I) 限制條件 $g_1, \cdots g_m$ 彼此間不能互相矛盾，例: $g_1(x,y) = 2x+3y$ 和 $g_2(x,y) = 4x+6y-1$, 則無法找到 $\mathbf{a} \in \mathbb{R}^n$ 使得 $g_1(\mathbf{a}) = g_2(\mathbf{a}) = 0$. 當兩函數的 *level sets* 相交可避免此狀況，即在滿足此兩限制條件下的點 $\mathbf{a}$, $\nabla g_1(\mathbf{a})$ 與 $\nabla g_2(\mathbf{a})$ 不會平行。因此，當設定

$$\frac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_m)}(\mathbf{a}) \neq 0.$$

條件下，可避免任兩 *level sets* 相切或平行狀況。亦可保證在 $\mathbf{a}$ 點附近的滿足所有限制條件的集合，即 *level sets* 的交集 $\bigcap_{j=1}^{m} \{\mathbf{x} \in V \mid g_j(\mathbf{x}) = 0\}$ 是一個 $n-m$ 維度的曲面。

(II) 幾何上來說，我們是在兩函數的 *level sets* 的交集上找滿足 f 的極值點，若 *constraints* 太多 $(m \geq n)$，則可能發生

(1) 無法找到能滿足所有 *constraints* 的可行點集;

(2) 限制條件 (*constraints*) 之間可能彼此相關 (即可移去部份條件);

(3) 每多一個條件，則 *level sets* 的交集少一個維度，當 $m = n$ 時，可能僅剩有限可行點。

(III) 在 $S := \bigcap_{j=1}^{m} \{\mathbf{x} \in V \mid g_j(\mathbf{x}) = 0\}$ 這個 $n-m$ 維度曲面上找 f 的極值點 $\mathbf{a}$，則 S 在 $\mathbf{a}$ 點的切空間 $T_{\mathbf{a}}S$ 的 *orthonormal space* $\left(T_{\mathbf{a}}S\right)^{\perp}$ 是一個 m 維的向量空間，由 $Span\{\nabla g_1(\mathbf{a}), \cdots, \nabla g_m(\mathbf{a})\}$ 所構成。因 f 在 $\mathbf{a}$ 有極值，f 在 $\mathbf{a}$ 這一層的 *level set* $\{\mathbf{x} \in V \mid f(\mathbf{x}) = f(\mathbf{a})\}$ 應在 $\mathbf{a}$ 點與 S 相切，則 $\nabla f(\mathbf{a})$ 會屬於 $\left(T_{\mathbf{a}}S\right)^{\perp} = Span\{\nabla g_1(\mathbf{a}), \cdots, \nabla g_m(\mathbf{a})\}$. 因此

$$\nabla f(\mathbf{a}) = \sum_{k=1}^{m} \lambda_k \nabla g_k(\mathbf{a}) = \mathbf{0}_m.$$

Note. Let M and N be two smooth manifolds with dimensions m and n, say $m \leq n$. Suppose M and N are tangent to each other at $\mathbf{a}$. Then $T_{\mathbf{a}}M \subseteq T_{\mathbf{a}}N$. This implies $\left(T_{\mathbf{a}}N\right)^{\perp} \subseteq \left(T_{\mathbf{a}}M\right)^{\perp}$. Hence, if $\mathbf{u} \perp N$ at $\mathbf{a}$, then $\mathbf{u} \in \left(T_{\mathbf{a}}N\right)^{\perp} \subseteq \left(T_{\mathbf{a}}M\right)^{\perp}$.

Proof. Equation (5.28) can be written as

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_1}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_1}(\mathbf{a}) = 0 \\ \dfrac{\partial f}{\partial x_2}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_2}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_2}(\mathbf{a}) = 0 \\ \vdots \\ \dfrac{\partial f}{\partial x_m}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_m}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_m}(\mathbf{a}) = 0 \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_n}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_n}(\mathbf{a}) = 0 \end{cases}$$

which is a system of n linear equations with m unknown variables $\lambda_1, \cdots, \lambda_m$. Since $\dfrac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_m)}(\mathbf{a}) \neq 0$, the first m equations in the system determines uniquely the λ_k's. Hence, it suffices to show that for those $\lambda_1, \cdots, \lambda_m$, the remaining system with $n - m$ equations

$$\begin{cases} \dfrac{\partial f}{\partial x_{m+1}}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_{m+1}}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_{m+1}}(\mathbf{a}) = 0 \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{a}) + \lambda_1 \dfrac{\partial g_1}{\partial x_n}(\mathbf{a}) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial x_n}(\mathbf{a}) = 0 \end{cases}$$

holds.

Let $p = n - m$. As in the proof of the Implicit Function Theorem, write vector in $\mathbb{R}^{m+p}$ int the form $\mathbf{x} = (\mathbf{y}, \mathbf{t}) = (y_1, \cdots, y_m, t_1, \cdots, t_p)$. We have to show that

$$\frac{\partial f}{\partial t_\ell}(\mathbf{a}) + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}) = 0$$

for $\ell = 1, \cdots, p$.

Let $\mathbf{g} = (g_1, \cdots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$. For $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (\mathbf{y}, \mathbf{t})$ where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{t} \in \mathbb{R}^p$. Choose $\mathbf{a} = (\mathbf{y}_0.\mathbf{t}_0)$ for some $\mathbf{y}_0 \in \mathbb{R}^m$ and $\mathbf{t}_0 \in \mathbb{R}^p$. Then $\mathbf{g}(\mathbf{y}_0, \mathbf{t}_0) = \mathbf{0}_m$ and $D_{\mathbf{y}}\mathbf{g}(\mathbf{y_0}, \mathbf{t}_0)$ is invertible.

By the Implicit Function Theorem, there exists an open set $W \subseteq \mathbb{R}^p$ which contains $\mathbf{t}_0$ and a function $\mathbf{h} : W \to \mathbb{R}^m$ such that $\mathbf{h}$ is continuously differentiable on W, $\mathbf{h}(t_0) = \mathbf{y}_0$, and

$$\mathbf{g}\big(\mathbf{h}(\mathbf{t}), \mathbf{t}\big) = \mathbf{0}_m \quad \text{for every } \mathbf{t} \in W.$$

For every $\mathbf{t} \in W$ and $k = 1, \cdots, m$, define

$$G_k(\mathbf{t}) = g_k\big(\mathbf{h}(\mathbf{t}), \mathbf{t}\big) \quad \text{and} \quad F(\mathbf{t}) = f\big(\mathbf{h}(\mathbf{t}), \mathbf{t}\big).$$

Since $\mathbf{g}\big(\mathbf{h}(\mathbf{t}),\mathbf{t}\big) = \mathbf{0}_m$ on W, $G_k(\mathbf{t})$ is identically zero on W for $k = 1,\cdots,k$ and hence $D_{\mathbf{t}}G_k(\mathbf{t}) \equiv \mathbf{0}_{1\times p}$ (the zero matrix $\big[0\big]_{1\times p}$).

Since $\mathbf{t}_0 \in W$ and $\big(\mathbf{h}(\mathbf{t}_0),\mathbf{t}_0\big) = (\mathbf{y}_0,\mathbf{t}_0) = \mathbf{a}$, by the Chain Rule,

$$\mathbf{0}_{1\times p} = D_{\mathbf{t}}G_k(\mathbf{t}_0) = \begin{bmatrix} \dfrac{\partial g_k}{\partial x_1}(\mathbf{a}) & \cdots & \dfrac{\partial g_k}{\partial x_n}(\mathbf{a}) \end{bmatrix}_{1\times n} \begin{bmatrix} \dfrac{\partial h_1}{\partial t_1}(\mathbf{t_0}) & \cdots & \dfrac{\partial h_1}{\partial t_p}(\mathbf{t_0}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial h_m}{\partial t_1}(\mathbf{t_0}) & \cdots & \dfrac{\partial h_m}{\partial t_p}(\mathbf{t_0}) \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n\times p}$$

Hence, the ℓth component of $DG_k(\mathbf{t}_0)$ is

$$\sum_{j=1}^{m} \frac{\partial g_k}{\partial x_j}(\mathbf{a})\frac{\partial h_j}{\partial t_\ell}(\mathbf{t}_0) + \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}) \tag{5.29}$$

for $k = 1,2,\cdots,m$. Multiplying (5.29) by λ_k and adding, we have

$$\begin{aligned} 0 &= \sum_{k=1}^{m}\sum_{j=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_j}(\mathbf{a})\frac{\partial h_j}{\partial t_\ell}(\mathbf{t}_0) + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}) \\ &= \sum_{j=1}^{m} \Big[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_j}(\mathbf{a})\Big] \frac{\partial h_j}{\partial t_\ell}(\mathbf{t}_0) + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}). \end{aligned}$$

Therefore,

$$0 = -\sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(\mathbf{a})\frac{\partial h_j}{\partial t_\ell}(\mathbf{t}_0) + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}). \tag{5.30}$$

Suppose that $f(\mathbf{a})$ is a local maximum subject to the constraints $\mathbf{g}(\mathbf{a}) = \mathbf{0}_m$. Let $E_0 = \big\{\mathbf{x} \in V \,\big|\, \mathbf{g}(\mathbf{x}) = \mathbf{0}\big\}$, and choose an n-dimensional open ball $B_n(\mathbf{a}, r)$ such that

$$f(\mathbf{x}) \le f(\mathbf{a}) \quad \text{for every } \mathbf{x} \in B_n(\mathbf{a}, r) \cap E_0.$$

Since $\mathbf{h}$ is continuous, choose a p-dimensional open ball $B_p(\mathbf{t}_0, \varepsilon)$ scuh that $\big(\mathbf{h}(\mathbf{t}),\mathbf{t}\big) \in B_n(\mathbf{a}, r)$ for every $\mathbf{t} \in B_p(\mathbf{t}_0, \varepsilon)$. Since $F(\mathbf{t}_0)$ is a local maximum of F on $B_p(\mathbf{t}_0)$, $\nabla F(\mathbf{t}_0) = \mathbf{0}_p$. Applying the Chain Rule as above, we obtain

$$0 = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(\mathbf{a})\frac{\partial h_j}{\partial t_\ell}(\mathbf{t}_0) + \frac{\partial f}{\partial t_\ell}(\mathbf{a}) \tag{5.31}$$

Adding (5.30) and (5.31), we conclude that

$$0 = \frac{\partial f}{\partial t_\ell}(\mathbf{a}) + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial t_\ell}(\mathbf{a}).$$

[Note that the proof is refered to the book “*Introduction to Analysis 4th Ed.*”, William R. Wade, page 443-445.] □

Example 5.12.8. Find all extrema of $x^2 + y^2 + z^2$ subject to the constraints $x - y = 1$ and $y^2 - z^2 = 1$.

Proof. Let $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x - y - 1$ and $h(x, y, z) = y^2 - z^2 - 1$. Then

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle, \quad \nabla g(x, y, z) = \langle 1, -1, 0\rangle \quad \text{and} \quad \nabla h(x, y, z) = \langle 0, 2y - 2z\rangle.$$

Consider $\nabla f + \lambda\nabla g + \mu\nabla h = \mathbf{0}$. That is,

$$\langle 2x + \lambda, 2y - \lambda + 2\mu y, 2z - 2\mu z\rangle = \langle 0, 0, 0\rangle.$$

To solve

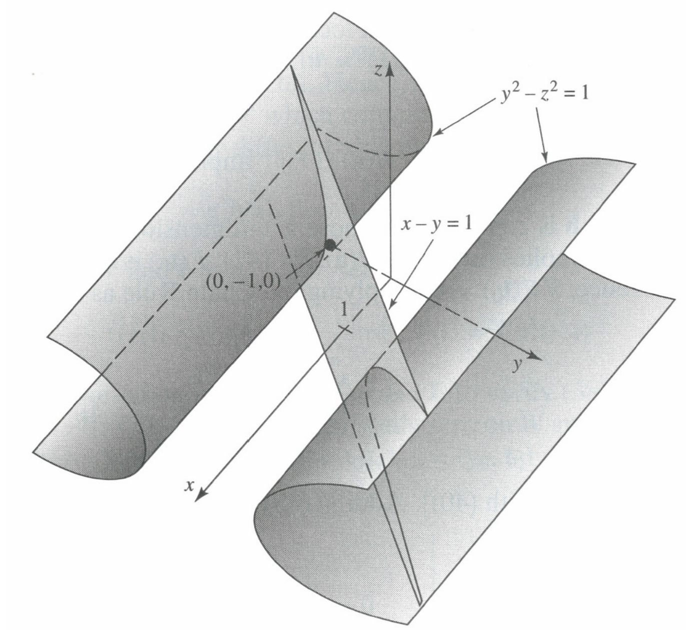
$$\begin{cases} 2x + \lambda = 0 & (5.32) \\ 2y - \lambda + 2\mu y = 0 & (5.33) \\ 2z - 2\mu z = 0 & (5.34) \end{cases}$$

By (5.34), either $z = 0$ or $\mu = 1$

(1) If $\mu = 1$, by (5.32) and (5.33), $\lambda = -2x = 4y$. Thus, $x = -2y$. From $g(x, y) = x - y - 1 = 0$, we have $(x, y) = (\frac{2}{3}, -\frac{1}{3})$. But it cannot make $h(x, y, z) = y^2 - z^2 - 1 = 0$.

(2) If $z = 0$, by $h(x, y, z) = y^2 - z^2 - 1 = 0$ and $g(x, y, z) = x - y - 1 = 0$, we have $(x, y) = (2, 1)$ or $(0, -1)$. Therefore, the only possible extreme points are $(2, 1, 0)$ and $(0, -1, 0)$. The only candidates for extrema of f subject to the constraints $g = 0 = h$ are $f(2, 1, 0) = 5$ and $f(0, -1, 0) = 1$.

Geometrically, this problem is to find the points on the intersection of the plane $x - y = 1$ and the hyperbolic cylinder $y^2 - z^2 = 1$ which lie closest to the origin. both of these points correspond to local minima, and there is no maxima. In particular, the minimum of $x^2 + y^2 + z^2$ subject to the given constraints is 1, attained at the point $(0, -1, 0)$.



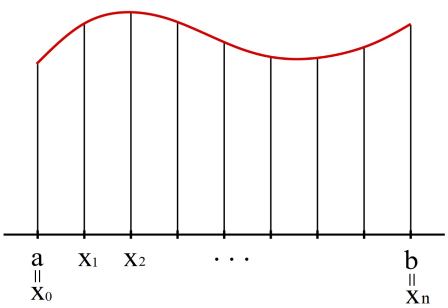
□

Chapter 6

Integration of Functions of Several Variables

6.1 Integrable Functions

Review: Let $f : [a,b] \to \mathbb{R}$ be a bounded function.



Let $P = \{x_0 < x_1 < \cdots < x_n\}$ be a partition of $[a,b]$. The upper and lower sums of P for f are

$$U(P,f) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$$

$$L(P,f) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$$

If P_1 and P_2 are two partitions of $[a,b]$ and $P_1 \subseteq P_2$, then

$$L(P_1,f) \le L(P_2,f) \le U(P_2,f) \le U(P_1,f).$$

The lower and upper integrals are

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f) \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf_P(P,f)$$

and

$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f).$$

If $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$, we call f is "*(Darboux) integrable*" on $[a,b]$ and denote the number $\int_a^b f(x)\,dx$.

Remark. A function f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

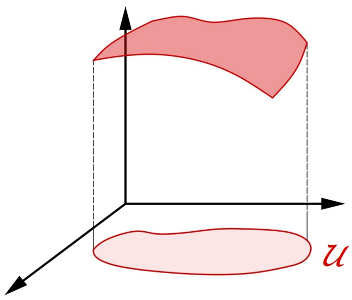
Definition 6.1.1. Let P be a partition of $[a,b]$ and $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$.

(1) We call the form $\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$ the "*Riemann sum for f over* $[a,b]$".

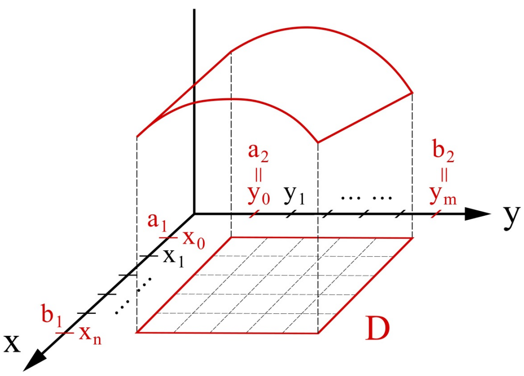
(2) If $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$ exists where $\|P\| = \max_{1\le i\le n}(x_i - x_{i-1})$, we say f is "*(Riemann) integrable on* $[a,b]$" and hence $\int_a^b f(x)\,dx$ exists.

■ **Multi-variable Functions**

Question: Let $\mathcal{U} \subseteq \mathbb{R}^2$ be a bounded set and $f : \mathcal{U} \to \mathbb{R}$. How to compute the volume below the graph of f?



Let $D = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$ be a bounded function. Denote



$$\begin{aligned} P_x &= \{a_1 = x_0 < x_1 < \cdots < x_n = b_1\}, \\ P_y &= \{a_2 = y_0 < y_1 < \cdots < y_m = b_2\} \text{ and} \\ P &= \left\{\triangle_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \;\middle|\; 1 \le i \le n,\ 1 \le j \le m\right\}. \end{aligned}$$

The lower and upper sums of P for f are

$$\begin{aligned} U(P,f) &= \sum_{\substack{1\le i\le n \\ 1\le j\le m}} \sup_{\mathbf{x}\in\triangle_{ij}} f(\mathbf{x})\mathbb{A}(\triangle_{ij}) \\ L(P,f) &= \sum_{\substack{1\le i\le n \\ 1\le j\le m}} \inf_{\mathbf{x}\in\triangle_{ij}} f(\mathbf{x})\mathbb{A}(\triangle_{ij}) \end{aligned}$$

where $\mathbb{A}(\triangle_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$ is the area of $\triangle_{ij}$.

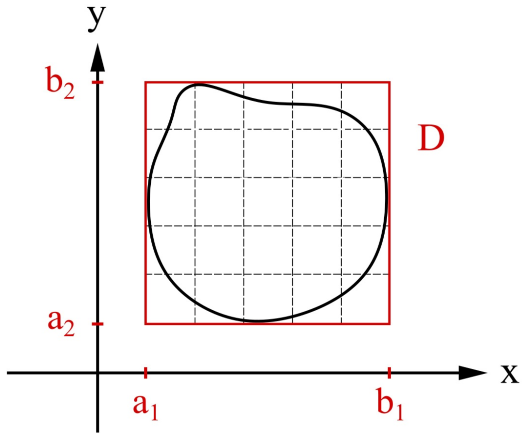
Suppose that P' be a refinement of P ($P' \subseteq P$). Then

$$L(P, f) \le L(f, P') \le U(f, P') \le U(f, P).$$

We want to ask whether $\sup\limits_P L(P, f) \overset{?}{=} \inf\limits_P U(P, f)$.

Question: How about if D is not a rectangle?

Let $D \subseteq \mathbb{R}^2$ be a bounded set and $f : D \to \mathbb{R}$ be a bounded function.

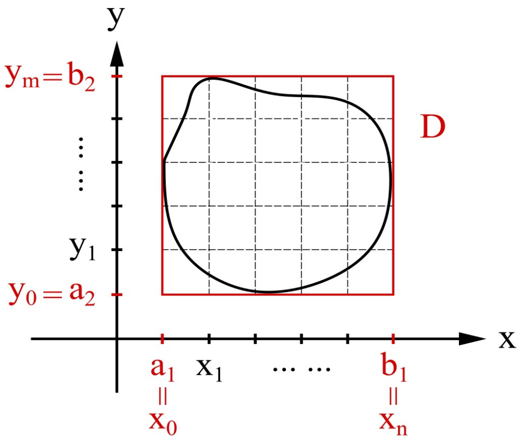


In order to consider the integral, we may deal with two things

(i) Compute the area of domain which is not rectange.

(ii) Set a new function from f which is defined on a rectangle covering D and has the same integral as f.

Definition 6.1.2. Let $D \subseteq \mathbb{R}^2$ be a bounded set.



Define

$$\begin{aligned} a_1 &= \inf \left\{ x \in \mathbb{R} \;\middle|\; (x, y) \in D \text{ for some } y \in \mathbb{R} \right\} \\ b_1 &= \sup \left\{ x \in \mathbb{R} \;\middle|\; (x, y) \in D \text{ for some } y \in \mathbb{R} \right\} \\ a_2 &= \inf \left\{ y \in \mathbb{R} \;\middle|\; (x, y) \in D \text{ for some } x \in \mathbb{R} \right\} \\ b_2 &= \sup \left\{ y \in \mathbb{R} \;\middle|\; (x, y) \in D \text{ for some } x \in \mathbb{R} \right\} \end{aligned}$$

Let

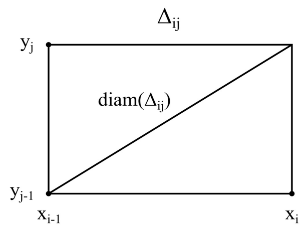
$$\begin{aligned} P_x &= \left\{ a_1 = x_0 < x_1 < \cdots < x_n = b_1 \right\} \text{ be a partition of } [a_1, b_1], \\ P_y &= \left\{ a_2 = x_0 < y_1 < \cdots < y_m = b_2 \right\} \text{ be a partition of } [a_2, b_2] \text{ and} \\ P &= \left\{ \triangle_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \;\middle|\; 1 \le i \le n,\ 1 \le j \le m \right\}. \end{aligned}$$

The mesh size of the partition P, denoted by $\|P\|$, is defined by

$$\|P\| = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$

Remark.

Note. The number $\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$ is called "*the diameter of* $\triangle_{ij}$" and is denoted by $diam(\triangle_{ij})$.



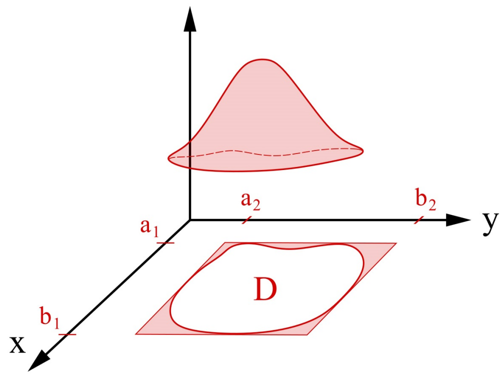
We try to define the upper sums and the lower sums corresponding to partitions.

Problem: f may not be defined on some subrectangles.

To extend f from A to $[a_1, b_1] \times [a_2, b_2]$ by

$$\overline{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D. \end{cases}$$

Then we can compute the volume of the region below $\overline{f}$ on $[a_1, b_1] \times [a_2, b_2]$.



Definition 6.1.3. Let $D \subseteq \mathbb{R}^2$ be a bounded set and $f : D \to \mathbb{R}$ be a bounded function. Let $P = \left\{ \triangle_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \,\middle|\, 1 \le i \le n,\ 1 \le j \le m \right\}$.

(1) The upper sum and the lower sum of f with respect to P are defined by

$$U(P,f) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \sup_{(x,y)\in\triangle_{ij}} \overline{f}(x,y)\mathbb{A}(\triangle_{ij})$$

and

$$L(P,f) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \inf_{(x,y)\in\triangle_{ij}} \overline{f}(x,y)\mathbb{A}(\triangle_{ij})$$

where $\mathbb{A}(\triangle_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$ is the area of $\triangle_{ij}$ and

$$\overline{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D. \end{cases}$$

The upper integral and lower integral of f over D are defined by

$$\overline{\int_D} f(x,y)\, d\mathbb{A} = \inf_{P:\ partition} U(P,f)$$

and

$$\underline{\int_D} f(x,y)\, d\mathbb{A} = \sup_{P:\ partition} U(P,f).$$

We say that a function f is Riemann (Darboux) integrable (over D) if

$$\overline{\int}_D f(x,y)\, d\mathbb{A} = \underline{\int}_D f(x,y)\, d\mathbb{A}.$$

The number is denoted by $\int_D f(x,y)\, d\mathbb{A}$ and is called "*the integral of f over D*".

Question: How about higher dimensional cases?

Definition 6.1.4. Let $D \subseteq \mathbb{R}^n$ be a bounded subset. Define $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ by

$$\begin{aligned} a_k &= \inf\left\{x_k \in \mathbb{R} \,\middle|\, (x_1, \cdots, x_n) \in D \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots x_n \in \mathbb{R}\right\} \\ b_k &= \sup\left\{x_k \in \mathbb{R} \,\middle|\, (x_1, \cdots, x_n) \in D \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots x_n \in \mathbb{R}\right\} \end{aligned}$$

Let $P^{(k)} = \left\{a_k = x_0^{(k)} < x_1^{(k)} < \cdots < x_{N_k}^{(k)} = b_k\right\}$ for $k = 1, \cdots, n$ and

$$P = \left\{\triangle_{i_1\cdots i_n} = [x_{i_1-1}^{(1)},\ x_{i_1}^{(1)}] \times \cdots \times [x_{i_n-1}^{(n)},\ x_{i_n}^{(n)}] \,\middle|\, 1 \le i_k \le N_k \text{ for } k = 1, \cdots, n\right\}.$$

The mesh size of the partition P, denoted by $\|P\|$, is defined by

$$\|P\| = \max_{\substack{1\le i\le N_k \\ k=1,\cdots,n}} \sqrt{(x_{i_1}^{(1)} - x_{i_1-1}^{(1)})^2 + \cdots + (x_{i_n}^{(n)} - x_{i_n-1}^{(n)})^2}.$$

The number $\max\limits_{\substack{1\le i\le N_k \\ k=1,\cdots,n}} \sqrt{(x_{i_1}^{(1)} - x_{i_1-1}^{(1)})^2 + \cdots + (x_{i_n}^{(n)} - x_{i_n-1}^{(n)})^2}$ is called "*the diameter of* $\triangle_{i_1\cdots i_n}$" and is denoted by $diam(\triangle_{i_1\cdots i_n})$.

Definition 6.1.5. Let $D \subseteq \mathbb{R}^n$ be a bounded set and $f : D \to \mathbb{R}$ be a bounded function. Let $P = \left\{\triangle_{i_1\cdots i_n} \,\middle|\, 1 \le i_k \le N_k, k = 1, \cdots, n\right\}$ be a partition of $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

(1) The (Darboux) upper sum and the (Darboux) lower sum of f with respect to P are defined by

$$\begin{aligned} U(P, f) &= \sum_{\triangle_{i_1\cdots i_n}\in P} \sup_{\mathbf{x}\in\triangle_{i_1\cdots i_n}} \overline{f}(\mathbf{x}) V(\triangle_{i_1\cdots i_n}) \\ L(P, f) &= \sum_{\triangle_{i_1\cdots i_n}\in P} \inf_{\mathbf{x}\in\triangle_{i_1\cdots i_n}} \overline{f}(\mathbf{x}) V(\triangle_{i_1\cdots i_n}) \end{aligned}$$

where $\overline{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in D \\ 0 & \text{if } \mathbf{x} \notin D \end{cases}$ and $V(\triangle_{i_1\cdots i_n}) = (x_{i_1}^{(1)} - x_{i_1-1}^{(1)}) \times \cdots \times (x_{i_n}^{(n)} - x_{i_n-1}^{(n)})$ is the volume of the rectangle $\triangle_{i_1\cdots i_n}$.

(2) The (Darboux) upper integral and the (Darboux) lower integral of f over D are defined by

$$\begin{aligned} \Big(\overline{\int}_D f(\mathbf{x})\, dV(\mathbf{x}) = \Big)\ \overline{\int}_D f(\mathbf{x})\, d\mathbf{x} &= \inf_{P:partition} U(P, f) \\ \Big(\underline{\int}_D f(\mathbf{x})\, dV(\mathbf{x}) = \Big)\ \underline{\int}_D f(\mathbf{x})\, d\mathbf{x} &= \sup_{P:partition} L(P, f) \end{aligned}$$

(3) We say that a function is Riemann (Darboux) integrable over D if

$$\overline{\int}_D f(\mathbf{x})\,d\mathbf{x} = \underline{\int}_D f(\mathbf{x})\,d\mathbf{x}$$

and the number is denoted by $\int_D f(\mathbf{x})\,d\mathbf{x}$.

Remark. (1) $U(P, f)$ and $L(P, f)$ are Darboux upper sum and lower sum. Let $P = \{\triangle_1, \cdots, \triangle_N\}$ be a partition of D. We called the sum $\sum_{k=1}^{N} f(\xi_k)V(\triangle_k)$ for some $\xi_k \in \triangle_k$ "*the Riemann sum of f over D*".

(2) f is Riemann integrable over D if

$$\lim_{\|P\|\to 0} \sum_{\triangle_k \in P} \bar{f}(\xi_k)V(\triangle_k)$$

converges to a number I.

(3) f is Riemann integrable over D if and only if f is Darboux integrable over D.

Remark. Let $K \subseteq \mathbb{R}^n$ be compact and f is continuous on K. Then f is integrable over K. (How to prove? Is it true?)

Definition 6.1.6. Let $D \subseteq \mathbb{R}^n$ be a bounded set $P, P', P_1, \cdots, P_k$ be partitions of D.

(1) We say that P' is a refinement of P if for any $\triangle' \subseteq P'$, there exists $\triangle \in P$ such that $\triangle' \subseteq \triangle$.

(2) We say that P is a common refinement of $P_1, \cdots, P_k$ if P is a refinement of P_j for $j = 1, \cdots, k$.

Proposition 6.1.7. *Let $D \subseteq \mathbb{R}^n$ be a bounded set, P_1 and P_2 be partitions of D, and $f : D \to \mathbb{R}$ be a bounded function.*

(1) Suppose that P is a common refinement of P_1 and P_2. Then

$$L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f)$$

(2) By (1), any upper sum $U(P, f)$ is an upper bound of all lower sums and any lower sum $L(P, f)$ is an lower bound of all upper sums. Hence, for any partition P,

$$\sup_{P':partiition} L(P', f) \le U(P, f)$$

and

$$\inf_{P':partiition} U(P', f) \ge L(P, f)$$

(3) *If* $\inf_{P:partition} U(P, f) = \sup_{P:partition} L(P, f) = c$*, then* c *is the unique number which is less than any upper sum and greater than any lower sum.*

(4) *It is possible that*

$$\sup_{P:partition} L(P, f) < \inf_{P:partition} U(P, f).$$

For example, $D = [0, 1] \times [0, 1]$ *and* $f(x, y) = \begin{cases} 1 & x \in \mathbb{Q}, y \in \mathbb{Q} \\ 0 & otherwise \end{cases}$

Proposition 6.1.8. *Let* $D \subseteq \mathbb{R}^n$ *be a bounded set and* $f : D \to \mathbb{R}$ *be a bounded function. Then* f *is Darboux integrable over* D *if and only if for every* $\varepsilon > 0$*, there exists a partition* P *of* D *such that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof. (Exercise) □

Proposition 6.1.9. *Let* $D \subseteq \mathbb{R}^n$ *be a bounded set and* $f : D \to \mathbb{R}$ *be a bounded function with extension* $\bar{f}$*. Then* f *is Riemann integrable if and only if there exists a number* $I \in \mathbb{R}$ *such that for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $P = \{\triangle_1, \cdots, \triangle_N\}$ *be a partition of* D *with* $||P|| < \delta$*, then*

$$\Big| \sum_{k=1}^{N} \bar{f}(\xi_k) V(\triangle_k) - I \Big| < \varepsilon.$$

Proof. (Exercise) □

Theorem 6.1.10. *Let* $D \subseteq \mathbb{R}^n$ *be a bounded set and* $f : D \to \mathbb{R}$ *be a bounded function. Then* f *is Darboux integrable over* D *if and only if for given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any partition* P *of* D *with* $||P|| < \delta$*, then*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof. (Exercise)

□

Theorem 6.1.11. *Let* $D \subseteq \mathbb{R}^n$ *be a bounded set and* $f : D \to \mathbb{R}$ *be a bounded function. Then* f *is Darboux integrable over* D *if and only if* f *is Riemann integrable over* D*.*

Proof. ($\Longrightarrow$) Since f is bounded, there exists $M > 0$ such that $|f(\mathbf{x})| < M$ for every $\mathbf{x} \in D$. W.L.O.G, we may assume that $f(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in D$. Otherwise, we can replacing f by $f + M$.

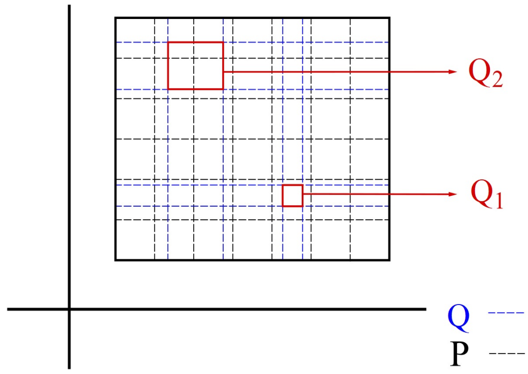
Since D is bounded, $D \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$. Set $L = \max_{1 \leq i \leq n}(b_i - a_i)$. Also, since f is Darboux integrable over D, there exist $I \in \mathbb{R}$ and partitions

$$\begin{aligned} P_1 &= \{a_1 = x_0^{(1)} < \cdots < x_{N_1}^{(1)} = b_1\} \\ &\vdots \\ P_n &= \{a_n = x_0^{(n)} < \cdots < x_{N_n}^{(n)} = b_n\} \end{aligned}$$

and

$$P = \left\{\triangle_{ij} = [x^{(1)}_{i_1-1}, x^{(1)}_{i_1}] \times \cdots \times [x^{(n)}_{i_n-1}, x^{(n)}_{i_n}] \;\middle|\; 1 \le i_k \le N_k,\ k = 1, \cdots, n\right\}$$

such that $U(P, f) - L(P, f) < \dfrac{\varepsilon}{2}$ and $L(P, f) \le I \le U(P, f)$.

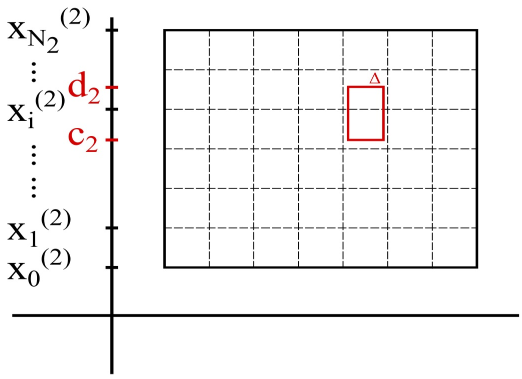


Let $\sharp(P) = N\ (= \prod_{k=1}^{n} N_k)$ and set $\delta = \dfrac{\varepsilon}{8M(L+1)^n N^n}$. For $Q = \{\square_1, \cdots, \square_K\}$ be a partition of D such that $\|Q\| < \delta$.

Separate Q into two classes, say Q_1 and Q_2. Let Q_1 be the subset of Q such that if $\square \in Q_1$ then $\square$ is contained in a single $\triangle_{ij} \in P$ and $Q_2 = Q \backslash Q_1$.

We obtain

$$\sum_{\square_i \in Q_1} \Big(\sup_{\mathbf{x}\in\square_i} \bar{f}(\mathbf{x}) - \inf_{\mathbf{x}\in\square_i} \bar{f}(\mathbf{x})\Big) V(\square_i) \le U(P, f) - L(P, f) < \frac{\varepsilon}{2}. \tag{6.1}$$



For $\square = [c_1, d_1] \times \cdots \times [c_n, d_n] \in Q_2$, there exist $k \in \{1, \cdots, n\}$ and $i_k \in \{1, \cdots, N_k\}$ such that for $x^{(k)}_{i_k} \in [c_k, d_k]$, $\square \subseteq [a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}] \times [x^{(k)}_{i_k} - \delta, x^{(k)}_{i_k} + \delta] \times [a_{k+1}, b_{k+1}] \times \cdots \times [a_n, b_n]$.

Then

$$\sum_{\square_j \in Q_2} V(\square_j) \le 2\delta L^{n-1} N^n < \frac{\varepsilon}{4M}.$$

Hence,

$$\Big|\sum_{\square_j \in Q_2} \sup_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) V(\square_j)\Big| \le M \sum_{\square_j \in Q_2} V(\square_j) < \frac{\varepsilon}{4} \quad \text{and} \quad \Big|\sum_{\square_j \in Q_2} \inf_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) V(\square_j)\Big| \le M \sum_{\square_j \in Q_2} V(\square_j) < \frac{\varepsilon}{4}.$$

We have

$$\begin{aligned} U(Q, f) - L(Q, f) &= \sum_{\square_i \in Q_1} \Big(\sup_{\mathbf{x}\in\square_i} \bar{f}(\mathbf{x}) - \inf_{\mathbf{x}\in\square_i} \bar{f}(\mathbf{x})\Big) V(\square_i) + \sum_{\square_j \in Q_2} \Big(\sup_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) - \inf_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x})\Big) V(\square_j) \\ &\le U(P, f) - L(P, f) + \Big|\sum_{\square_j \in Q_2} \sup_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) V(\square_j)\Big| + \Big|\sum_{\square_j \in Q_2} \inf_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) V(\square_j)\Big| \\ &< \varepsilon. \end{aligned}$$

Also

$$U(Q, f) = \sum_{\square_i \in Q_1} \sup_{\mathbf{x}\in\square_i} \bar{f}(\mathbf{x}) V(\square_i) + \sum_{\square_j \in Q_2} \sup_{\mathbf{x}\in\square_j} \bar{f}(\mathbf{x}) V(\square_j) \le U(P, f) + \frac{\varepsilon}{4}.$$

Let $\xi_i \in \square_i$ for $i = 1, \cdots, K$, we have

$$\sum_{\square_i \in Q} \bar{f}(\xi_i)V(\square_i) - I \leq U(Q, f) - I \leq U(P, f) + \frac{\varepsilon}{4} - I \leq U(P, f) - L(P, f) + \frac{\varepsilon}{4} < \varepsilon.$$

and

$$\sum_{\square_i \in Q} \bar{f}(\xi_i)V(\square_i) - I \geq L(Q, f) - I \geq U(Q, f) - \varepsilon - I \leq L(P, f) - I - \varepsilon \geq -2\varepsilon.$$

Therefore,

$$\Big| \sum_{\square_i \in Q} \bar{f}(\xi_i)V(\square_i) - I \Big| < 2\varepsilon$$

and f is Riemann integrable over D.

($\Longleftarrow$) Since f is bounded and Riemann integrable, there exists $I \in \mathbb{R}$ and for given $\varepsilon > 0$ there exists $\delta > 0$ such that if a partition $P = \{\triangle_1, \cdots, \triangle_N\}$ of D with $\|P\| < \delta$,

$$\Big| \sum_{k=1}^{N} \bar{f}(\xi_k)V(\triangle_k) - I \Big| < \frac{\varepsilon}{4} \quad \text{for any } \xi_k \in \triangle_k,\ k = 1, \cdots, N.$$

Define $M_i = \sup_{\mathbf{x} \in \triangle_i} \bar{f}(\mathbf{x})$ and $m_i = \inf_{\mathbf{x} \in \triangle_i} \bar{f}(\mathbf{x})$. There are $\mathbf{T}_i, \mathbf{t}_i \in \triangle_i$ such that

$$M_i < \bar{f}(\mathbf{T}_i) + \frac{\varepsilon}{4V(D)} \quad \text{and} \quad m_i \geq \bar{f}(\mathbf{t}_i) - \frac{\varepsilon}{4V(D)}.$$

Then

$$\begin{aligned} U(P, f) &= \sum_{k=1}^{N} M_k V(\triangle_k) < \sum_{k=1}^{N} \Big(\bar{f}(\mathbf{T}_k) + \frac{\varepsilon}{4V(D)}\Big) V(\triangle_k) \\ &< I + \frac{\varepsilon}{4} + \frac{\varepsilon}{4V(D)} \sum_{k=1}^{N} V(\triangle_k) \\ &= I + \frac{\varepsilon}{2}. \end{aligned}$$

and

$$\begin{aligned} L(P, f) &= \sum_{k=1}^{N} m_k V(\triangle_k) > \sum_{k=1}^{N} \Big(\bar{f}(\mathbf{t}_k) - \frac{\varepsilon}{4V(D)}\Big) V(\triangle_k) \\ &> I - \frac{\varepsilon}{4} - \frac{\varepsilon}{4V(D)} \sum_{k=1}^{N} V(\triangle_k) \\ &= I - \frac{\varepsilon}{2}. \end{aligned}$$

Hence, $U(P, f) - L(P, f) < \varepsilon$ and f is Darboux integrable over D. □

Remark. From now on, we will also call the above integrals $\int_D f(\mathbf{x})\, d\mathbf{x}$ "*Riemann-Darboux integral*".

Recall: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$, then

$$\int_a^b f_n(\mathbf{x})\, d\mathbf{x} \to \int_a^b f(\mathbf{x})\, d\mathbf{x}.$$

Theorem 6.1.12. *Let $D \subseteq \mathbb{R}^n$ be a bounded set and $f_k : D \to \mathbb{R}$ be a sequence of Riemann integrable functions over D such that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on D. Then f is Riemann integrable over D and*

$$\lim_{k\to\infty} \int_D f_k(\mathbf{x})\, d\mathbf{x} = \int_D f(\mathbf{x})\, d\mathbf{x}.$$

Proof. (Exercise) □

Remark. There are other definition of Darboux integral. We can divide D into serveral pieces of subregions such that

(i) $D = \bigcup_{i=1}^{n} D_i$

(ii) $Int(D_i) \cap Int(D_j) = \emptyset$

(iii) each D_i has nonnegative volume $V(D_i)$

Define

$$U(\triangle, f) = \sum_{i=1}^{n} \sup_{\mathbf{x}\in D_i} f(\mathbf{x}) V(D_i) \quad \text{and} \quad L(\triangle, f) = \sum_{i=1}^{n} \inf_{\mathbf{x}\in D_i} f(\mathbf{x}) V(D_i)$$

Then

$$\underline{\int_D} f(\mathbf{x})\, d\mathbf{x} = \sup L(\triangle, f) \quad \text{and} \quad \overline{\int_D} f(\mathbf{x})\, d\mathbf{x} = \inf U(\triangle, f)$$

By using this method, we need to compute the volume of D_i in advance. But we don't need to extend f to $\bar{f}$ and D_i need not be a rectangle.

❑ Volume of Sets

Definition 6.1.13. Let $E \subseteq \mathbb{R}^n$ be a bounded set.

(1) The "*characteristic function* $\mathbb{1}_E$ *(or* χ_E*)*" is defined by

$$\mathbb{1}_E(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in E \\ 0 & \mathbf{x} \in \mathbb{R}\backslash E. \end{cases}$$

(2) E is said to have volume if $\mathbb{1}_E$ is Riemann integrable (over E), and the volume of E is denoted by $V(E)$ where

$$V(E) = \int_E \mathbb{1}_E(\mathbf{x})\, \mathbf{dx}.$$

(3) E is said to have volume zero if $V(E) = \int_E \mathbb{1}(\mathbf{x})\, d\mathbf{x} = 0$.

Remark. That "*a set does not have volume*" ("a set has no volume") is different from that "*a set has volume zero*". Not all bounded sets have volume. For example, $E := \mathbb{Q} \cap [0,1]$ has no volume. It does NOT mean that E has volume $V(E) = 0$ since $\mathbb{1}_E$ is not Riemann integrable.

Remark. (1) A rectangle $S = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n \left[\text{or } (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n\right]$ has volume

$$V(S) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

(2) An open rectangle(set) has nonzero volume.

(3) If E_1 and E_2 have volumes and $E_1 \subseteq E_2$, then $V(E_1) \leq V(E_2)$.

Proposition 6.1.14. *Let $E \subseteq \mathbb{R}^n$ be bounded. The E has volume zero if and only if for every $\varepsilon > 0$, there exists finite (open) rectangles $S_1, \cdots, S_N$ such that*

$$E \subseteq \bigcup_{k=1}^{N} S_k \quad \textit{and} \quad \sum_{k=1}^{N} V(S_k) < \varepsilon.$$

Proof. ($\Longrightarrow$) Since $0 = V(E) = \int_E \mathbb{1}_E(\mathbf{x})\, d\mathbf{x} = \overline{\int}_E \mathbb{1}_E(\mathbf{x})\, d\mathbf{x}$, for given $\varepsilon > 0$, there exists a partition $P = \{\triangle_1, \cdots, \triangle_N\}$ of E such that

$$\sum_{k=1}^{N} \sup_{\mathbf{x} \in \triangle_k} \mathbb{1}_E(\mathbf{x})\, d\mathbf{x} = U(\mathbb{1}_E, P) < \overline{\int}_E \mathbb{1}_E(\mathbf{x})\, d\mathbf{x} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

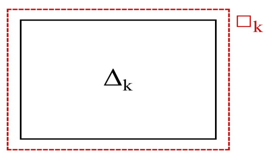
Since $\sup_{\mathbf{x} \in \triangle_k} \mathbb{1}_E(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \triangle_k \cap E \\ 0 & \text{otherwise} \end{cases}$,

$$\sum_{\substack{\triangle_k \in P \\ \triangle_k \cap E \neq \emptyset}} V(\triangle_k) = U(P, \mathbb{1}_E) < \frac{\varepsilon}{2}.$$

Moreover, for every $\triangle_k \in P$ with $\triangle_k \cap E \neq \emptyset$, we can find an open rectangle $\square_k$ such that $\triangle_k \subsetneq \square_k$ and $V(\square_k) \leq 2V(\triangle_k)$.

Then

$$E \subseteq \bigcup_{k=1}^{N} \triangle_k \subseteq \bigcup_{k=1}^{N} \square_k$$

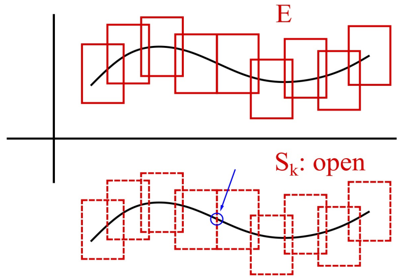


and

$$\sum_{k=1}^{N} V(\square_k) \leq 2 \sum_{k=1}^{N} V(\triangle_k) < \varepsilon.$$

($\Longleftarrow$) Let $S_1, \cdots, S_N$ be rectangles such that

$$E \subseteq \bigcup_{k=1}^{N} S_k \quad \text{and} \quad \sum_{k=1}^{N} V(S_k) < \varepsilon.$$

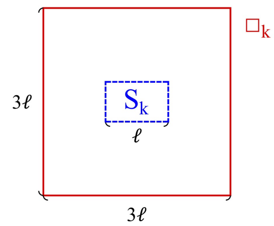


W.L.O.G, we may assume that for each k,

$$\frac{\text{max length of side of } S_k}{\text{min length of side of } S_k} \leq 2.$$

Otherwise, we can divide S_k such that it satisfies the above preperty.

For each S_k, we can choose a (cubic) rectangle $\square_k$ side length of 3 times multple of max side length of S_k such that $S_k \subseteq \square_k$. Then $V(\square_k) \le 2^{n-1} \cdot 3^n V(S_k)$.



Let P be a partition of E such that for each $\triangle \in P$ with $\triangle \cap E \neq \emptyset$, then $\triangle \subseteq \square_k$ for some $k = 1, \cdots, N$. Hence,

$$U(P, \mathbb{1}_E) = \sum_{\substack{\triangle \in P \\ \triangle \cap E \neq \emptyset}} V(\triangle) \le \sum_{k=1}^{N} V(\square_k) \le 2^{n-1} \cdot 3^n \cdot \sum_{k=1}^{N} V(S_k) < 2^{n-1} \cdot 3^n \varepsilon.$$

Since ε is an arbitrary positive number, $\overline{\int}_E \mathbb{1}_E(\mathbf{x})\, d\mathbf{x} = 0$ and therefore $V(E) = 0$. □

Example 6.1.15. (1) A set consisting of finite points is volume zero. (finite set)

(2) The set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subseteq [0, 1]$ is volume zero. (infinitely countable set)

(3) The Cantor set is volume zero. (uncountable set)

(4) If $f : [a, b] \to \mathbb{R}^n$ for $n > 1$ is of class C^1, then $f\big([a, b]\big)$ has volume zero. (at most 1-dimensional set in $\mathbb{R}^n$)

Proof. Let $P_k = \{a = x_0 < x_1 < \cdots < x_N = b\}$ be a partition of $[a, b]$ with $x_i - x_{i-1} = \frac{b-a}{N} = \delta$. Since f is of class C^1, there exists $M > 0$ such that $\|\nabla f(x)\|_{\mathbb{R}^n} < M$ for every $x \in [a, b]$.

By the Mean Value Theorem, for $\mathbf{f} = (f_1, \cdots, f_n)$ and let $t \in [x_{i-1}, x_i]$,

$$f_j(x_i) - f_j(t) = f_j'\big(c_{ij}(t)\big)(x_i - t) \quad \text{for some } c_{ij}(t) \in [t, x_i].$$

Then

$$\|\mathbf{f}(x_i) - \mathbf{f}(t)\|_{\mathbb{R}^n} < \sum_{j=1}^{n} \big|f_j(x_i) - f_j(t)\big| \le \sum_{j=1}^{n} \big|f_j'\big(c_{ij}(t)\big)\big| |x_i - x_{i-1}| \le nM\delta.$$

Since t is an arbitrary point in $[x_{i-1}, x_i]$, we have $f\big([x_{i-1}, x_i]\big) \subseteq B\big(\mathbf{f}(x_i), nM\delta\big)$ and moreover $\mathbf{f}\big([a, b]\big) \subseteq \bigcup_{i=1}^{N} B\big(\mathbf{f}(x_i), nM\delta\big)$.

Also, since

$$\begin{aligned} V\Big(\bigcup_{i=1}^{N} B\big(\mathbf{f}(x_i), nM\delta\big)\Big) &\le \sum_{i=1}^{N} V\Big(B\big(\mathbf{f}(x_i), nM\delta\big)\Big) \le \underbrace{C}_{\substack{\text{some constant} \\ \text{depending on } n}} N n^n M^n \Big(\frac{b-a}{N}\Big)^n \\ &= \underbrace{C[nM(b-a)]^n}_{\text{constant}} N^{1-n} \to 0 \quad \text{as } N \to 0. \end{aligned}$$

We have $V\Big(\mathbf{f}\big([a.b]\big)\Big) = 0$. □

Remark. We can extend the proposition to countable cover of E and obtain "meazero zero". We will skip this argument.

Remark. Let $D = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a bounded function. If f is continuous on D except on a volume zero subset $E \subseteq D$, then f is integrable over D.

6.2 Properties of the Integrals

Proposition 6.2.1. *Let $A \subseteq \mathbb{R}^n$ be bounded and $f, g : A \to \mathbb{R}$ be bounded. Then*

(1) If $B \subseteq A$, then

$$\underline{\int}_A \big(f\mathbb{1}_B\big)(\mathbf{x})\, d\mathbf{x} = \underline{\int}_B f(\mathbf{x})\, d\mathbf{x} \quad \textit{and} \quad \overline{\int}_A \big(f\mathbb{1}_B\big)(\mathbf{x})\, d\mathbf{x} = \overline{\int}_B f(\mathbf{x})\, d\mathbf{x}.$$

(2)

$$\underline{\int}_A f(\mathbf{x})\, d\mathbf{x} + \underline{\int}_A g(\mathbf{x})\, d\mathbf{x} \le \underline{\int}_A \big(f+g\big)(\mathbf{x})\, d\mathbf{x}$$

and

$$\overline{\int}_A f(\mathbf{x})\, d\mathbf{x} + \overline{\int}_A g(\mathbf{x})\, d\mathbf{x} \ge \overline{\int}_A \big(f+g\big)(\mathbf{x})\, d\mathbf{x}$$

(3) If $c \ge 0$, then

$$\underline{\int}_A \big(cf\big)(\mathbf{x})\, d\mathbf{x} = c\underline{\int}_A f(\mathbf{x})\, d\mathbf{x} \quad \textit{and} \quad \overline{\int}_A \big(cf\big)(\mathbf{x})\, d\mathbf{x} = c\overline{\int}_A f(\mathbf{x})\, d\mathbf{x}.$$

If $c < 0$, then

$$\underline{\int}_A \big(cf\big)(\mathbf{x})\, d\mathbf{x} = c\overline{\int}_A f(\mathbf{x})\, d\mathbf{x} \quad \textit{and} \quad \overline{\int}_A \big(cf\big)(\mathbf{x})\, d\mathbf{x} = c\underline{\int}_A f(\mathbf{x})\, d\mathbf{x}.$$

(4) If $f \le g$ on A, then

$$\underline{\int}_A f(\mathbf{x})\, d\mathbf{x} \le \underline{\int}_A g(\mathbf{x})\, d\mathbf{x} \quad \textit{and} \quad \overline{\int}_A f(\mathbf{x})\, d\mathbf{x} \le \overline{\int}_A g(\mathbf{x})\, d\mathbf{x}$$

(5) If A has volume zero, then f is Riemann integrable over A and $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x} = 0.$

Proof. (5) Since f is bounded, there exist $m, M \in \mathbb{R}$ such that $m \le f(\mathbf{x}) \le M$ for every $\mathbf{x} \in A$. Then

$$m\mathbb{1}_A(\mathbf{x}) \le f(\mathbf{x}) \le M\mathbb{1}_A(\mathbf{x}) \quad \text{for every } \mathbf{x} \in A.$$

We have

$$\begin{aligned} 0 = mV(A) &= m\int_A \mathbb{1}_A(\mathbf{x})\,d\mathbf{x} = \int_A m\mathbb{1}_A(\mathbf{x})\,d\mathbf{x} \\ &\leq \underbrace{\underline{\int}_A f(\mathbf{x})\,d\mathbf{x} \leq \overline{\int}_A f(\mathbf{x})\,d\mathbf{x}}_{\text{we don't know whether or not } f \text{ is integrable yet.}} \\ &\leq \int_A M\mathbb{1}(\mathbf{x})\,d\mathbf{x} = M\int_A \mathbb{1}(\mathbf{x})\,d\mathbf{x} = MV(A) = 0. \end{aligned}$$

Hence, f is integrable over A and $\int_A f(\mathbf{x})\,d\mathbf{x} = 0$.

□

Remark. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $f, g : A \to \mathbb{R}$ be bounded functions. Then

$$\underline{\int}_A \big(f - g\big)(\mathbf{x})\,d\mathbf{x} \leq \underline{\int}_A f(\mathbf{x})\,d\mathbf{x} - \underline{\int}_A g(\mathbf{x})\,d\mathbf{x}$$

and

$$\overline{\int}_A f(\mathbf{x})\,d\mathbf{x} - \overline{\int}_A g(\mathbf{x})\,d\mathbf{x} \leq \overline{\int}_A \big(f - g\big)(\mathbf{x})\,d\mathbf{x}.$$

Proof. (Exercise) □

Corollary 6.2.2. *Let $A, B \subseteq \mathbb{R}^n$ be bounded such that $A \cap B$ has volume zero, and $f : A \cup B \to \mathbb{R}$ be bounded. Then*

(1)

$$\underline{\int}_A f(\mathbf{x})\,d\mathbf{x} + \underline{\int}_B f(\mathbf{x})\,d\mathbf{x} \leq \underline{\int}_{A\cup B} f(\mathbf{x})\,d\mathbf{x}$$

and

(2)

$$\overline{\int}_{A\cup B} f(\mathbf{x})\,d\mathbf{x} \leq \overline{\int}_A f(\mathbf{x})\,d\mathbf{x} + \overline{\int}_B f(\mathbf{x})\,d\mathbf{x}.$$

Proof. (1)

$$\begin{aligned} \underline{\int}_A f(\mathbf{x})\,d\mathbf{x} + \underline{\int}_B f(\mathbf{x})\,d\mathbf{x} &= \underline{\int}_{A\cup B} \big(f\mathbb{1}_A\big)(\mathbf{x})\,d\mathbf{x} + \underline{\int}_{A\cup B} \big(f\mathbb{1}_B\big)(\mathbf{x})\,d\mathbf{x} \\ &\leq \underline{\int}_{A\cup B} \big(f\mathbb{1}_A + f\mathbb{1}_B\big)(\mathbf{x})\,d\mathbf{x} \\ &= \underline{\int}_{A\cup B} \Big(f\mathbb{1}_{A\cup B} - \big(-f\mathbb{1}_{A\cap B}\big)\Big)(\mathbf{x})\,d\mathbf{x} \\ &\leq \underline{\int}_{A\cup B} \big(f\mathbb{1}_{A\cup B}\big)(\mathbf{x})\,d\mathbf{x} - \underbrace{\overline{\int}_{A\cup B} -\big(f\mathbb{1}_{A\cap B}\big)(\mathbf{x})\,d\mathbf{x}}_{=0 \text{ since } V(A\cap B)=0} \\ &= \underline{\int}_{A\cup B} f(\mathbf{x})\,d\mathbf{x} \end{aligned}$$

(2) (Exercise)

□

Theorem 6.2.3. *Let $A \subseteq \mathbb{R}^n$ be bounded, $c \in \mathbb{R}$ and $f, g : A \to \mathbb{R}$ be Riemann integrable. Then*

(1) $f \pm g$ is Riemann integrable and $\displaystyle\int_A \big(f \pm g\big)(\mathbf{x})\, d\mathbf{x} = \int_A f(\mathbf{x})\, d\mathbf{x} \pm \int_A g(\mathbf{x})\, d\mathbf{x}$.

(2) cf is Riemann integrable and $\displaystyle\int_A \big(cf\big)(\mathbf{x})\, d\mathbf{x} = c\int_A f(\mathbf{x})\, d\mathbf{x}$.

(3) $|f|$ is Riemann integrable and $\displaystyle\Big|\int_A f(\mathbf{x})\, d\mathbf{x}\Big| \le \int_A \big|f(\mathbf{x})\big|\, d\mathbf{x}$.

(4) If $f \le g$, then $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x} \le \int_A g(\mathbf{x})\, d\mathbf{x}$.

(5) If A has volume and $|f| \le M$, then

$$\Big|\int_A f(\mathbf{x})\, d\mathbf{x}\Big| \le MV(A).$$

Proof. (Exercise) □

Theorem 6.2.4. *(Mean value Theorem for Integrals) Let $A \subseteq \mathbb{R}^n$ be connected and compact, and have volume. Suppose that $f : A \to \mathbb{R}$ is continuous, then there exists $\mathbf{x}_0 \in A$ such that*

$$\int_A f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{x}_0)V(A)$$

If $V(A) \neq 0$, we call the number $\displaystyle\frac{1}{V(A)}\int_A f(\mathbf{x})\, d\mathbf{x}$ *"the average of f over A".*

Proof. It suffices to show the case $V(A) \neq 0$. Since A is compact and f is continuous on A, there exists $m, M \in \mathbb{R}$ such that $m = \min_{\mathbf{x}\in A} f(\mathbf{x})$ and $M = \max_{\mathbf{x}\in A} f(\mathbf{x})$. Then

$$m\mathbb{1}_A(\mathbf{x}) \le f(\mathbf{x}) \le M\mathbb{1}_A(\mathbf{x}) \quad \text{for every } \mathbf{x} \in A.$$

Hence,

$$mV(A) = \int_A m\mathbb{1}_A(\mathbf{x})\, d\mathbf{x} \le \int_A f(\mathbf{x})\, d\mathbf{x} \le \int_A M\mathbb{1}_A(\mathbf{x})\, d\mathbf{x}.$$

and we obtain $\displaystyle m \le \frac{1}{V(A)}\int_A f(\mathbf{x})\, d\mathbf{x} \le M$.

Since A is connected and f is continuous on A, $f(A)$ is a connected subset in $\mathbb{R}$ and hence $f(A)$ is an interval. Since $m = \min_{\mathbf{x}\in A} f(\mathbf{x})$ and $M = \max_{\mathbf{x}\in A} f(\mathbf{x})$, $f(A) = [m, M]$. There exists $\mathbf{x}_0$ such that

$$f(\mathbf{x}_0) = \frac{1}{V(A)}\int_A f(\mathbf{x})\, d\mathbf{x}.$$

□

Definition 6.2.5. Let $B \subseteq A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be a function. The restriction of f to B, denoted by $f\big|_B$, is defined by

$$f\big|_B(\mathbf{x}) = f(\mathbf{x}) \quad \text{for every } \mathbf{x} \in B \quad (f\big|_B : B \to \mathbb{R}).$$

Lemma 6.2.6. *Let $B \subseteq A \subseteq \mathbb{R}^n$ be bounded and $f : A \to \mathbb{R}$ be a bounded function. Suppose that $f\mathbb{1}_B$ is Riemann integrable over A. Then f is integrable over B and*

$$\int_A (f\big|_B)(\mathbf{x})\, d\mathbf{x} = \int_B f(\mathbf{x})\, d\mathbf{x}.$$

Remark. There exist $B \subseteq A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ such that f is integrable over A but not integrable over B. For example $f \equiv 1$ on $A = [0, 1]$ and $B = \mathbb{Q} \cap [0, 1]$. (Consider the exmaple again!)

6.3 The Fubini Theorem

Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be continuous (Riemann integrable) over A.

Question: How to compute $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x}$?

Recall that $f : [a, b] \to \mathbb{R}$ is continuous. By the Fundamental Theorem of Calculus, if $F(x)$ satisfies $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

But there is no F.T.C for multi-variables functions. Can we rewrite a Riemann integral for a multi-variable function into several one dimensional Riemann integrals by iterating?

For example, let $D = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and consider the three integrals

$$\int_D f(x, y)\, d\mathbb{A}, \quad \int_0^1 \Big(\int_0^1 f(x, y)\, dx \Big)\, dy, \quad \int_0^1 \Big(\int_0^1 f(x, y)\, dy \Big)\, dx.$$

Are those integrals equal?

Example 6.3.1. Let $D = [0, 1] \times [0, 1]$ and $f(x, y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$ Then

$$\int_D f(x, y)\, d\mathbb{A} = 0 \quad \text{(Check it!)}$$

For any $y \in [0, 1]$, the function $f^y(x) := f(x, y) = 0$ (except perhaps at a single point $x = \frac{1}{2}$). Hence, $\displaystyle\int_0^1 f^y(x, y)\, dx = 0$ for any $y \in [0, 1]$. Then

$$\int_0^1 \Big(\int_0^1 f(x, y)\, dx \Big) = \int_0^1 0\, dy = 0.$$

For any $x \in [0,1]$, consider the function $f_x(y) := f(x,y)$. If $x = \frac{1}{2}$, $f_{1/2}(y) = f(\frac{1}{2}, y) = \begin{cases} 1 & y \in \mathbb{Q} \\ 0 & y \in [0,1]\backslash\mathbb{Q}. \end{cases}$

Hence $f_{1/2}(y)$ is not (Riemann) integrable and $\int_0^1 f(x,y)\,dy$ is not defined when $x = \frac{1}{2}$. Thus, we cannot compute $\int_0^1 \Big(\int_0^1 f(x,y)\,dy \Big) dx$.

Note that for a function, the lower and upper integrals are always defined. We will solve the problem of undefined integrals by using upper and lower integrals. Let's start with the case $n = 2$ and $D = [a,b] \times [c,d]$.

Definition 6.3.2. Let $D = [a,b] \times [c,d]$ and $f : D \to \mathbb{R}$ be bounded. For a fixed $x \in [a,b]$, $f(x,\cdot)$ is a function from $[c,d]$ into $\mathbb{R}$.

$$\underline{\int_c^d} f(x,y)\,dy := \text{the lower integral of } f(x,\cdot).$$

and

$$\overline{\int_c^d} f(x,y)\,dy := \text{the upper integral of } f(x,\cdot).$$

If $\underline{\int_c^d} f(x,y)\,dy = \overline{\int_c^d} f(x,y)\,dy$, we write

$$\int_c^d f(x,y)\,dy = \text{the integral of } f(x,\cdot) \text{ over } [c,d].$$

Similarly, we can also define

$$\underline{\int_a^b} f(x,y)\,dx, \quad \overline{\int_a^b} f(x,y)\,dx \quad \text{and} \quad \int_a^b f(x,y)\,dx.$$

Lemma 6.3.3. *Let $D = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be bounded. Then*

$$\underline{\int_D} f(x,y)\,d\mathbb{A} \overset{(*)}{\leq} \underline{\int_a^b} \Big(\underline{\int_c^d} f(x,y)\,dy \Big) dx \leq \overline{\int_a^b} \Big(\overline{\int_c^d} f(x,y)\,dy \Big) dx \leq \overline{\int_D} f(x,y)\,d\mathbb{A}$$

and

$$\underline{\int_D} f(x,y)\,d\mathbb{A} \leq \underline{\int_c^d} \Big(\underline{\int_a^b} f(x,y)\,dx \Big) dy \leq \overline{\int_c^d} \Big(\overline{\int_a^b} f(x,y)\,dx \Big) dy \leq \overline{\int_D} f(x,y)\,d\mathbb{A}$$

Proof. It suffices to prove $(*)$. By the definition of the lower integral $\underline{\int_D} f(x,y)\,d\mathbb{A} = \sup_P L(P,f)$. For given $\varepsilon > 0$, there exist partitions $P_x = \big\{a = x_0 < \cdots < x_n = b\big\}$ of $[a,b]$, $P_y = \big\{c = y_0 < \cdots < y_m = d\big\}$ of $[c,d]$ and $P = \big\{\triangle_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \;\big|\; 1 \leq i \leq n,\ 1 \leq j \leq m\big\}$ of D such that

$$\underline{\int_D} f(x,y)\,d\mathbb{A} - \varepsilon < L(P,f).$$

Then

$$\begin{aligned}
\underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)\,dy\Big)dx &= \underline{\int_a^b}\Big(\sum_{j=1}^{m}\underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy\Big)dx \quad \text{(Check!)}\\
&= \sum_{i=1}^{n}\underline{\int_{x_{i-1}}^{x_i}}\Big(\sum_{j=1}^{m}\underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy\Big)dx\\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{m}\underline{\int_{x_{i-1}}^{x_i}}\Big(\underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy\Big)dx\\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{m}\underline{\int_{x_{i-1}}^{x_i}}\Big(\underline{\int_{y_{j-1}}^{y_j}} \underbrace{\inf_{(x,y)\in\triangle_{ij}} f(x,y)}_{constant}\,dy\Big)dx\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\inf_{(x,y)\in\triangle_{ij}} f(x,y)\underbrace{(x_i-x_{i-1})(y_j-y_{j-1})}_{=V(\triangle_{ij})}\\
&= L(P,f)\\
&> \underline{\int_D} f(x,y)\,d\mathbb{A}-\varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\underline{\int_D} f(x,y)\,d\mathbb{A} \leq \underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)\,dy\Big)dx.$$

□

Theorem 6.3.4. *Let $D=[a,b]\times[c,d]$ be a rectangle in $\mathbb{R}^2$ and $f: D\to\mathbb{R}$ be Riemann integrable. Then*

(1) the functions $\phi(x)=\underline{\int_c^d} f(x,y)\,dy$ and $\psi(x)=\overline{\int_c^d} f(x,y)\,dy$ are Riemann integrable over $[a,b]$;

(2) the functions $\rho(y)=\underline{\int_a^b} f(x,y)dx$ and $\sigma(y)=\overline{\int_a^b} f(x,y)\,dx$ are Riemann integrable over $[c,d]$, and

(3) The integral of f over D

$$\begin{aligned}
\int_D f(x,y)\,d\mathbb{A} &= \int_a^b\Big(\underline{\int_c^d} f(x,y)\,dy\Big)dx = \int_a^b\Big(\overline{\int_c^d} f(x,y)\,dy\Big)dx\\
&= \int_c^d\Big(\underline{\int_a^b} f(x,y)\,dx\Big)dy = \int_c^d\Big(\overline{\int_a^b} f(x,y)\,dx\Big)dy
\end{aligned}$$

Proof. (1) To prove $\phi(x)=\underline{\int_c^d} f(x,y)\,dy$ is integrable over $[a,b]$. That is, to prove

$$\underline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)\,dy\Big)dx = \overline{\int_a^b}\Big(\underline{\int_c^d} f(x,y)\,dy\Big)dx.$$

By Lemma6.3.3,

$$\underline{\int}_D f(x,y)\,d\mathbb{A} \overset{6.3.3}{\le} \underline{\int}_a^b\Big(\underline{\int}_c^d f(x,y)\,dy\Big)dx \le \overline{\int}_a^b\Big(\underline{\int}_c^d f(x,y)\,dy\Big)dx \tag{6.2}$$

$$\le \overline{\int}_a^b\Big(\overline{\int}_c^d f(x,y)\,dy\Big)dx \overset{6.3.3}{\le} \overline{\int}_D f(x,y)\,d\mathbb{A}.$$

Since f is Riemann integrable over D,

$$\underline{\int}_a^b\Big(\underline{\int}_c^d f(x,y)\,dy\Big)dx = \overline{\int}_a^b\Big(\underline{\int}_c^d f(x,y)\,dy\Big)dx.$$

(2) By the similar results for $\overline{\int}_c^d f(\cdot,y)\,dy$, $\underline{\int}_a^b f(x,\cdot)\,dx$, $\overline{\int}_a^b f(x,\cdot)\,dx$, the statement (2) is proved.

(3) The proof of (3) is direct from (6.2).

□

Theorem 6.3.5. *(Fubini's Theorem) Let $D = [a,b]\times[c,d] \subseteq R^2$ and f be Riemann integrable over D. Suppose that for each $x \in [a,b]$, the function $f(x,\cdot)$ is integrable on $[c,d]$ and $\phi(x) = \int_c^d f(x,y)\,dy$ is integrable on $[a,b]$. Then*

$$\underbrace{\Big(\iint_D f(x,y)\,d\mathbb{A}\Big)}_{\text{double integrals}} = \int_D f(x,y)\,d\mathbb{A} = \underbrace{\int_a^b\Big(\int_c^d f(x,y)\,dy\Big)dx}_{\text{iterated integrals}}.$$

Likewise, if $f(\cdot,y)$ is integrable on $[a,b]$ and the function $\psi(y) = \int_a^b f(x,y)\,dx$ is integrable on $[c,d]$, then

$$\iint_D f(x,y)\,d\mathbb{A} = \int_c^d\Big(\int_a^b f(x,y)\,dx\Big)dy.$$

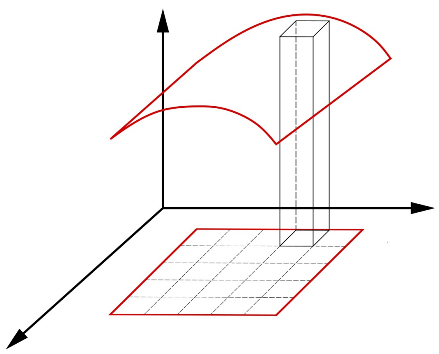
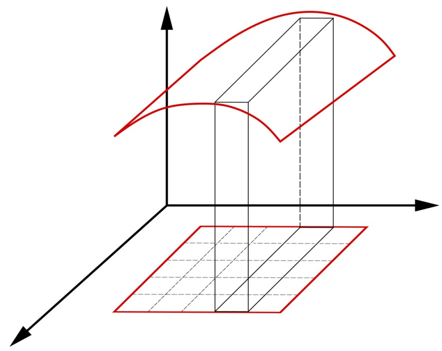
Remark. (1) We usually use

$$\begin{array}{lcl} \int_a^b\int_c^d f(x,y)\,dydx & & \int_a^b\Big(\int_c^d f(x,y)\,dy\Big)dx \\ \underline{\int}_a^b\overline{\int}_c^d f(x,y)\,dydx & \text{to denote} & \underline{\int}_a^b\Big(\overline{\int}_c^d f(x,y)\,dy\Big)dx \\ \overline{\int}_a^b\underline{\int}_c^d f(x,y)\,dydx & & \overline{\int}_a^b\Big(\underline{\int}_c^d f(x,y)\,dy\Big)dx \\ \vdots & & \end{array}$$

and so on

(2) In the viewpoint of the concept of integral

$$\iint_D f(x,y)\,d\mathbb{A} = \int_D f(\mathbf{x})\,d\mathbb{A} = \int_D f(x,y)\,\underbrace{d(x,y)}_{d\mathbb{A}} \ne \int_a^b\int_c^d f(x,y)\,dydx$$

Remark.

$$f(x,y) \text{ is integrable over } D \quad \Longrightarrow \quad \underline{\int_c^d} f(x,y)\,dy \text{ and } \overline{\int_c^d} f(x,y)\,dy \text{ are integrable over } [a,b] \text{ and}$$

$$\not\Uparrow \qquad \underline{\int_a^b} f(x,y)\,dy \text{ and } \overline{\int_a^b} f(x,y)\,dy \text{ are integrable over } [c,d].$$

But $f(x,\cdot)$ is integrable over $[c,d]$
or $f(\cdot,y)$ is integrable over $[a,b]$.

Example 6.3.6. Let $f:[0,1]\times[0,1]\to\mathbb{R}$ by $f(x,y)=\begin{cases} 1/p & \text{if } x,y\in\mathbb{Q},\ 0\neq x=\dfrac{q}{p} \text{ with } (p,q)=1 \\ 0 & \text{otherwise} \end{cases}$.

Then $f(x,y)$ is integrable over D (Skip, not easy to prove) and $\displaystyle\iint_D f(x,y)\,d\mathbb{A}=0$.

(1) For $y\in\mathbb{Q}^c$, $f^y(x)=f(x,y)\equiv 0$ for every $x\in[0,1]$. Then $f(\cdot,y)$ is integrable.

For $y\in\mathbb{Q}$, $f^y(x)=f(x,y)=\begin{cases} 1/p & \text{if } x=\dfrac{q}{p} \\ 0 & \text{if } y\in\mathbb{Q}^c \end{cases}$. Then $f(\cdot,y)$ is integrable over $[0,1]$.

For $x\in\mathbb{Q}^c\cup\{0\}$, $f_x(y)=f(x,y)=0$ for every $y\in[0,1]$. Then $f(x,\cdot)$ is integrable.

(2) For $x=\dfrac{q}{p}$ with $(p,q)=1$, $f(x,y)=f(\dfrac{q}{p},y)=\begin{cases} 1/p & \text{if } y\in\mathbb{Q} \\ 0 & \text{if } y\in\mathbb{Q}^c \end{cases}$. Then $f(x,\cdot)$ is not integrable.

Remark. Suppose that $f(x,\cdot)$ and $f(\cdot,y)$ are Riemann integrable over $[c,d]$ and $[a,b]$ respectively. It cannot imply that f is Riemann integrable over D. For example

$$f(x,y)=\begin{cases} 1 & \text{if } (x,y)=(\dfrac{k}{2^n},\dfrac{\ell}{2^n}),\ 0<k,\ell<2^n \text{ are odd numbers and } n\in\mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

on $D=[0,1]\times[0,1]$. For $x\in[0,1]$,
if $x\neq\dfrac{k}{2^n}$ for some $n\in\mathbb{N}$ and $0<k<2^n$ is odd, then $f(x,\cdot)\equiv 0$.
if $x=\dfrac{k}{2^n}$ for some $n\in\mathbb{N}$ and $0<k<2^n$ is odd, then

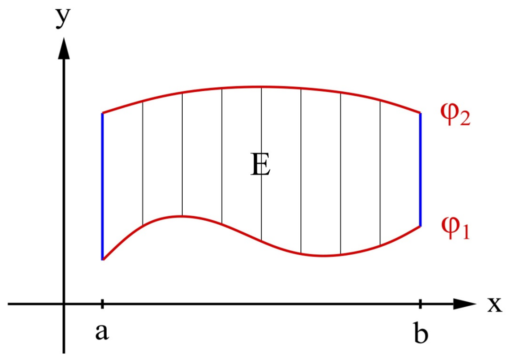
$$f(x,y)=\begin{cases} 1 & \text{if } (x,y)=(\dfrac{k}{2^n},\dfrac{1}{2^n}),(\dfrac{k}{2^n},\dfrac{3}{2^n}),\cdots,(\dfrac{k}{2^n},\dfrac{2^n-1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$$

Then $f(x,\cdot)$ is integrable over $[0,1]$. Similarly, $f(\cdot,y)$ is integrable over $[0,1]$.

Also, $\displaystyle\int_0^1 f(x,y)\,dy = 0 = \int_0^1 f(x,y)\,dx$. But f is not Riemann integrable over $[0,1]\times[0,1]$.

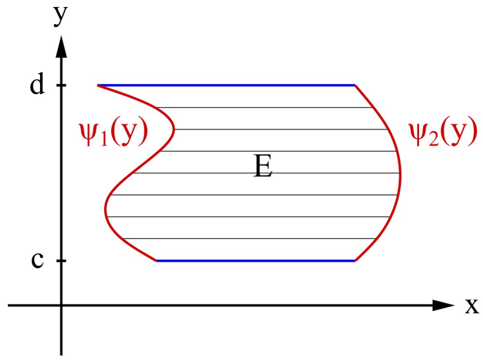
Corollary 6.3.7. *(1) Let $\phi_1,\phi_2 : [a,b] \to \mathbb{R}$ be of class C^1 such that $\phi_1(x) \le \phi_2(x)$ for every $x \in [a,b]$, $E = \{(x,y) \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$ and $f : E \to \mathbb{R}$ be continuous. Then f is Riemann integrable over E and*

$$\int_E f(x,y)\,d\mathbb{A} = \int_a^b \Big(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\Big)dx.$$



(2) Let $\psi_1,\psi_2 : [c,d] \to \mathbb{R}$ be of class C^1 such that $\psi_1(y) \le \psi_2(y)$ for every $y \in [c,d]$, $E = \{(x,y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$ and $f : E \to \mathbb{R}$ be continuous. Then f is Riemann integrable over E and

$$\int_E f(x,y)\,d\mathbb{A} = \int_c^d \Big(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\Big)dy.$$



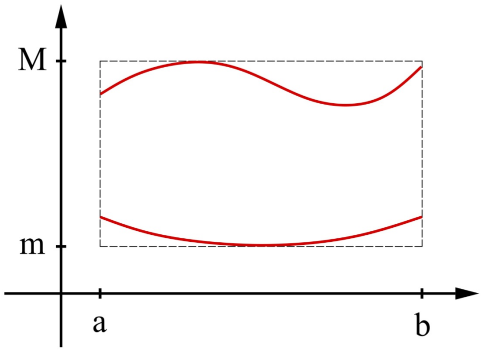
Proof. (1) Since ϕ_1 and ϕ_2 are of class C^1, then graphs of ϕ_1 and ϕ_2, $\{(x,\phi_1(x)) \mid a \le x \le b\}$ and $\{(x,\phi_2(x)) \mid a \le x \le b\}$ have volume zero.

Also, the left and right sides of E, $\{a\}\times[\phi_1(a),\phi_2(a)]$ and $\{b\}\times[\phi_1(b),\phi_2(b)]$, have volume zero. Then boundary of E has volume zero.

Let $M = \max_{a\le x\le b} \phi_2(x)$ and $m = \min_{a\le x\le b} \phi_1(x)$. Hence, $\bar{f}^E$ is continuous on $[a,b]\times[m,M]\backslash\partial E$ and then $\bar{f}^E$ is integrable over $[a,b]\times[m,M]$.

On the other hand, for every $x \in [a,b]$, $\bar{f}^E(x,\cdot)$ is continuous on $[m,M]$ except two points $\phi_1(x)$ and $\phi_2(x)$. Hence, $\bar{f}^E(x,\cdot)$ is integrable over $[m,M]$ and

$$\int_m^M \bar{f}^E(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy.$$



By the Fubini Theorem,

$$\int_E f(x,y)\,dxdy = \int_{[a,b]\times[m,M]} \bar{f}^E(x,y)\,dxdy = \int_a^b \Big(\int_m^M \bar{f}^E(x,y)\,dy\Big)dx = \int_a^b \Big(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\Big)dx.$$

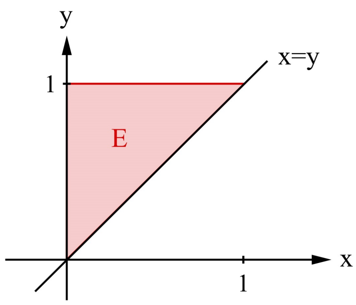
(2) Similar as proof of (1)

□

Remark. The corollary is also true if $\phi_1, \phi_2, \psi_1, \psi_2$ are of class C instead of C^1. (Skip the proof)

Example 6.3.8.

Let $E = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x \le y \le 1\}$ and $f(x,y) = xy$. Since f is continuous on E, for every $x \in [0,1]$, $f(x,\cdot)$ is continuous on $[x,1]$. By Fubini's Theorem,



$$\int_E f(x,y)\,d\mathbb{A} = \int_0^1 \Big(\int_x^1 xy\,dy\Big)dx = \int_0^1 x\Big(\int_x^1 y\,dy\Big)dx = \int_0^1 x\Big(\frac{1}{2} - \frac{1}{2}x^2\Big)\,dx = \frac{1}{8}.$$

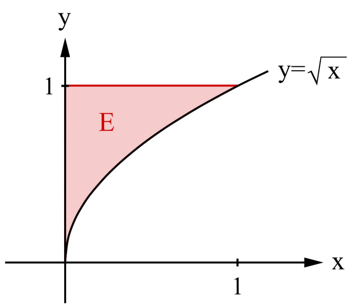
On the other hand, $E\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le y,\ 0 \le y \le 1\}$.

$$\int_E f(x,y)\,d\mathbb{A} = \int_0^1 \Big(\int_0^y xy\,dy\Big)dy = \int_0^1 y\Big(\int_0^y x\,dx\Big)\,dy = \frac{1}{8}.$$

Example 6.3.9.

Let $E = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$ and $f(x,y) = e^{y^3}$. Since f is continuous on E, by Fubini's Theorem,

$$\int_E e^{y^3}\,d\mathbb{A} = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dydx = ??$$



We don't know how to integrate e^{y^3}. On the other hands, $E = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le y^2,\ 0 \le y \le 1\}$,

$$\int_E e^{y^3}\, d\mathbb{A} = \int_0^1 \Big(\int_0^{y^2} e^{y^3}\, dx \Big)\, dy = \int_0^1 y^2 e^{y^3}\, dy = \frac{e-1}{3}.$$

Theorem 6.3.10. *(Fubini's Theorem) Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be rectangles, and $f : A \times B \to \mathbb{R}$ be bounded. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, write $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. Then*

$$\underline{\int}_{A\times B} f(\mathbf{z})\, d\mathbf{z} \le \underline{\int}_A \Big(\underline{\int}_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} \le \overline{\int}_A \Big(\overline{\int}_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} \le \overline{\int}_{A\times B} f(\mathbf{z})\, d\mathbf{z}$$

and

$$\underline{\int}_{A\times B} f(\mathbf{z})\, d\mathbf{z} \le \underline{\int}_B \Big(\underline{\int}_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \Big) d\mathbf{y} \le \overline{\int}_A \Big(\overline{\int}_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \Big) d\mathbf{y} \le \overline{\int}_{A\times B} f(\mathbf{z})\, d\mathbf{z}.$$

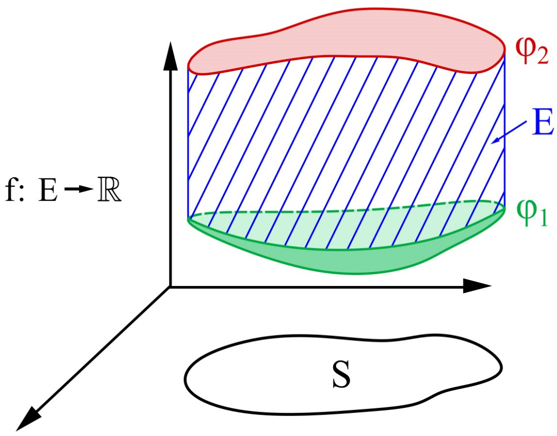
In particular, if f is Riemann integrable over $A \times B$, then

$$\begin{aligned} \int_{A\times B} f(\mathbf{z})\, d\mathbf{z} &= \int_A \Big(\underline{\int}_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} = \int_A \Big(\overline{\int}_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} \\ &= \int_B \Big(\underline{\int}_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \Big) d\mathbf{y} = \int_B \Big(\overline{\int}_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \Big) d\mathbf{y} \end{aligned}$$

Proof. (Ignore)(see 2-dimensional case) □

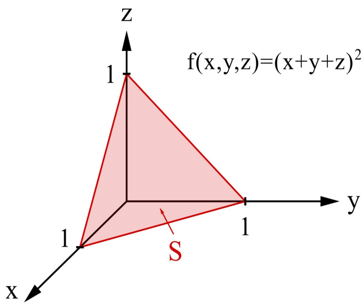
Corollary 6.3.11. *Let $S \subseteq \mathbb{R}^n$ be a bounded set with volume, $\phi_1, \phi_2 : S \to \mathbb{R}$ be continuous such that $\phi_1(\mathbf{x}) \le \phi_2(\mathbf{x})$ for every $\mathbf{x} \in S$. Let $E = \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid \mathbf{x} \in S\ \phi_1(\mathbf{x}) \le y \le \phi_2(\mathbf{x})\}$ and $f : E \to \mathbb{R}$ be continuous. Then f is Riemann integrable over E and*

$$\int_E f(\mathbf{x}, y)\, d(\mathbf{x}, y) = \int_S \Big(\int_{\phi_1(\mathbf{x})}^{\phi_2(\mathbf{x})} f(\mathbf{x}, y)\, dy \Big) d\mathbf{x}$$



Proof. (Ignore) □

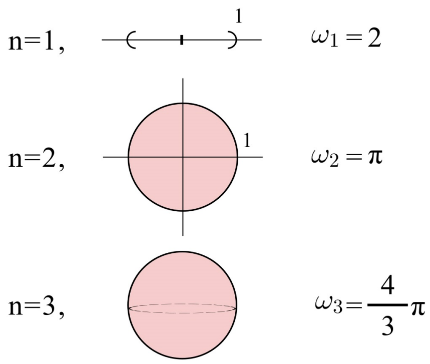
Example 6.3.12. Let $E = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x \ge 0,\ y \ge 0,\ z \ge 0,\ x + y + z \le 1\right\}$ and $f(x, y, z) = (x + y + z)^2$. Then $E = \left\{(x, y, z) \,\middle|\, 0 \le z \le 1 - x - y,\ 0 \le y \le 1 - x,\ 0 \le x \le 1\right\}$.



Let $S = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0 \le y \le 1 - x,\ 0 \le x \le 1\right\}$ and define $\phi_1(x, y) = 0$ and $\phi_2(x, y) = 1 - x - y$. We have

$$\begin{aligned}
\int_E f(x,y,z)\, d(x,y,z) &= \int_S \left(\int_0^{1-x-y} (x+y+z)^2\, dz \right) d(x,y) \\
&= \int_S \frac{1}{3}\left[1 - (x+y)^3\right] d(x,y) \\
&= \int_0^1 \left(\int_0^{1-x} \frac{1}{3}\left[1 - (x+y)^3\right] dy \right) dx \\
&= \int_0^1 \frac{1}{4} - \frac{x}{3} + \frac{x^4}{12}\, dx = \frac{1}{10}.
\end{aligned}$$

Example 6.3.13. Let ω_n be the volume of the n-dimensional unit ball. Find the formula of ω_n.



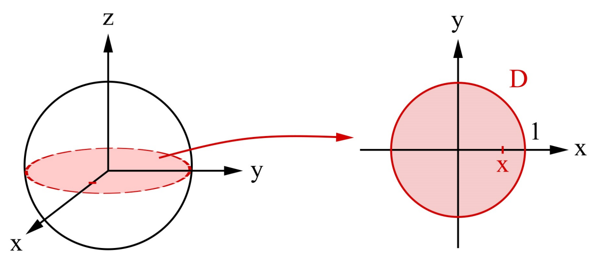
For $n = 1$, $\omega_1 = 2$

For $n = 2$, $\omega_2 = \pi$

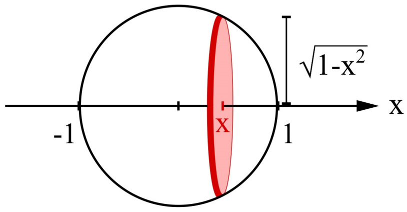
For $n = 3$, let $D = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \le 1\right\}$.

Let

$$\begin{aligned}
E &= \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, 0 \le x^2 + y^2 + z^2 \le 1\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, -\sqrt{1 - x^2 - y^2} \le z \le \sqrt{1 - x^2 - y^2},\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}, -1 \le x \le 1\right\}
\end{aligned}$$



$$\begin{aligned}\omega_3 &= \int_E \mathbb{1}_E(x,y,z)\, d(x,y,z) = \int_D \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz d(x,y) \\ &= \int_{-1}^{1} \underbrace{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dzdy}_{\substack{\text{the volume(area) of the 2-dimensional} \\ \text{disk with radius } \sqrt{1-x^2} = \omega_2(\sqrt{1-x^2})^2}}\ dx = \int_{-1}^{1} \omega_2(1-x^2\, dx \\ &= \omega_2(x - \frac{1}{3}x^3)\Big|_{-1}^{1} = \frac{4}{3}\omega_2 = \frac{4}{3}\pi.\end{aligned}$$



How about ω_n?

Consdier

$$\begin{aligned}E_n &= \left\{(x_1,\cdots,x_n) \in \mathbb{R}^n \,\middle|\, 0 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le 1\right\} \\ &= \Big\{(x_1,\cdots,x_n) \in \mathbb{R}^n \,\Big|\, -\sqrt{1-x_1^2-\cdots-x_{n-1}^2} \le x_n \le \sqrt{1-x_1^2-\cdots-x_{n-1}^2}, \\ &\qquad -\sqrt{1-x_1^2-\cdots-x_{n-2}^2} \le x_{n-1} \le \sqrt{1-x_1^2-\cdots-x_{n-2}^2}, \cdots, \\ &\qquad -\sqrt{1-x_1^2} \le x_2 \le \sqrt{1-x_1^2},\ -1 \le x_1 \le 1\Big\}\end{aligned}$$

$$\begin{aligned}\omega_n &= \int_{E_n} \mathbb{1}_{E_n}(x_1,\cdots,x_n)\, d(x_1,\cdots,x_n) \\ &= \int_{-1}^{1} \underbrace{\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \int_{-\sqrt{1-x_1^2-x_2^2}}^{\sqrt{1-x_1^2-x_2^2}} \int \cdots \int_{-\sqrt{1-x_1^2-\cdots-x_{n-1}^2}}^{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}} 1\, dx_n dx_{n-1} \cdots dx_2}_{\substack{\text{the volume of } (n-1)\text{-dimensional ball with radius } \sqrt{1-x_1^2} \\ = \omega_{n-1}(\sqrt{1-x_1^2})^{n-1} = \omega_{n-1}(1-x_1^2)^{\frac{n-1}{2}}}}\ dx_1\end{aligned}$$

Then

$$\begin{aligned}\omega_n &= \omega_{n-1} \int_{-1}^{1} (1-x^2)^{\frac{n-1}{2}}\, dx \\ &= \omega_{n-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n\theta\, d\theta \\ &= 2\omega_{n-1} \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = 2\omega_{n-1} \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2}\theta\, d\theta \\ &= 2 \cdot \underbrace{\left(2\omega_{n-2} \int_0^{\frac{\pi}{2}} \cos^{n-1}\theta\, d\theta\right)}_{=\omega_{n-1}} \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta\end{aligned}$$

Since

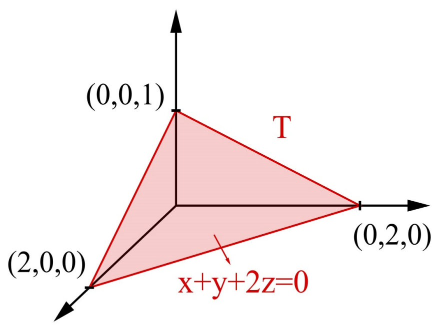
$$\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \begin{cases} \dfrac{(n-1)(n-3)\cdots 2}{n(n-2)} \cdots 3 \displaystyle\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta & \text{if } n \text{ is odd} \\ \dfrac{(n-1)(n-3)\cdots 1}{n(n-2)} \cdots 2 \displaystyle\int_0^{\frac{\pi}{2}} 1\, d\theta & \text{if } n \text{ is even} \end{cases},$$

we have $\omega_n = \dfrac{2\omega_{n-2}}{n}\pi$. Therefore,

$$\omega_n = \begin{cases} \dfrac{(2\pi)^{\frac{n-1}{2}}}{n(n-2)\cdots 3}\omega_1 & \text{if } n \text{ is odd} \\ \dfrac{(2\pi)^{\frac{n-2}{2}}}{n(n-2)\cdots 4}\omega_2 & \text{if } n \text{ is even} \end{cases}$$

Example 6.3.14. Find the mass of the tetrahedron T formed by the three coordinate planes and the plane $x + y + 2z = 2$ if the mass density is $\rho(x, y, z) = e^{-z}$.

$$M = \int_T e^{-z}\, dV.$$



$$\int_0^2 \int_0^{2-x} \int_0^{1-(x+y)/2} e^{-z}\, dzdydx \qquad (6.3)$$

$$\text{or} \quad \int_0^1 \int_0^{2-2z} \int_0^{2-y-2z} e^{-z}\, dxdydz$$

$$\text{or} \quad \int_0^2 \int_0^{1-(y/2)} \int_0^{2-y-2z} e^{-z}\, dxdzdy$$

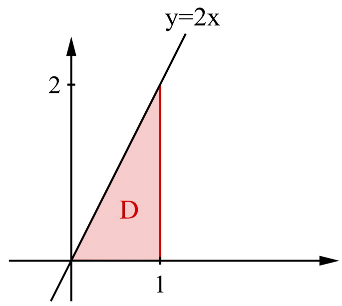
$$(6.3) = \int_0^2 \int_0^{2-x} 1 - e^{\frac{x+y}{2}-1}\, dydx = \int_0^2 2e^{\frac{x}{2}-1} - x\, dx = 2 - 4e^{-1}.$$

Example 6.3.15. Evaluate

$$\int_0^2 \int_{y/2}^1 ye^{-x^3}\, dxdy = \int_D ye^{-x^3}\, d\mathbb{A}$$

$$= \int_0^1 \int_0^{2x} ye^{-x^3}\, dydx = \int_0^1 2x^2 e^{-x^3}\, dx$$

$$= \frac{2}{3}(1 - e^{-1})$$



wher D is the region bounded by $x = 1$, $y = 2x$ and x-axis.

Remark. In general, $\int_a^b \int_c^d f(x, y)\, dydx \neq \int_c^d \int_a^b f(x, y)\, dxdy$ (See exercise13 in "Folland", page 176 or lecture note Problem 7.8)

■ **Some Applicaitons**

(1) If $f(x, y) \geq 0$, $\iint_S f\, d\mathbb{A}$ can be interpreted as the volume of the region in $\mathbb{R}^3$ between the graph of f and the xy-plane that lies over the base region S.

(2) Evaluate a quantity of some substance (ex: mass, electric charge, chemical compound)

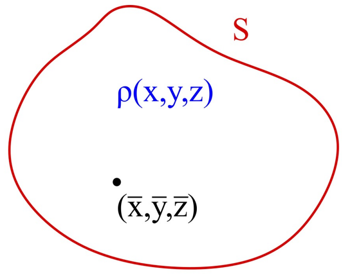
$$M = \int_S \rho(x,y,z)\, d(x,y,z), \quad S \subseteq \mathbb{R}^k$$

(3) centroid of the region S. For a region (or an object) $S \subseteq \mathbb{R}^3$ and the density function $\rho(x,y,z)$, the mass of the object is

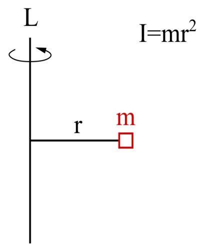
$$M = \int_S \rho(x,y,z)\, d(x,y,z)$$

and the centroid $(\bar{x},\bar{y},\bar{z})$ of S is

$$\begin{aligned} \bar{x} &= \frac{1}{M}\int_S x\rho(x,y,z)\, d(x,y,z), \\ \bar{y} &= \frac{1}{M}\int_S y\rho(x,y,z)\, d(x,y,z), \\ \bar{z} &= \frac{1}{M}\int_S z\rho(x,y,z)\, d(x,y,z), \end{aligned}$$



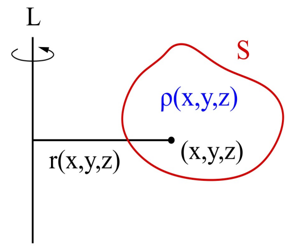
(4) moment of inertia(轉動慣量)



Let $r(x,y,z)$ = distance from (x,y,z) to L.

$$I = \int_S r^2(x,y,z)\rho(x,y,z)\, d(x,y,z)$$

For example, L is z-axis, then $r(x,y,z) = \sqrt{x^2+y^2}$.



6.4 Change of Variables

Recall:

Differentiation		**Integration**
Product Rule	$\longleftrightarrow$	Integration by parts
Chain Rule	$\overset{F.T.C}{\longleftrightarrow}$	Change of variables

For $h(x) = f\big(g(x)\big)$, let $u = g(x)$, then $du = g'(x)\,dx$. By the change of variables and F.T.C,

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f\big(g(x)\big)g'(x)\,dx = \int_a^b h(x)g'(x)\,dx.$$

For example,

$$\int_1^2 x^2 e^{x^3}\,dx \overset{u=x^3}{=} \int_1^8 e^u\,\frac{1}{3}du = \frac{1}{3}\int_{u(1)}^{u(2)} f(u)\,du$$
$$\Big(f(x) = e^x,\ u(x) = x^3,\ h(x) = f\big(u(x)\big) = e^{x^3}\Big)$$

Note. If $g : [a,b] \to \mathbb{R}$ is differentiable and increasing, then $g'(x) \geq 0$ and $g(a) < g(b)$.

$$\int_a^b f\big(g(x)\big)|g'(x)|\,dx = \int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

If $g : [a,b] \to \mathbb{R}$ is differentiable and decreasing, then $g'(x) \leq 0$ and $g(a) > g(b)$.

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

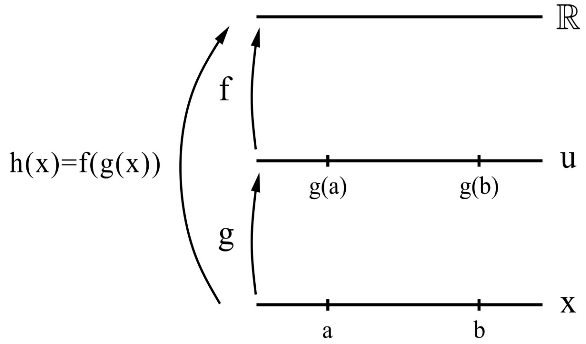
We have

$$\int_a^b f\big(g(x)\big)|g'(x)|\,dx = \int_{g(b)}^{g(a)} f(u)\,du.$$

Hence, in each case,

$$\int_{[a,b]} f\big(g(x)\big)|g'(x)|\,dx = \int_{g\left([a,b]\right)} f(u)\,du.$$

Geometrically, in 1-dimension,



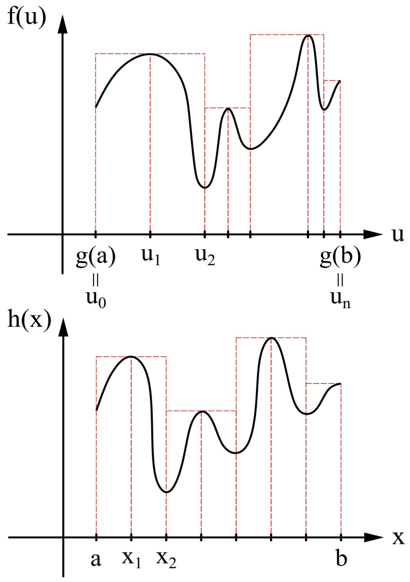
$$\int_{g(a)}^{g(b)} f(u)\,du \;\approx\; \sum_{i=1}^{n} f(u_i^*)\,\underbrace{(u_i - u_{i-1})}$$

$$\boxed{\text{where } u_i^* = g(x_i^*)} \;=\; \sum_{i=1}^{n} f\big(g(x_i^*)\big)\;\underbrace{g'(t_i)(x_i - x_{i-1})}_{\substack{u_i - u_{i-1} = g(x_i) - g(x_{i-1}) \\ (M.V.T) \;= g'(t_i)(x_i - x_{i-1})}}$$

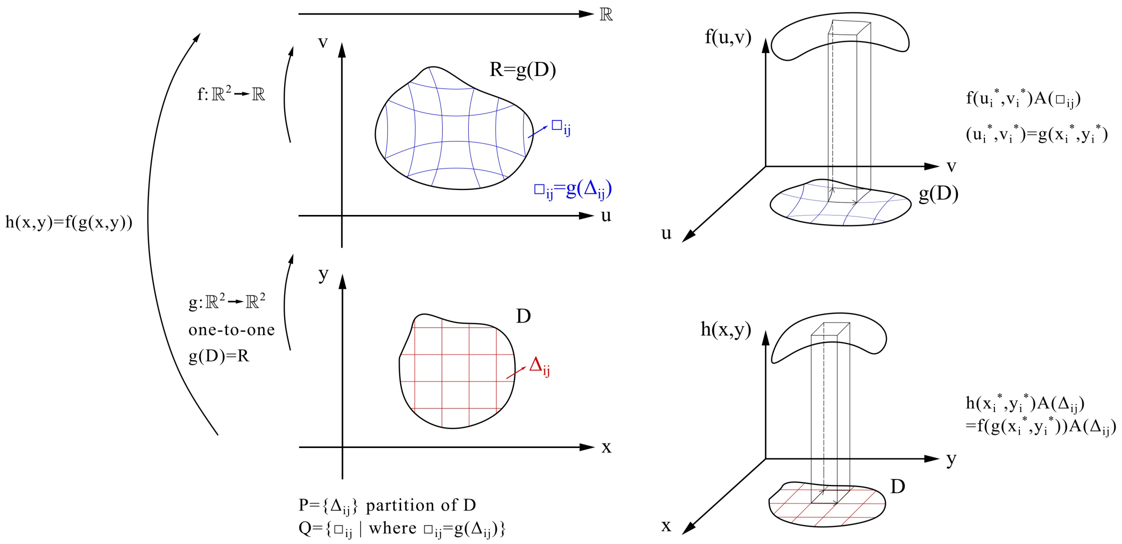
$$\int_a^b h(x)\,dx \;\approx\; \sum_{i=1}^{n} h(x_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} f\big(g(x_i^*)\big)(x_i - x_{i-1})$$

$$\frac{\triangle u_i}{\triangle x_i} \approx g'(t_i)$$



In 2-dimensions,



$$\int_D h(x,y)\,d(x,y) \;\approx\; \sum_{\triangle\in P} h(x^*,y^*)\,\mathbb{A}(\triangle) \quad \text{where } (x^*,y^*) \in \triangle$$

$$\int_{g(D)} f(u,v)\,d(u,v) \;\approx\; \sum_{\square\in Q} f(u^*,v^*)\,\mathbb{A}(\square) \quad \text{where } (u^*,v^*) = g(x^*,y^*) \in \square$$

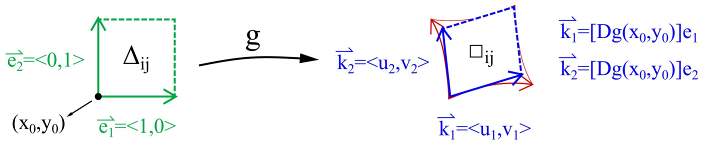
$$= \sum_{\triangle\in P} f\big(g(x^*,y^*)\big)\frac{\mathbb{A}(\square)}{\mathbb{A}(\triangle)}\,\mathbb{A}(\triangle).$$

$$\frac{\mathbb{A}(u,v)}{\mathbb{A}(x,y)} = \frac{\mathbb{A}(\square)}{\mathbb{A}(\triangle)} = \frac{\diamond}{\square}$$

$$\Downarrow$$

$$\lim_{\|P\|\to 0}\frac{\mathbb{A}(u,v)}{\mathbb{A}(x,y)} = \left|\frac{\partial(g_1,g_2)}{\partial(x,y)}\right|$$

As $\|P\| \to 0$, $\|Q\| \to 0$.



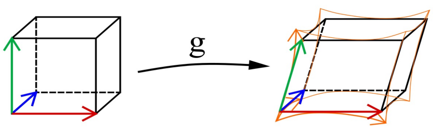
Heurestically,

$$\int_{g(D)} f(u,v)\, d(u,v) = \int_D f\big(g(x,y)\big)\left|J_g(x,y)\right|\, d(x,y).$$

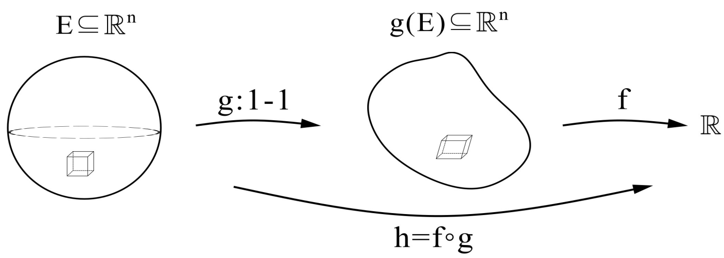
Note: Some situations need to be avoided. For example, g cannot be degenerated.

In general, for $E \subseteq \mathbb{R}^n$, $g : E \to g(E)$ (one-to-one) and $f : g(E) \to \mathbb{R}$. We may guess

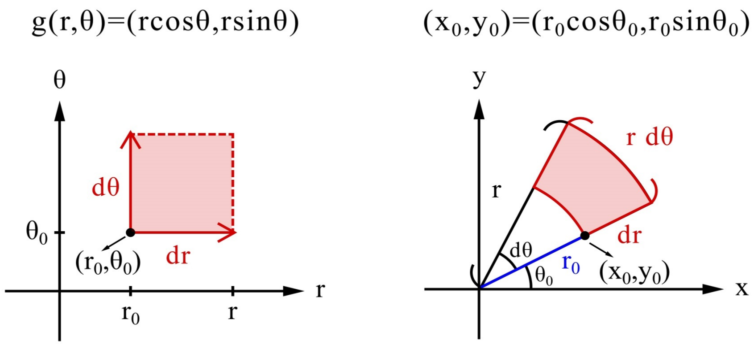
$$\int_{g(E)} f(\mathbf{u})\, d\mathbf{u} = \int_E f\big(g(\mathbf{x})\big)\ \underline{\ \ ??\ \ }\ d\mathbf{x}.$$

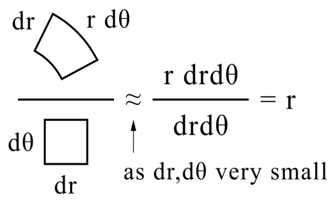


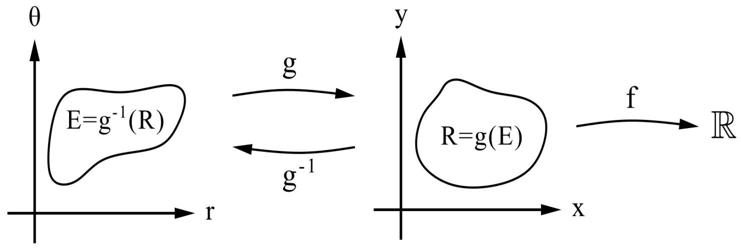
The term $\underline{\ \ ??\ \ }$ is supposed to represent "the rate of change of volumes between $\square$ and $g(\square)$ under the transform g".



Example 6.4.1. Let $g(r,\theta) = (r\cos\theta, r\sin\theta)$.







Then

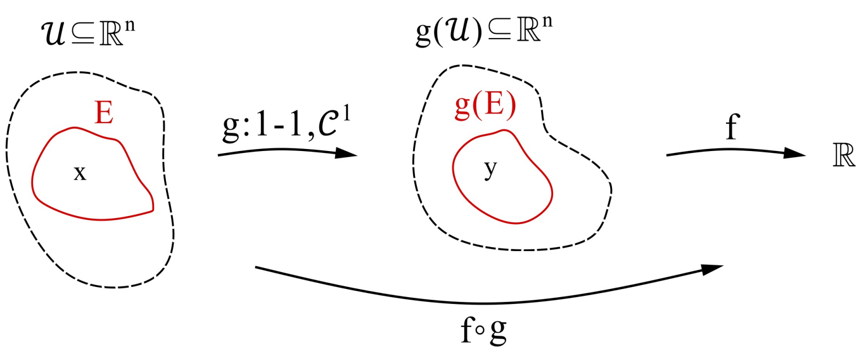
$$\int_{g(E)} f(x,y)\,dxdy = \int_E f(r\cos\theta, r\sin\theta)\,r\,drd\theta$$

or

$$\int_R f(x,y)\,dxdy = \int_{g^{-1}(R)} f(r\cos\theta, r\sin\theta)\,r\,drd\theta.$$

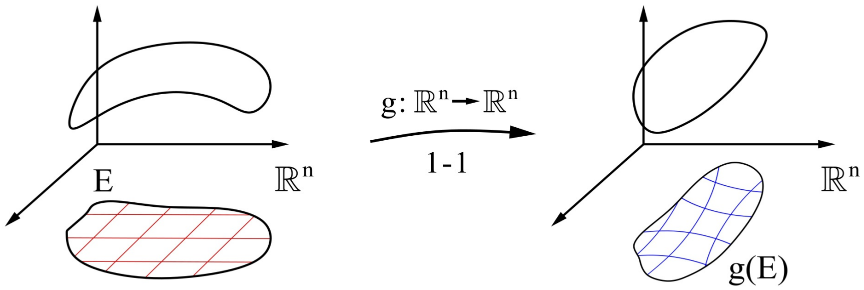
Theorem 6.4.2. *(Change of Variables Formula) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open bounded set, and $\mathbf{g} : \mathcal{U} \to \mathbb{R}^n$ be an one-to-one C^1 mapping with C^1 inverse; that is, $\mathbf{g}^{-1} : \mathbf{g}(\mathcal{U}) \to \mathcal{U}$ is also continuously differentiable. Assume that the Jacobian of $\mathbf{g}$, $J_{\mathbf{g}} = \det\left(\left[D\mathbf{g}\right]\right)$, does not vanish in $\mathcal{U}$, and $E \subset\subset \mathcal{U}$ has volume. Then $\mathbf{g}(E)$ has volume. Moreover, if $f : \mathbf{g}(E) \to \mathbb{R}$ is bounded and integrable, then $(f \circ \mathbf{g})|J_{\mathbf{g}}|$ is integrable over E and*

$$\int_{\mathbf{g}(E)} f(\mathbf{y})\,d\mathbf{y} = \int_E (f\circ\mathbf{g})(\mathbf{x})\left|J_{\mathbf{g}}(\mathbf{x})\right|\,d\mathbf{x} = \int_E (f\circ\mathbf{g})(\mathbf{x})\left|\frac{\partial(g_1,\cdots,g_n)}{\partial(x_1,\cdots,x_n)}\right|\,d\mathbf{x}$$



Proof. (Skip the proof) we will only show the special case $\mathbf{g} = A \in M_n(\mathbb{R})$. □

Sketch the proof (Consider the case that f is a continuous function)

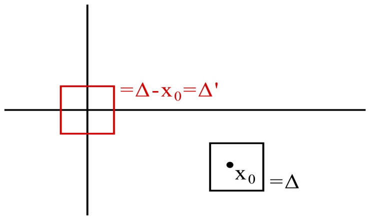


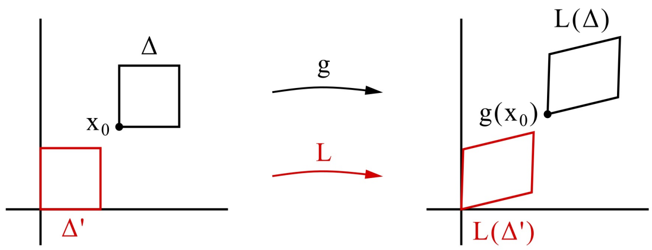
- If the mesh size is sufficiently small, f is (almost) a constant function on each $\triangle$ (e.g. let $f \equiv ave(f)$ on $\triangle$). Hence, we may assume f is a constant function and prove that

$$V\big(\mathbf{g}(E)\big) = \int_{\mathbf{g}(E)} \mathbb{1}_{\mathbf{g}(E)}(\mathbf{y})\, d\mathbf{y} = \int_E \mathbb{1}_{\mathbf{g}(E)}\big(\mathbf{g}(\mathbf{x})\big)|J_{\mathbf{g}}(\mathbf{x})|\, d\mathbf{x} = \int_E |J_{\mathbf{g}}(\mathbf{x})|\, d\mathbf{x}$$

- If the mesh size is sufficiently small, since $g \in C^1$, for $\mathbf{x}, \mathbf{x}_0 \in \triangle$,

$$\mathbf{g}(\mathbf{x}) \approx \mathbf{g}(\mathbf{x}_0) + D\mathbf{g}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$
$$\Rightarrow \quad \mathbf{g}(\triangle) \approx \mathbf{g}(\mathbf{x}_0) + \underbrace{L(\triangle')}_{L=D\mathbf{g}(\mathbf{x}_0)} \quad \text{where } \triangle' = \triangle - \mathbf{x}_0$$
$$\Rightarrow \quad V\big(\mathbf{g}(\triangle)\big) \approx V\big(L(\triangle')\big) = \underset{L \text{ is linear}}{V\big(L(\triangle)\big)}$$





Since $\mathbf{g} \in C^1$,

$$J_{\mathbf{g}}(\mathbf{x}) \approx J_{\mathbf{g}}(\mathbf{x}_0) = \det\big[D\mathbf{g}(\mathbf{x}_0)\big].$$

Hence, we may assume that $\mathbf{g}$ is a linear map. That is, $\mathbf{g}(\mathbf{x}) = L\mathbf{x}$ for some $L \in \mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^n\big)$ and thus $D\mathbf{g}(\mathbf{x}) = L$ for every $\mathbf{x} \in \mathbb{R}^n$. To prove

$$V\big(\mathbf{g}(\square)\big) = V\big(L(\square)\big) = \int_{\square} |\det L|\, d\mathbf{x}.$$

Let $\mathbf{g}(\mathbf{x}) = L\mathbf{x}$ for some $L \in \mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^n\big)$. Then $D\mathbf{g}(\mathbf{x}) = L$ for every $\mathbf{x} \in \mathbb{R}$.

Question: What's the intuition of the rate of change of volumes under the transformation?
Question: Why is the rate equal to $|J_{\mathbf{g}}(\mathbf{x})| = \big| \det \big[D\mathbf{g}(\mathbf{x})\big] \big|$?

$$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

■ **Gaussian Elimination**

$$\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \longrightarrow \left[\begin{array}{ccc} \quad & \quad & \quad \\ \quad & \quad & \quad \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & g \\ 0 & 1 & h \end{array}\right]$$

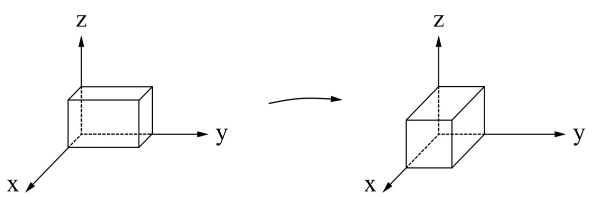
■ **Gaussian Elimination**

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & e & f \\ 0 & 1 & g & h \end{array}\right]$$

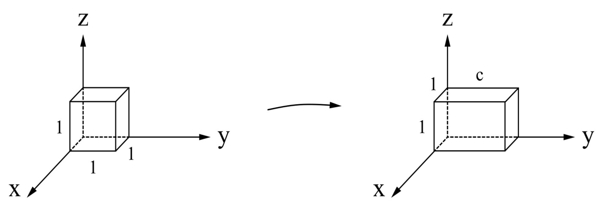
Note. By the observation of Gaussian elimination, we find that every linear map can be expressed as the composition of several "elementary transformations" as follows.

■ **Three elementary transformations:**

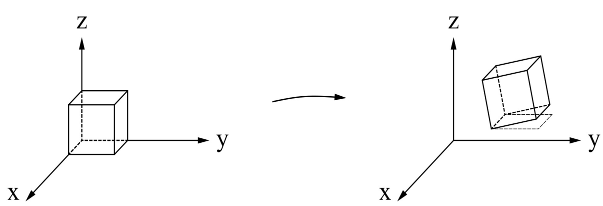
(1) $\mathbf{g}(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$



(2) $\mathbf{g}(x_1, \cdots, x_i, \cdots x_n) = (x_1, \cdots, cx_i, \cdots, x_n)$



(3) $\mathbf{g}(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_i + cx_j, \cdots, x_j, \cdots, x_n)$



For example, $\mathbf{g} \in \mathcal{B}(\mathbb{R}^2; \mathbb{R}^2)$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{②\leftrightarrows①} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{②\times\alpha} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \alpha c & \alpha d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{②\times\alpha+①} \Rightarrow \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+\alpha c & b+\alpha d \\ c & d \end{bmatrix}$$

Lemma 6.4.3. *Let $\mathbf{g} \in \mathcal{B}(\mathbb{R}^n; \mathbb{R})$ and $A \subseteq \mathbb{R}^n$ be a set which has volume. Then $\mathbf{g}(A)$ has volume and*

$$V\big(\mathbf{g}(A)\big) = \int_{\mathbf{g}(A)} \mathbb{1}_{\mathbf{g}(A)}(\mathbf{y})\, d\mathbf{y} = \int_A |J_{\mathbf{g}}(\mathbf{x})|\, d\mathbf{x}.$$

Proof. For every $\mathbf{g} \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^n)$, there exists $L \in M_n(\mathbb{R})$ such that $\mathbf{g}(\mathbf{x}) = L\mathbf{x}$ for every $\mathbf{x} \in A$.

(I) $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a rectangle.
Case (I-1):

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & & & & & & & \vdots \\ \vdots & & \ddots & & & & & & \vdots \\ \vdots & & & 0 & \cdots & 1 & & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & & \vdots \\ \vdots & & & 1 & \cdots & 0 & & & \vdots \\ \vdots & & & & & & \ddots & & \vdots \\ \vdots & & & & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{ith row} \\ \\ \leftarrow \text{jth row} \\ \\ \\ \\ \end{matrix}$$

$\uparrow$ *ith column* $\uparrow$ *jth column*

That is, $\mathbf{g}(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$. Then $\det(L) = -1$. Thus,

$$L(A) = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_j, b_j] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_{j-1}, b_{j-1}] \times [a_i, b_i] \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_n, b_n]$$

Hence, $V\big(L(A)\big) = V(A) = \big|\det(L)\big| V(A)$.

Case (I-2):

$$L = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \textit{ ith row} \\ \\ \\ \\ \end{matrix}$$

$$\uparrow$$
$$\textit{ith column}$$

That is, $\mathbf{g}(x_1, \cdots, x_n) = (x_1, \cdots, cx_i, \cdots, x_n)$. Then, $\det(L) = c$. Thus,

$$L(A) = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [ca_i, cb_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n].$$

Hence, $V\big(L(A)\big) = |c|V(A) = \big|\det(L)\big|V(A)$.

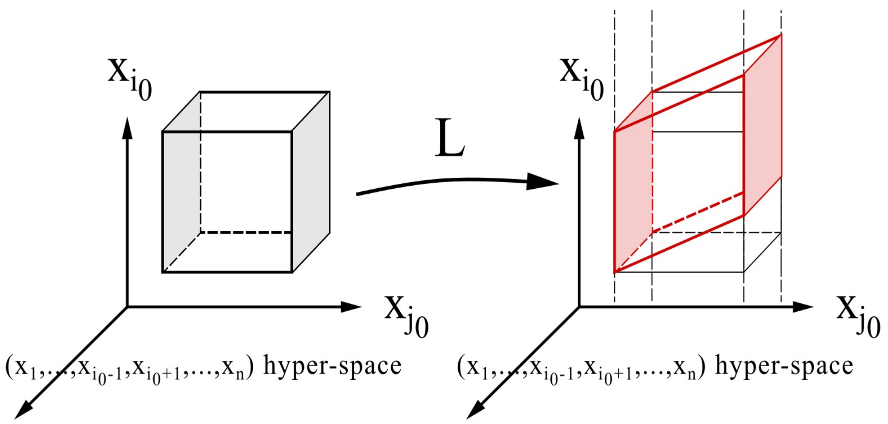
Case (I-3):

$$L = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \textit{ ith row} \\ \\ \\ \\ \\ \end{matrix}$$

$$\uparrow$$
$$\textit{jth column}$$

That is, $\mathbf{g}(x_1, \cdots, x_n) = (x_1, \cdots, x_i + cx_j, \cdots, x_n)$. Then, $\det(L) = 1$. Thus,

$$L(A) = \bigcup_{x_j \in [a_j, b_j]} [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i + cx_j, b_i + cx_j] \times [a_{i+1}, b_{i+1}]$$
$$\times \cdots \times [a_{j-1}, b_{j-1}] \times \{x_j\} \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_n, b_n].$$



Let $D = [a_1, b_1]\times\cdots\times[a_{i-1}, b_{i-1}]\times[a_{i+1}, b_{i+1}]\times\cdots\times[a_n, b_n]$ and $\hat{\mathbf{x}}_\mathbf{i} = (x_1, \cdots, x_{i-1}, x_{i+1} \cdots, x_n)$, by the Fubini's Theorem,

$$V\big(L(A)\big) = \int_D \Big(\int_{a_i+cx_j}^{b_i+cx_j} 1\ dx_i \Big) d\hat{\mathbf{x}}_i = V(A) = \big|\det(L)\big| V(A).$$

Let $\mathbf{g} \in \mathcal{B}\big(\mathbb{R}^n; \mathbb{R}^n\big)$, then $\mathbf{g} = \mathbf{g}_1 \circ \cdots \circ \mathbf{g}_k$ is a composition of several elementary transformations $\mathbf{g}_1, \cdots, \mathbf{g}_k$ where eahc $\mathbf{g}_i$ is one of the transformation in Case (I-1) - Case (I-3).

Let $L, L_1, \cdots, L_k \in M_n(\mathbb{R})$ be the matrics corresponding $\mathbf{g}, \mathbf{g}_1, \cdots, \mathbf{g}_k$. Then $\det(L_i) = J_{\mathbf{g}_i}(\mathbf{x})$ for every $\mathbf{x} \in A$.

$$\begin{aligned}
V\big(\mathbf{g}(A)\big) &= V\big(\mathbf{g}_1 \circ \cdots \circ \mathbf{g}_k(A)\big) = \big|\det(L_1)\big| V\big(\mathbf{g}_2 \circ \cdots \circ \mathbf{g}_k(A)\big) \\
&= \big|\det(L_1)\big|\big|\det(L_2)\big| V\big(\mathbf{g}_3 \circ \cdots \circ \mathbf{g}_k(A)\big) \\
&\ \vdots \\
&= \big|\det(L_1)\big| \cdots \big|\det(L_k)\big| V(A) \\
&= \big|\det(L_1 \circ L_2 \circ \cdots \circ L_k)\big| V(A) \\
&= \big|\det(L)\big| V(A) \\
&= \big|J_\mathbf{g}(\mathbf{x})\big| V(A) \\
&= \int_A \big|J_\mathbf{g}(\mathbf{x})\big|\ d\mathbf{x}.
\end{aligned}$$

(II) A is an arbitrary set with volume.

Case (II-1): $\det(L) = 0$.

Let R be a rectangle in $\mathbb{R}^n$ such that $A \subseteq R$. Then $L(A) \subseteq L(R)$. Thus

$$V\big(L(A)\big) \le V\big(L(R)\big) = \big|\det(L)\big| V(R) = 0.$$

We have

$$V\big(L(A)\big) = 0 = \big|\det(L)\big| V(A).$$

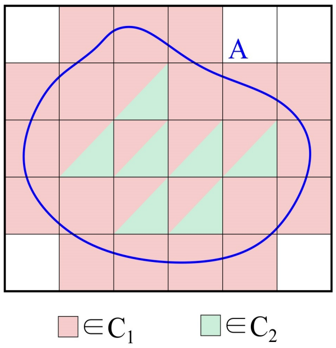
Case (II-2): $\det(L) \neq 0$.

Since A has volume, $\mathbb{1}_A$ is integrable and $\int_A \mathbb{1}_A(\mathbf{x})\ d\mathbf{x} = V(A)$. Then for given $\varepsilon > 0$, there exists a partition P of A such that

$$U(P, \mathbb{1}_A) - L(P, \mathbb{1}_A) < \frac{\varepsilon}{\big|\det(L)\big|}.$$

We have

$$\Big|U(P, \mathbb{1}_A) - V(A)\Big| < \frac{\varepsilon}{\big|\det(L)\big|} \quad \text{and} \quad \Big|L(P, \mathbb{1}_A) - V(A)\Big| < \frac{\varepsilon}{\big|\det(L)\big|}.$$



Let

$$\begin{aligned} C_1 &= \{\triangle \in P \mid \triangle \cap A \neq \emptyset\} \text{ and} \\ C_2 &= \{\triangle \in P \mid \triangle \subseteq A\}. \end{aligned}$$

Define $R_1 = \bigcup_{\triangle \in C_1} \triangle$ and $R_2 = \bigcup_{\triangle \in C_2} \triangle$. Then $R_2 \subseteq A \subseteq R_1$.

Since $\det(L) \neq 0$, L is one-to-one. Thus $L(\triangle_i)$ and $L(\triangle_j)$ are not overlapping if $\triangle_i \neq \triangle_j$ for every $\triangle_i, \triangle_j \in P$.

We have

$$\begin{aligned} V\big(L(R_1)\big) &= V\big(L(\bigcup_{\triangle \in C_1} \triangle)\big) = V\big(\bigcup_{\triangle \in C_1} L(\triangle)\big) = \sum_{\triangle \in C_1} V\big(L(\triangle)\big) \\ &= \big|\det(L)\big| \sum_{\triangle \in C_1} V(\triangle) = \big|\det(L)\big| U(P, \mathbb{1}_A) \\ &< \big|\det(L)\big| V(A) + \varepsilon. \end{aligned}$$

Also,

$$\begin{aligned} V\big(L(R_2)\big) &= \sum_{\triangle \in C_2} V\big(L(\triangle)\big) \\ &= \big|\det(L)\big| \sum_{\triangle \in C_2} V(\triangle) = \big|\det(L)\big| L(P, \mathbb{1}_A) \\ &\geq \big|\det(L)\big| V(A) - \varepsilon. \end{aligned}$$

Since $L(R_2) \subseteq L(A) \subseteq L(R_1)$,

$$\overline{\int}_{L(A)} 1 \, d\mathbf{x} - \underline{\int}_{L(A)} 1 \, d\mathbf{x} \leq V\big(L(R_1)\big) - V\big(L(R_2)\big) < 2\varepsilon.$$

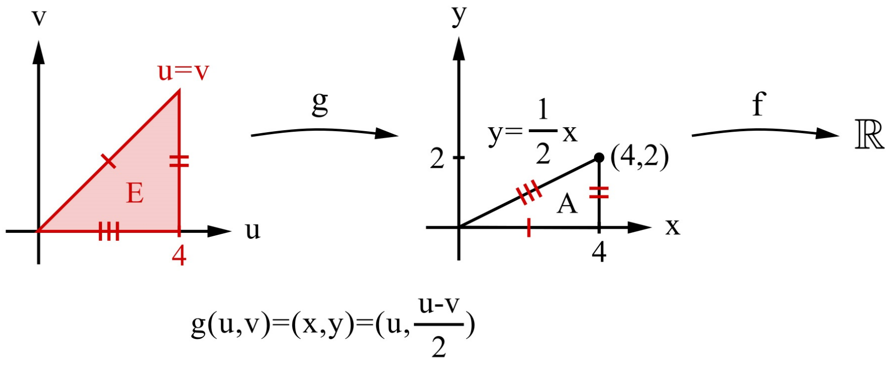
Since $\varepsilon > 0$, $\overline{\int}_{L(A)} 1 \, d\mathbf{x} = \underline{\int}_{L(A)} 1 \, d\mathbf{x}$ and hence $\mathbb{1}_{L(A)}$ is integrable over $L(A)$. Therefore, $L(A)$ has volume and $V\big(L(A)\big) = \int_{L(A)} \mathbb{1}_{L(A)} \, d\mathbf{x} = \big|\det(L)\big| V(A)$.

□

Example 6.4.4. Let $A \subseteq \mathbb{R}^2$ be the region which is bounded by $x = 4$, $y = \frac{1}{2}x$ and x-axis. $f : A \to \mathbb{R}$ be defined by $f(x, y) = y\sqrt{x - 2y}$. Find $\displaystyle\int_A f(x, y) \, d(x, y)$.

Method 1: Let $(u, v) = (x, x - 2y)$ and define $\mathbf{g}(u, v) = (u, \dfrac{u - v}{2}) = (x, y)$. Then $\mathbf{g}$ is defined on $E \subseteq \mathbb{R}^2$ which is bounded by $u = 4$, $u = v$ and u-axis. Thus, $\mathbf{g} : E \to A$ is bijective. The Jacobian of $\mathbf{g}$ is

$$J_{\mathbf{g}}(u, v) = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

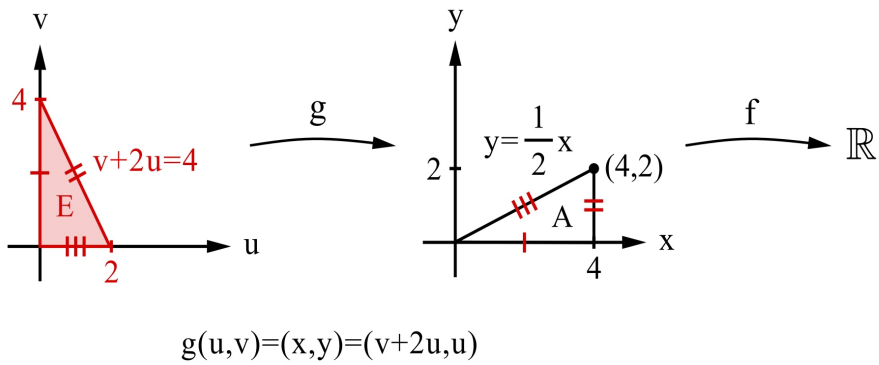


We have

$$\begin{aligned}\int_A f(x,y)\,d(x,y) &= \int_{\mathbf{g}(E)} f(x,y)\,d(x,y) = \int_E f\big(\mathbf{g}(u,v)\big)\big|J_{\mathbf{g}}(u,v)\big|\,d(u,v)\\ &= \frac{1}{4}\int_E (u-v)\sqrt{v}\,du,v = \frac{1}{4}\int_0^4\int_0^u (u-v)\sqrt{v}\,dvdu\\ &= \frac{1}{4}\int_0^4 \frac{1}{15}u^{\frac{5}{2}}\,du = \frac{256}{105}\end{aligned}$$

Method 2: Let $u = y$ and $v = x-2y$. Then $x = v+2u$. Define $\mathbf{g}(u,v) = (v+2u, u)$. The Jacobian is $J_{\mathbf{g}}(u,v) = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1$.

Consider $0 \le v + 2u \le 4$ and $0 \le u \le \frac{1}{2}v + u$. This implies $u \ge 0$, $v \ge 0$ and $0 \le v + 2u \le 4$. The set E is the region in uv-plane which is bounded by u-axis, v-axis and $v = -2u + 4$. Then $\mathbf{g} : E \to A$ is bijective.

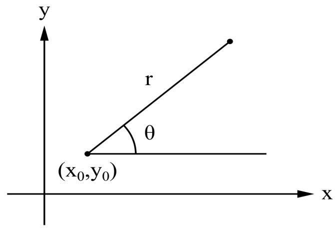


We have

$$\begin{aligned}\int_A f(x,y)\,d(x,y) &= \int_E u\sqrt{v}\,|-1|\,d(u,v) = \int_0^4\int_0^{2-\frac{1}{2}v} u\sqrt{v}\,dudv\\ &= \int_0^4 \sqrt{v}\Big(\int_0^{2-\frac{1}{2}v} u\,du\Big)dv\\ &= \int_0^4 \sqrt{v}\big(2 - v + \frac{1}{8}v^2\big)\,dv = \frac{256}{105}.\end{aligned}$$

Example 6.4.5. (Polar Coordinate)

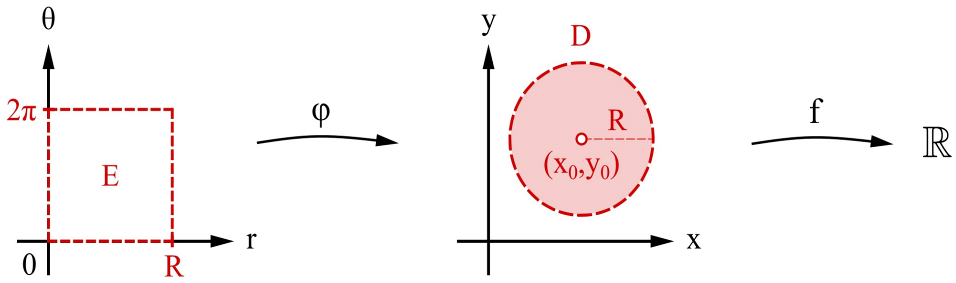
Let $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$.



Consider the function

$$\phi : \underbrace{(0,R)\times(0,2\pi)}_{E} \to \underbrace{\left\{(x,y)\ \middle|\ 0 < (x-x_0)^2 + (y-y_0)^2 < R^2\right\}\setminus\left\{(x,y_0)\ \middle|\ x_0 < x < x_0 + R\right\}}_{D}$$

where $\phi(r,\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$ is bijective from E to D.

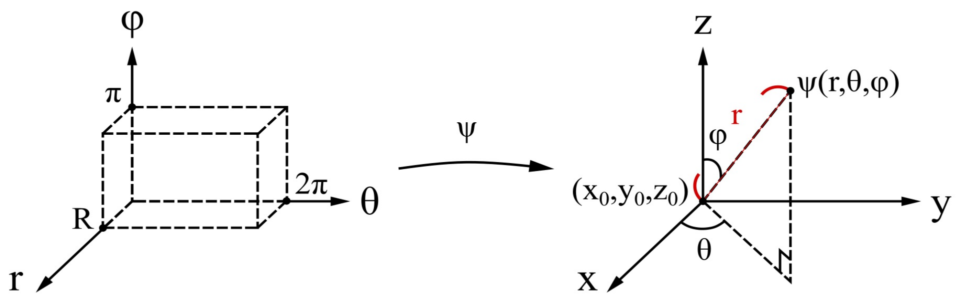


The Jacobian is $J_\phi(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. For $f : D \to \mathbb{R}$, we have

$$\begin{aligned}
\int_D f(x,y)\,d(x,y) &= \int_{\phi(E)} f(x,y)\,d(x,y) \\
&= \int_E f\big(\phi(r,\theta)\big)\left|J_\phi(r,\theta)\right|\,d(r,\theta) \\
&= \int_0^R\int_0^{2\pi} f\big(\phi(r,\theta)\big)r\,d\theta dr \\
&= \int_0^R\int_0^{2\pi} f(x_0 + r\cos\theta, y_0 + r\sin\theta)r\,d\theta dr.
\end{aligned}$$

Example 6.4.6. (Spherical Coordinate) Define $\psi(r,\theta,\phi) = (x_0 + r\cos\theta\sin\phi, y_0 + r\sin\theta\sin\phi, z_0 + r\cos\phi)$. Let $D = (0,R)\times(0,2\pi)\times(0,\pi)$ be a rectangle in (r,θ,ϕ)-space. Then ψ is a bijective from D to $B\big((x_0,y_0,z_0),R\big)$ a ball in $\mathbb{R}^3$. The Jacobian is

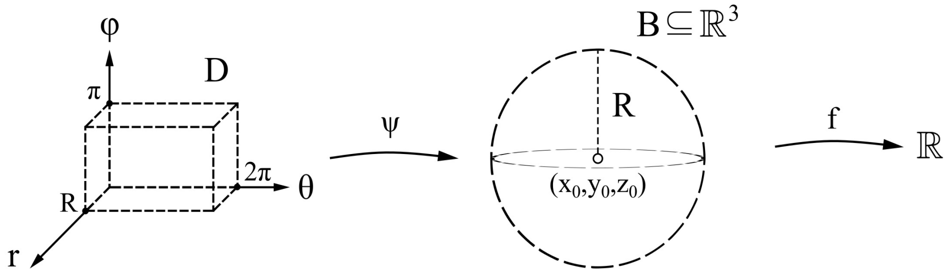
$$J_\psi(r,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & 0 & -r\sin\phi \end{vmatrix} = -r^2\sin\phi.$$



ψ(r,θ,φ)=(x0+rcosθsinφ, y0+rsinθcosφ, z0+rcosφ)

Let $f : B \to \mathbb{R}$ be Riemann integrable. Then

$$\begin{aligned}\int_B f(x,y,z)\, d(x,y,z) &= \int_D f\big(\psi(r,\theta,\phi)\big)\big|J_\psi(r,\theta,\phi)\big|\, d(r,\theta,\phi)\\ &= \int_0^R \int_0^{2\pi} \int_0^{\pi} f\big(x_0 r\cos\theta\sin\theta, y_0 + r\sin\theta\sin\phi, z_0 + r\cos\phi\big) r^2 \sin\phi \; d\phi d\theta dr\end{aligned}$$



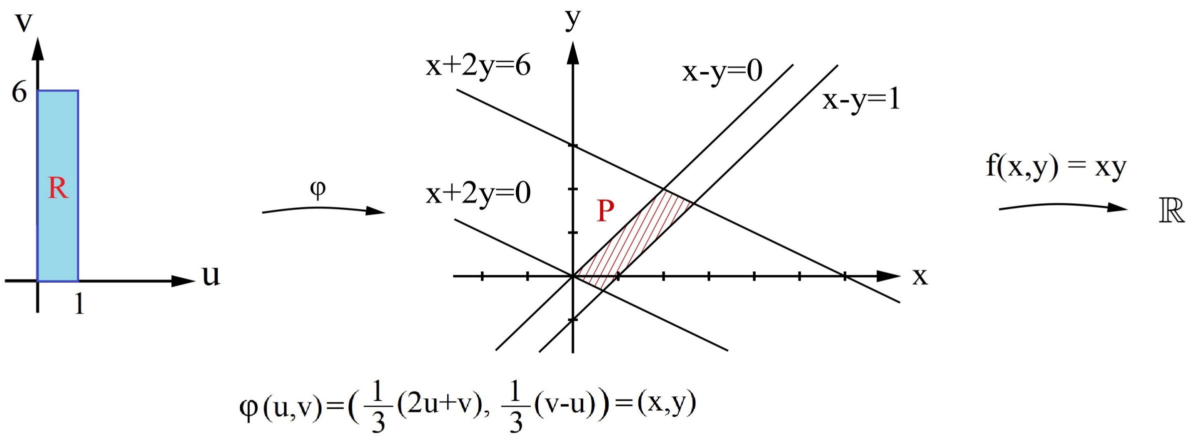
Example 6.4.7. Let P be the region bounded by $x-y=0$, $x+2y=0$, $x-y=1$ and $x+2y=6$. Find $\displaystyle\int_P xy\, d\mathbb{A}$.

Proof. Let $u = x-y$ and $v = x+2y$. Then $x = \frac{1}{3}(2u+v)$ and $y = \frac{1}{3}(v-u)$. Define $\phi(u,v) = \left(\frac{1}{3}(2u+v), \frac{1}{3}(v-u)\right)$ and $R = [0,1]\times[0,6]$. Then $\phi : R \to P$ is one-to-one and onto and the Jacobian is

$$J_\phi(u,v) = \begin{vmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

We have

$$\begin{aligned}\int_P xy\, d(x,y) &= \int_R \frac{1}{3}(2u+v)\cdot\frac{1}{3}(v-u)\cdot\frac{1}{3}\, d(u,v)\\ &= \frac{1}{27}\int_0^1\int_0^6 2-u^2+uv+v^2\, dvdu\\ &= \frac{77}{27}.\end{aligned}$$



□

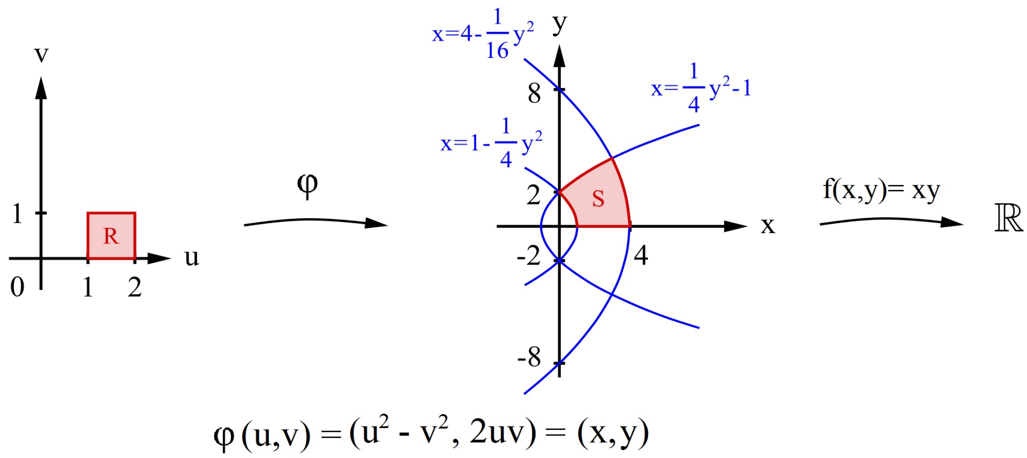
Example 6.4.8. Let S be the region bounded by x-axis, $x = 1-\frac{1}{4}y^2$, $x = \frac{1}{4}y^2-1$ and $x = 4-\frac{1}{16}y^2$. Find $\int_S xy\, d(x,y)$.

Proof. Let $\phi(u,v) = (u^2 - v^2, 2uv) = (x,y)$ and $R = \{(u,v) \mid 1 \le u \le 2,\ 0 \le v \le 1\}$. The Jacobian is

$$J_\phi(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2).$$

We have

$$\begin{aligned} \int_S xy\, d\mathbb{A}(x,y) &= \int_{[1,2]\times[0,1]} (u^2-v^2)\cdot 2uv \cdot 4(u^2+v^2)\, d\mathbb{A}(u,v) \\ &= 8\int_1^2 \int_0^1 uv(u^2-v^2)\, dvdu \\ &= 40. \end{aligned}$$



□

6.5 Improper Integrals

Recall: Let $f : [a, b] \to \mathbb{R}$ be bounded. We can define $\displaystyle\int_a^b f(x)\,dx$.

Question: How about the domain is unbounded or f is unbounded?

(1) Let $f : \mathbb{R} \to \mathbb{R}$ (or $f : (a, \infty) \to \mathbb{R}$ or $f : (-\infty, b) \to \mathbb{R}$). Then we define

$$\int_{\mathbb{R}} f(x)\,dx = \lim_{\substack{s\to-\infty \\ t\to\infty}} \int_s^t f(x)\,dx.$$

$$\left(\text{or } \int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \text{ or } \int_{-\infty}^b f(x)\,dx = \lim_{s\to-\infty} \int_s^b f(x)\,dx\right).$$

(2) Let $f(a, b) \to \mathbb{R}$ and $\lim\limits_{x\to a^+} f(x) = \infty$. We define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

How about the improper multiple integrals?

For example, $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded (and continuous). What is $\displaystyle\int_{\mathbb{R}^2} f(x, y)\,d\mathbb{A}$?

$$\int_{\mathbb{R}^2} f(x,y)\,d\mathbb{A} \overset{??}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \lim_{\substack{s_1,s_1\to-\infty \\ t_1,t_2\to\infty}} \int_{s_1}^{t_1}\int_{s_2}^{t_2} f(x,y)\,dxdy.$$

The values could be equal, but it is not the definition of $\int_{\mathbb{R}^2} f(x, y)\,d\mathbb{A}$.

Idea: $\displaystyle\int_{\mathbb{R}^2} f(\mathbf{x})\,d\mathbb{A} = \lim_{r\to\infty} \int_{D_r} f(\mathbf{x})\,d\mathbb{A}$ where D_r are a family of sets with volumes that fill out $\mathbb{R}^2$ as $r \to \infty$.

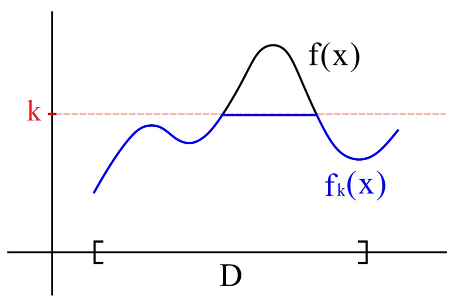
Difficulty: For different families D_r, the limit $\displaystyle\lim_{r\to\infty} \int_{D_r} f(\mathbf{x})\,d\mathbb{A}$ may not be equal.

In order to discuss the existence of the above limits and the integrals, we start with the assumption that all functions are nonnegative and all sets have volumes.

(1) Let $D \subseteq \mathbb{R}^n$ be bounded and $f : D \to \mathbb{R}$ be unbounded. Define $f_k : D \to \mathbb{R}$ by

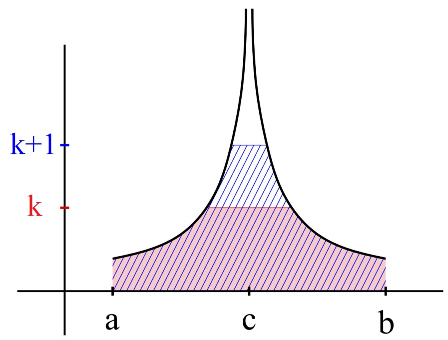
$$f_k(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) \le k \\ k & \text{otherwise} \end{cases} = \min\big(f(\mathbf{x}), k\big)$$

(f_k is usually denoted by $f \wedge k$)



Then f_k is bounded on D for each k and $f_{k+1}(\mathbf{x}) \geq f_k(\mathbf{x})$ for each $k \in \mathbb{N}$ and for every $\mathbf{x} \in D$. Suppose that each f_k is integrable over D. Then the sequence $\left\{ \int_D f_k(\mathbf{x})\, d\mathbf{x} \right\}_{k=1}^{\infty}$ is increasing and hence we can consider its limit and define the improper integral

$$\int_D f(\mathbf{x})\, d\mathbf{x} = \lim_{k\to\infty} \int_D f_k(\mathbf{x})\, d\mathbf{x}.$$



(2) Let $D \subseteq \mathbb{R}^n$ be unbounded, $f : D \to \mathbb{R}$. Let D be the union of an increasing sequence of sets $U_1, U_2, \cdots$ such that

$$D = \bigcup_{k=1}^{\infty} U_k \quad (U_1 \subseteq U_2 \cdots)$$

where each U_k has volume and f is integrable on each U_k. Then the sequence $\left\{ \int_{U_k} f(\mathbf{x})\, d\mathbf{x} \right\}_{k=1}^{\infty}$ is increasing. Hence the limit $\lim\limits_{k\to\infty} \int_{U_k} f(\mathbf{x})\, d\mathbf{x}$ exists, provided that we allow ∞ as a value.

Remark. Suppose that $\left\{ \tilde{U}_k \right\}_{k=1}^{\infty}$ is another sequence of sets satisfying the above conditions. Then

$$\lim_{k\to\infty} \int_{U_k} f(\mathbf{x})\, d\mathbf{x} = \lim_{k\to\infty} \int_{\tilde{U}_k} f(\mathbf{x})\, d\mathbf{x}.$$

Definition 6.5.1. Let $A \subseteq \mathbb{R}^n$ be a set with volume and $f : A \to \mathbb{R}$ be nonnegative. Let $\{B_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ be any sequence of bounded sets with volumes satisfying

(i) $B_k \subseteq B_{k+1}$ for every $k \in \mathbb{N}$

(ii) for every $R > 0$, $B(\mathbf{0}, R) \subseteq B_k$ when k is sufficiently large.

We define the integral of f over A by

$$\int_A f(\mathbf{x})\, d\mathbf{x} = \lim_{k\to\infty} \int_{A\cap B_k} \big(f \wedge k\big)(\mathbf{x})\, d\mathbf{x}$$

provided the limit exists (we allow ∞ as a limit) and where the limit is independint of the choice of the sequence $\{B_k\}_{k=1}^{\infty}$. We say that "*f is integrable over A*" if the integral converges. That is, $\int_A f(\mathbf{x})\, d\mathbf{x} < \infty$.

Note that we can use the same indices $B_k, f \wedge k$ since f is nonnegative. It may not be true if f is a general function.

Remark. Let $A \subseteq \mathbb{R}^n$ be a set with volume* and $f : A \to \mathbb{R}$ be nonnegative.

(1) If f is continuous on A or (at most) discontinuous on a volume zero subset B of A, then $\underbrace{f \text{ is integrable over } A}_{??wrong??}$.

(2) Suppose that f is integrable over A. To evaluate the improper integral $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x}$, one can choose $B_k = [-k,k] \times \cdots \times [-k,k]$ for convenience.

Example 6.5.2. Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$. Consider

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, d\mathbb{A} &= \lim_{k\to\infty} \int_{[-k,k]\times[-k,k]} e^{-(x^2+y^2)}\, d\mathbb{A} \overset{(Fubini)}{=} \lim_{k\to\infty} \int_{-k}^{k}\int_{-k}^{k} e^{-(x^2+y^2)}\, dxdy \\
&= \lim_{k\to\infty} \int_{-k}^{k}\int_{-k}^{k} e^{-x^2}\cdot e^{-y^2}\, dxdy = \lim_{k\to\infty} \Big[\Big(\int_{-k}^{k} e^{-x^2}\, dx\Big)\Big(\int_{-k}^{k} e^{-y^2}\, dy\Big)\Big] \\
&= \Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}\, dy\Big) = \Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Big)^2.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, d\mathbb{A} = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\cdot r\, d\theta dr = \pi \quad (x = r\cos\theta,\ y = r\sin\theta),$$

$\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$

Example 6.5.3. Let $E_1 = \big\{\mathbf{x} \in \mathbb{R}^n \;\big|\; 0 \le \|\mathbf{x}\|_{\mathbb{R}^n} \le 1\big\}$ and $E_2 = \big\{\mathbf{x} \in \mathbb{R}^n \;\big|\; \|\mathbf{x}\|_{\mathbb{R}^n} \ge 1\big\}$. For $p \in \mathbb{R}$, find the range of p such that

(1) $\displaystyle\int_{E_1} \|\mathbf{x}\|_{\mathbb{R}^n}^{-p}\, d\mathbf{x}$ converges

(2) $\displaystyle\int_{E_2} \|\mathbf{x}\|_{\mathbb{R}^n}^{-p}\, d\mathbf{x}$ converges.

(Exercise)

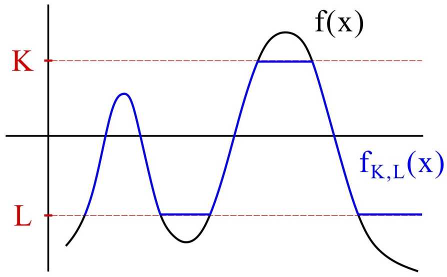
■ **Nonnegative functions**

Question: What about the improper integral of functions that are not nonnegative? Let $A \subseteq \mathbb{R}^n$

*If A is unbounded or A has volume ∞, then A is suppose to be replaced by $A \cap B(0,R)$ which has volume.

be a set with volume and $f : A \to \mathbb{R}$. For $L \le K$ and define

$$f_{K,L}(\mathbf{x}) = \begin{cases} K & \text{if } f(\mathbf{x}) \ge k \\ f(\mathbf{x}) & \text{if } L \le f(\mathbf{x}) \le K \\ L & \text{if } f(\mathbf{x}) \le L \end{cases}$$



Let A be the union of an increasing sequence of sets $U_1, U_2, \cdots$. That is,

$$A = \bigcup_{i=1} U_i \quad (U_1 \subseteq U_2 \subseteq \cdots)$$

where each U_i has volume and $f_{K,L}$ is integrable over U_i for every $L \le K$ and $i \in \mathbb{N}$. Consider $\int_{U_i} f_{K,L}(\mathbf{x})\, d\mathbf{x}$. If f is integrable over A, then

$$\left| \int_{U_i} f_{K,L}(\mathbf{x})\, d\mathbf{x} - \int_A f(\mathbf{x})\, d\mathbf{x} \right| \to 0 \quad \text{as } K, i \to \infty \text{ and } L \to -\infty$$

Definition 6.5.4. Let $A \subseteq \mathbb{R}^n$ be a set with volume and $f : A \to \mathbb{R}$ be a function. We say that "*f is integrable over A*" if for each sequence $\{B_i\}_{i=1}^{\infty} \subseteq \mathbb{R}^n$of bounded sets with volume satisfying

(1) $B_i \subseteq B_{i+1}$ for every $i \in \mathbb{N}$

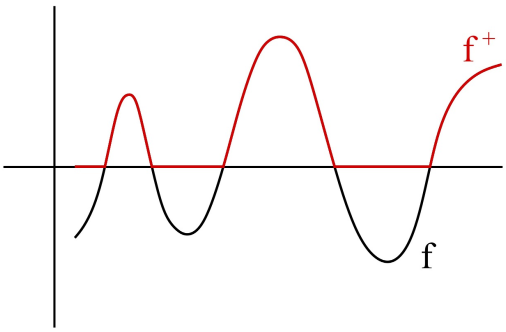
(2) for every $R > 0$, $B(\mathbf{0}, R) \subseteq B_i$ when i is sufficiently large then the limit

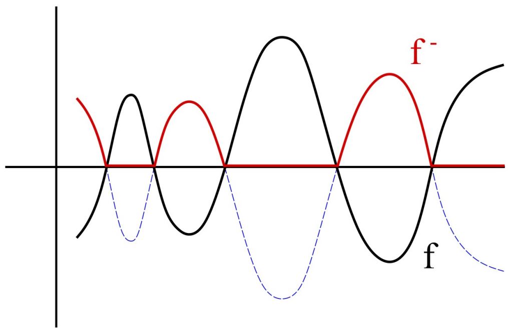
$$\lim_{\substack{i\to\infty \\ K\to\infty \\ L\to-\infty}} \int_{A\cap B_i} f_{K,L}(\mathbf{x})\, d\mathbf{x}$$

exists.

(**Another viewpoint for nonnegative functions**) Let $f : A \to \mathbb{R}$. Define $f^+, f^- : A \to \mathbb{R}$ by

$$f^+(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) \ge 0 \\ 0 & \text{if } f(\mathbf{x}) \le 0 \end{cases} \quad \text{and} \quad f^-(\mathbf{x}) = \begin{cases} 0 & \text{if } f(\mathbf{x}) \ge 0 \\ -f(\mathbf{x}) & \text{if } f(\mathbf{x}) \le 0 \end{cases}$$





Remark. (1) f^+, f^- are nonnegative.

(2) $f^+(\mathbf{x}) = \max\{f(\mathbf{x}), 0\}$ and $f^-(\mathbf{x}) = \max\{-f(\mathbf{x}), 0\}$.

(3) $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

(4) If f is continuous, then so are f^+ and f^-.

Definition 6.5.5. Let $A \subseteq \mathbb{R}^n$ be a set with volume and $f : A \to \mathbb{R}$ be integrable over A. The improper integral $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x}$ is said to be "*absolutely convergent*" if $\displaystyle\int_A |f(\mathbf{x})|\, d\mathbf{x}$ converges.

Lemma 6.5.6. $\displaystyle\int_A f(\mathbf{x})\, d\mathbf{x}$ *absolutely converges if and only if f^+ and f^- are integrable over A.* *(That is,* $\displaystyle\int_A f^+(\mathbf{x})\, d\mathbf{x} < \infty$ *and* $\displaystyle\int_a f^-(\mathbf{x})\, d\mathbf{x} < \infty$ *.)*

Theorem 6.5.7. *(Comparison Test) Let $A \subseteq \mathbb{R}^n$ be a set with volume and $f, g : A \to \mathbb{R}$ be continuous (except possibly on a volume zero set). If $|f| \le g$ on A and g is integrable over A, then f is integrable over A.*

Proof. Since $|f| = f^+ + f^- \le g$ on A and f^+ and f^- are nonnegative, $0 \le f^+(\mathbf{x}), f^-(\mathbf{x}) \le g(\mathbf{x})$ for every $\mathbf{x} \in A$.

For every $k \in \mathbb{N}$ and $D_k = [-k, k] \times \cdots \times [-k, k]$,

$$\int_{A\cap D_k} \big(f^+ \wedge k\big)(\mathbf{x})\, d\mathbf{x} \le \int_{A\cap D_k} g(\mathbf{x})\, d\mathbf{x} \le \underbrace{\int_A g(\mathbf{x})\, d\mathbf{x} < \infty}_{g \text{ is integrable over } A} .$$

Since $\displaystyle\int_{A\cap D_k} \big(f^+ \wedge k\big)(\mathbf{x})\, d\mathbf{x}$ is increasing in k and bounded above, $\displaystyle\lim_{k\to\infty} \int_{A\cap D_k} \big(f^+ \wedge k\big)(\mathbf{x})\, d\mathbf{x}$ converges. Hence, f^+ is integrable over A.

Similarly, f^- is integrable over A and then f is integrable over A. □

Example 6.5.8. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \dfrac{\sin x}{x^2 + 1}$. Then $|f(x)| \le \dfrac{1}{x^2 + 1}$. Since $\displaystyle\int_0^\infty \frac{1}{x^2 + 1}\, dx$ converges, by the comparison test, f is integrable over $[0, \infty)$.

Question: Are "Fubini's Theorem" and "change of variables" still true for improper integrals?

6.6 Fubini's Theorem and Tonelli's Theorem

In order to prove the Fubini's Theorem for improper integral, we will introduce the "Monotone Convergence Theorem".

Theorem 6.6.1. *(Monotone Convergence Theorem) Let $A \subseteq \mathbb{R}^n$ be a set with volume, $f : A \to \mathbb{R}$ be a continuous function (except possibly on a volume zero set) and $f_n : A \to \mathbb{R}$ be integrable functions such that*

(i) $f_n \geq f_{n+1}$ *(or* $f_n \leq f_{n+1}$*) for every* $n \in \mathbb{N}$.

(ii) f_n *converges pointwise to* f

Then

$$\int_A f(\mathbf{x})\, d\mathbf{x} = \lim_{n\to\infty} \int_A f_n(\mathbf{x})\, d\mathbf{x}.$$

Theorem 6.6.2. *(Fubini) Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be sets with volumes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $f : A \times B \to \mathbb{R}$ be a function such that $f(\mathbf{x}, \cdot)$ is integrable over B for every $\mathbf{x} \in A$ and $f(\cdot, \mathbf{y})$ is integrable over A for every $\mathbf{y} \in B$. If f is absolutely integrable over $A \times B$, then*

$$\int_{A\times B} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \int_A \Big(\int_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} = \int_B \Big(\int_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \Big) d\mathbf{y}$$

Proof. (Sketch) Since $f(\mathbf{x}, \cdot)$ is integrable over B for every $\mathbf{x} \in A$, $f^+(\mathbf{x}, \cdot)$ and $f^-(\mathbf{x}, \cdot)$ are integrable over B for every $\mathbf{x} \in A$.

By the Fubini's Theorem, for $f^+ \wedge k$ and $f^- \wedge k$ on $D_k = [-k, k] \times \cdots \times [-k, k] = [-k, k]^{n+m}$,

$$\int_{A\cap[-k,k]^n} \Big(\int_{B\cap[-k,k]^m} \big(f^+ \wedge k\big)(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} = \int_{A\times B\cap[-k,k]^{n+m}} \big(f^+ \wedge k\big)(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}). \tag{6.4}$$

Since $f^+(\mathbf{x}, \cdot)$ is integrable over B for every $\mathbf{x} \in A$ and f^+ is integrable over $A \times B$,

$$\int_B f^+(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = \lim_{k\to\infty} \int_{B\cap[-k,k]^m} \big(f^+ \wedge k\big)(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \quad \text{for every } \mathbf{x} \in A$$

and

$$\int_{A\times B} f^+(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \lim_{k\to\infty} \int_{A\times B\cap[-k,k]^{n+m}} \big(f^+ \wedge k\big)(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y})$$

Moreover,

$$\int_A \Big(\int_B f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x} = \lim_{k\to\infty} \int_{A\cap[-k,k]^n} \Big(\int_{B\cap[-k,k]^m} \big(f^+ \wedge k\big)(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big) d\mathbf{x}$$

since f^+ is nonnegative. (Check!) Then the theorem is proved. □

(**Proof of the Fubini's Theorem (Improper Integral)**)

Since $f(\mathbf{x}, \cdot)$ is integrable over B for every $\mathbf{x} \in A$, $f^+(\mathbf{x}, \cdot)$ and $f^-(\mathbf{x}, \cdot)$ are integrable over B for every $\mathbf{x} \in A$. Then

$$\big(f^+ \wedge k\big)(\mathbf{x}, \cdot) \nearrow f^+(\mathbf{x}, \cdot) \quad \text{and} \quad \big(f^- \wedge k\big)(\mathbf{x}, \cdot) \nearrow f^-(\mathbf{x}, \cdot)$$

for every $\mathbf{x} \in A$ and as $k \to \infty$. Therefore,

$$\int_{B\cap[-k,k]^m} \big(f^+ \wedge k\big)(\mathbf{x},\mathbf{y})\, d\mathbf{y} = \int_B \big(f^+ \wedge k\big)(\mathbf{x},\mathbf{y})\mathbb{1}_{[-k,k]^m}(\mathbf{y})\, d\mathbf{y} \overset{(M.C.T)}{\nearrow} \int_B f^+(\mathbf{x},\mathbf{y})\, d\mathbf{y}.$$

Also,

$$\int_{B\cap[-k,k]^m} \big(f^- \wedge k\big)(\mathbf{x},\mathbf{y})\, d\mathbf{y} \nearrow \int_B f^-(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad \text{as } k \to \infty.$$

Define $f_k^+(\mathbf{x},\mathbf{y}) := \Big(\big(f^+ \wedge k\big)\,\mathbb{1}_{[-k,k]^n\times[-k,k]^m}\Big)(\mathbf{x},\mathbf{y})$. Then $f_k^+ \nearrow f^+$. Hence,

$$\begin{aligned}
\int_{A\times B} f^+(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) &\overset{M.C.T}{=} \lim_{k\to\infty} \int_{A\times B} f_k^+(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) \\
&= \lim_{k\to\infty} \int_{(A\times B)\cap([-k,k]^n\times[-k,k]^m)} \big(f^+ \wedge k\big)(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) \\
&\overset{(Fubini)}{=} \lim_{k\to\infty} \int_{A\cap[-k,k]^n} \Big(\int_{B\cap[-k,k]^m} \big(f^+ \wedge k\big)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x} \\
&= \lim_{k\to\infty} \int_A \underbrace{\Big(\int_B \big(f^+ \wedge k\big)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big)\mathbb{1}_{[-k,k]^n}(\mathbf{x})}_{\text{increasing to } \int_B f^+(\mathbf{x},\mathbf{y})\, d\mathbf{y}}\, d\mathbf{x} \\
&\overset{M.C.T}{=} \int_A \Big(\int_B f^+(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x}.
\end{aligned}$$

Similarly, $\displaystyle\int_{A\times B} f^-(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) = \int_A \Big(\int_B f^-(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x}$. Therefore,

$$\begin{aligned}
\int_{A\times B} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) &= \int_{A\times B} f^+(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) - \int_{A\times B} f^-(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) \\
&= \int_A \Big(\int_B f^+(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x} - \int_A \Big(\int_B f^-(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x} \\
&= \int_A \Big(\int_B f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\Big) d\mathbf{x}.
\end{aligned}$$

Remark. To apply Fubini's Theorem, the integrability of f is a necessary condition. That is,

$$\int_{A\times B} \big|f(\mathbf{x},\mathbf{y})\big|\, d(\mathbf{x},\mathbf{y}) < \infty.$$

Counterexample Let $R = [0,1]\times[0,1]$ and define

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \in R\backslash(0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let $A(x) = \int_0^1 f(x,y)\, dy$. For $x \neq 0$, set $u = x^2 + y^2$,

$$A(x) = \int_{x^2}^{x^2+1} \frac{x(2x^2-u)}{2u^2}\, du = \frac{x}{2(x^2+1)^2}.$$

Note that this formula is true for $x = 0$. Then

$$\int_0^1 \int_0^1 f(x,y)\, dydx = \int_0^2 A(x)\, dx = \frac{1}{8}.$$

On the other hand, let $B(y) = \int_0^1 f(x,y)\, dx$. For $y \neq 0$, set $u = x^2+y^2$, we have $B(y) = \dfrac{-y}{2(y^2+1)^2}$. This formula remains valid for $y = 0$.

$$\int_0^1 \int_0^1 f(x,y)\, dxdy = \int_0^1 B(y)\, dy = -\frac{1}{8}.$$

Therefore, $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dxdy \neq \int_0^1 \int_0^1 f(x,y)\, dydx.$

Question: What happened here?
Answer: The function f is not (absolutely) integrable over R. In fact, f has a bad discontinunity at $(0,0)$.

$$\int_R \big|f(x,y)\big|\, d(x,y) = 2\int_0^1 \int_0^x \frac{xy(x^2-y^2)}{(x^2+y^2)^3} = \int_0^1 \frac{1}{8x}\, dx = \infty.$$

Theorem 6.6.3. *(Change of Variables) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set with volume, and $\mathbf{g} : \mathcal{U} \to \mathbb{R}^n$ be an one-to-one C^1 mapping with C^1 inverse (that is, $\mathbf{g}^{-1} : g(\mathcal{U}) \to \mathcal{U}$ is also continuously differentiable). Suppose that the Jacobian of g, $J_{\mathbf{g}}(\mathbf{x})$, does not vanish in $\mathcal{U}$ and f is absolutely integrable over $\mathbf{g}(\mathcal{U})$. Then $(f \circ \mathbf{g})J_{\mathbf{g}}$ is absolutely integrable over $\mathcal{U}$ and*

$$\int_{\mathbf{g}(\mathcal{U})} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathcal{U}} \big(f \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} = \int_{\mathcal{U}} \big(f \circ \mathbf{g}\big)\left|\frac{\partial(g_1,\cdots g_n)}{\partial(x_1,\cdots,x_n)}\right| d\mathbf{x}.$$

Proof. Let $\{\mathcal{U}_k\}_{k=1}^{\infty}$ be a sequence of bounded open sets with volumes such that

$$(i)\ \mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}_k \quad (ii)\ \mathcal{U}_k \subset\subset \mathcal{U} \quad (iii)\ \mathcal{U}_k \subseteq \mathcal{U}_{k+1} \text{ for every } k \in \mathbb{N}.$$

Define $f_k^+ = f^+ \wedge k$ and $f_k^- = f^- \wedge k$. By the change of variables formula for bounded sets and bounded fucntions,

$$\int_{\mathcal{U}_k} \big(f_k^\pm \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} = \int_{\mathbf{g}(\mathcal{U}_k)} f_k^\pm(\mathbf{y})\, d\mathbf{y}.$$

Since $f_k^\pm\big(\mathbf{g}(\mathbf{x})\big)\mathbb{1}_{\mathcal{U}_k}(\mathbf{x}) \nearrow f^\pm\big(\mathbf{g}(\mathbf{x})\big)\mathbb{1}_{\mathcal{U}}(\mathbf{x})$ and $f_k^\pm(\mathbf{y})\mathbb{1}_{\mathbf{g}(\mathcal{U})}(\mathbf{y}) \nearrow f^\pm(\mathbf{y})\mathbb{1}_{\mathbf{g}(\mathcal{U})}(\mathbf{y})$ as well as

$$\int_{\mathcal{U}_k} \big(f_k^\pm \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} = \int_{\mathbf{g}(\mathcal{U}_k)} f_k^\pm(\mathbf{y}) \leq \int_{\mathbf{g}(\mathcal{U})} f^\pm(\mathbf{y})\, d\mathbf{y} < \infty.$$

(Hence, $\big(f_k^\pm \circ \mathbf{g}\big)(\mathbf{x})\mathbb{1}_{\mathcal{U}_k}(\mathbf{x})$ is integrable over $\mathcal{U}$.) By the monotone convergence theorem,

$$\begin{aligned}
\int_{\mathbf{g}(U)} f^\pm(\mathbf{y})\, d\mathbf{y} &\overset{M.C.T}{=} \lim_{k\to\infty} \int_{\mathbf{g}(\mathcal{U})} f_k^\pm(\mathbf{y})\mathbb{1}_{\mathbf{g}(\mathcal{U})}(\mathbf{y})\, d\mathbf{y} \\
&\overset{C.O.V}{=} \int_{\mathcal{U}} \big(f_k^\pm \circ \mathbf{g}\big)(\mathbf{x})\mathbb{1}_{\mathcal{U}_k}(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} \\
&\overset{M.C.T}{=} \int_{\mathcal{U}} \big(f^\pm \circ \mathbf{g}\big)(\mathbf{x})\, \big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x}.
\end{aligned}$$

Hence,

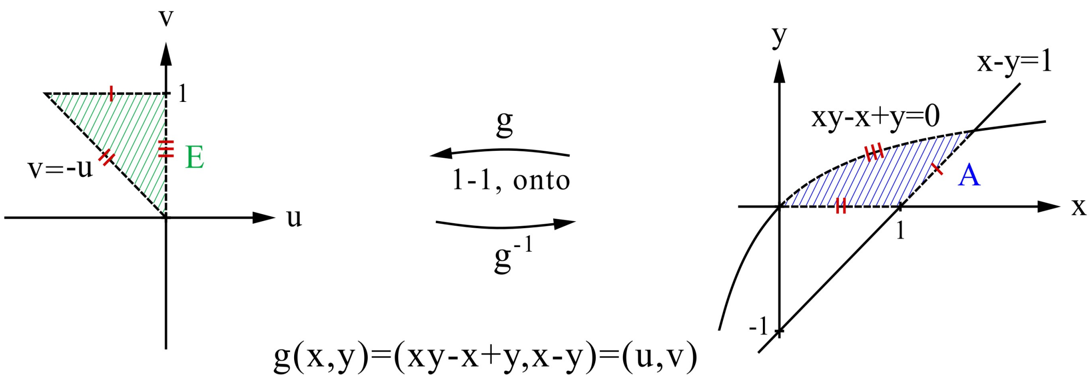
$$\begin{aligned}\int_{\mathbf{g}(\mathcal{U})} f(\mathbf{y})\, d\mathbf{y} &= \int_{\mathbf{g}(\mathcal{U})} f^+(\mathbf{y})\, d\mathbf{y} - \int_{\mathbf{g}(\mathcal{U})} f^-(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathcal{U}} \big(f^+ \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} - \int_{\mathcal{U}} \big(f^- \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x} \\ &= \int_{\mathcal{U}} \big(f \circ \mathbf{g}\big)(\mathbf{x})\big|J_{\mathbf{g}}(\mathbf{x})\big|\, d\mathbf{x}.\end{aligned}$$

□

Example 6.6.4. Let A be the region in the first quadrant and bounded by $xy - x + y = 0$ and $x - y = 1$ and $f(x, y) = x^2y^2(x + y)e^{-(x-y)^2}$. Find $\displaystyle\int_A f(x, y)\, d(x, y)$.

Proof. Let $g(x, y) = (u, v) = (xy - x + y, x - y)$ and $E = \big\{(u, v) \in \mathbb{R}^2 \;\big|\; 0 < v < 1,\ -v < u < 0\big\}$. Then the map $g : A \to E$ is one-to-one and onto (hence $g^{-1} : E \to A$ is one-to-one and onto).

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} y - 1 & x + 1 \\ 1 & -1 \end{vmatrix} = -(x + y).$$



For $(x, y) \in Int(A)$, $\dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$ and $J_{g^{-1}}(x, y) = \dfrac{\partial(x, y)}{\partial(u, v)} = -\dfrac{1}{x + y}$. We have

$$\begin{aligned}\int_A f(x, y)\, d(x, y) &= \int_E (u + v)^2 \cancel{(x + y)} e^{-v^2} \left|\frac{1}{\cancel{x + y}}\right| d(u, v) \\ &= \int_0^1 \int_{-v}^0 (u + v)^2 e^{-v^2}\, dudv = -\frac{1}{6}\left(\frac{2}{e} - 1\right)\end{aligned}$$

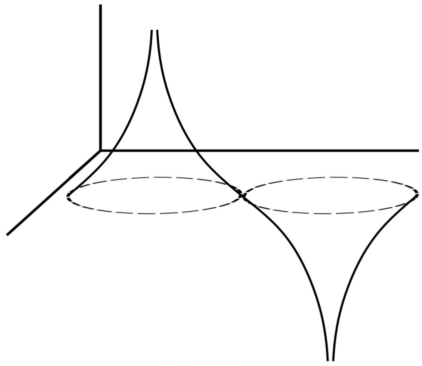
□

Remark. $J_g(x, y) \to \infty$ as $(x, y) \to (0, 0)$. Hence, there exists no open set $\mathcal{U} \subseteq \mathbb{R}^2$ such that $A \subset\subset \mathcal{U}$ and g is of class C^1 in $\mathcal{U}$.

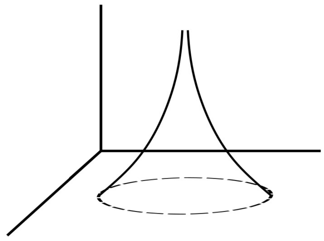
Remark. The lecture note also introduces another theorem, called "*Tornelli's Theorem*", which involves the identity of multiple integrals and iterated integrals. In our course, we skip the Tornelli's theorem and students can take advacecd course to learn it.

The main difference between Fubini's Thereom and Tornelli's Theorem is:

- The Fubini's Theorem needs that $f(\mathbf{x})$ is absolutely integrable. Hence, the integral $\int_D f(\mathbf{x})\, d\mathbf{x}$ must be a real value.
- The Tornelli's Theorem needs that $f(\mathbf{x})$ is nonnegative (nonpositive). Hence, the integral $\int_D f(\mathbf{x})\, d\mathbf{x}$ could be $\pm\infty$.



Fubini
(absolutely integrable)



Tornelli
(positive)

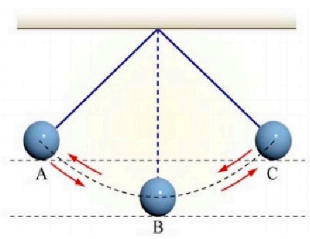
Chapter 7

Fourier Series*

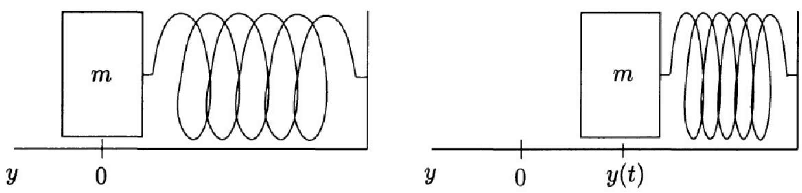
7.1 Physical Examples

❏ Simple Harmonic Motion

Simple harmonic motion describes the behavior of the most basic oscillatory system and is a natural place to start the study of vibrations. For example, simple pendulum, horizaontal spring.



Simple pendulum



Horizontal spring

Simple harmonic oscillator

*The content of this chapter is referred to *Fourier Analysis; E. Stein, R. Shakarchi*.

Consider the horizontal spring and let $y(t)$ denote the displacement of the mass at time t. Applying Newton's law, we have

$$-ky(t) = my''(t),$$

where $k > 0$ is a given physical quantity called the spring constant and m is the mass. Let $c = \sqrt{k/m}$. Then the equation becomes

$$y''(t) + c^2 y(t) = 0.$$

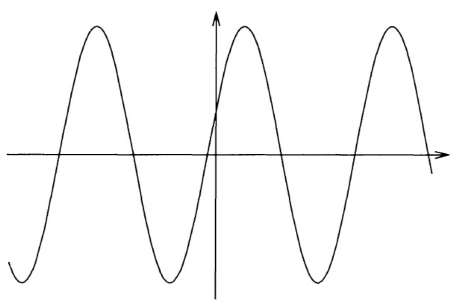
The equation can be solved by

$$y(t) = y(0)\cos ct + \frac{y'(0)}{c}\sin ct.$$

Consider

$$a\cos ct + b\sin ct = A\cos(ct - \phi)$$

where $A = \sqrt{a^2 + b^2}$ is called "amplitude" of the motion, c is its "natural frequency", ϕ is its "phase", and $2\pi/c$ is the "period" of the motion.



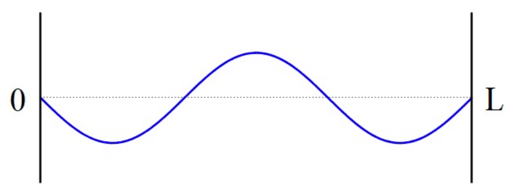
The graph of $A\cos(ct - \varphi)$

❑ Standing and Traveling Waves

■ Wave Equation

$$u_{tt} - c^2 u_{xx} = 0$$

where $c = \sqrt{\tau/\rho} > 0$ is the velocity of the spring, τ is the tension of the spring, and ρ is the density of the spring.



By changing of "units" in space, $x \to ax$, the spatial scale becomes $0 \le x \le L \;\to\; 0 \le x \le \frac{L}{a}$. Let $v(t, x) = u(t, ax)$, then

$$v_{tt} - \frac{c^2}{a^2} v_{xx} = 0.$$

Similarly, we also change the unit in time, $t \to bt$, the temporal scale becomes $0 \le t \le T \;\to\; 0 \le t \le \frac{T}{b}$. Let $v(t, x) = u(bt, x)$.

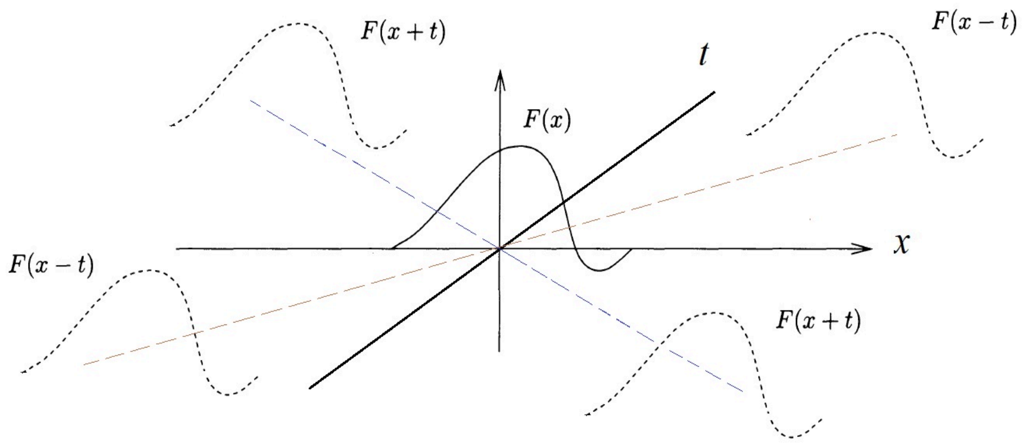
$$v_{tt} - b^2 c^2 v_{xx} = 0.$$

Hence, by choosing appropriate constants $a, b > 0$ such that $x \to ax$ and $t \to bt$, we may assume that the wave equation is

$$u_{tt} - u_{xx} = 0 \quad \text{on } 0 \le x \le \pi,\ t \ge 0.$$

• **Traveling Wave**

Observe that if F is any twice differentiable function, then $u(x,t) = F(x+t)$ and $u(x,t) = F(x-t)$ solve the wave equation. The speed of $u(x,t) = F(x-t)$ is 1 and more forward to the right.



Waves traveling in both directions

Since $u_{tt} - u_{xx} = 0$ is linear, for every $F, G \in C^2(\mathbb{R})$,

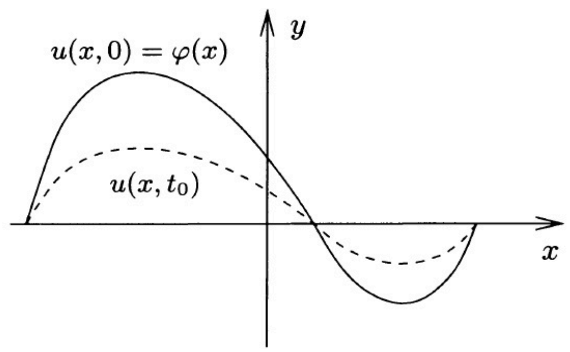
$$u(t,x) = F(x+t) + G(x-t)$$

is a solution. For given initial data, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$, the d'Alembert's formula gives

$$u(t,x) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy.$$

• **Superposition of standing waves**

First of all, we try to look for special solutions to the wave equation which are of the form $u(x,t) = \phi(x)\psi(t)$. In mathematics, this procedure is also called "*separation of variables*" and constructs solutions that are called "*pure tones*"(純音).



A standing wave at different moments in time: $t = 0$ and $t = t_0$

Then by the linearity of the wave equation, we can expect to combine these pure tones into a more complex combination of sound.

Note that the method of separation of variables gives rise to reduce the PDE problem to an ODE problem. Plugging $\phi(x)\psi(t)$ into the wave equation, we have

$$\phi(x)\psi''(t) = \phi''(x)\psi(t)$$

Thus,

$$\frac{\psi''(t)}{\psi(t)} = \frac{\phi''(x)}{\phi(x)} = \lambda$$

Note that λ is a constant. The wave equation reduece to

$$\begin{cases} \psi''(t) - \lambda\psi(t) = 0 \\ \phi''(x) - \lambda\phi(x) = 0 \end{cases}$$

If the constant $\lambda \geq 0$, the solution ϕ will not oscillate as time varies. Hence, we assume $\lambda = -m^2 < 0$. Then we can solve

$$\psi(t) = A\cos mt + B\sin mt$$

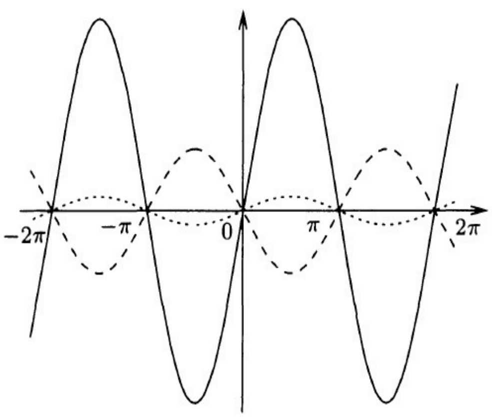
and

$$\phi(x) = \tilde{A}\cos mx + \tilde{B}\sin mx.$$

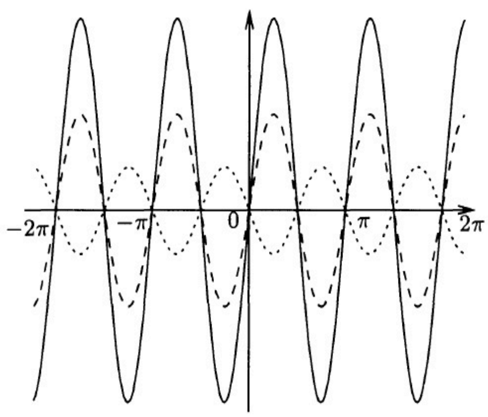
We take into account that the string is attached at $x = 0$ and $x = \pi$. The boundary condition gives $\phi(0) = \phi(\pi) = 0$. Hence, $\tilde{A} = 0$, and if $\tilde{B} \neq 0$ then $m \in \mathbb{Z}$. Moreover, we can absorb the cases $m \leq 0$ into the cases $m \geq 0$ and reduce the solution to

$$u_m(t,x) = \big(A_m\cos mt + B_m\sin mt\big)\sin mx$$

which is of the form of standing wave.†



(a) Fundamental tone or first harmonic of the vibrating string (m=1)



(b) First overtone or second harmonic (m=2)

†The readers could browse some websites listed below to figure out the overtone.
https://phet.colorado.edu/sims/html/wave-on-a-string/latest/wave-on-a-string_zh_TW.html
https://www.youtube.com/watch?v=0iJmDhNocaQ

Since the wave equation is linear, we can construct more solutions by taking linear combinations of the standing waves u_m. This technique is called "*superposition*" and gives the solution of the wave equation

$$u(t,x) = \sum_{m=1}^{\infty} \big(A_m \cos mt + B_m \sin mt\big) \sin mx.$$

Suppose that the initial data is given. That is, $u(x,0) = f(x)$ for $f(0) = f(\pi) = 0$. Then

$$\sum_{m=1}^{\infty} A_m \sin mx = f(x).$$

Question: Given $f(x)$ on $[0,\pi]$ with $f(0) = f(\pi) = 0$, can we find coefficients A_m such that

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \; ?$$

Question: If yes, how to find A_m?

Observe that

$$\int_0^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{2} & \text{if } m = n \end{cases}$$

Then, formally,

$$\begin{aligned} \int_0^{\pi} f(x) \sin nx \, dx &= \int_0^{\pi} \Big(\sum_{m=1}^{\infty} A_m \sin mx \Big) \sin nx \, dx \\ &= \sum_{m=1}^{\infty} A_m \int_0^{\pi} \sin mx \sin nx \, dx = A_n \cdot \frac{\pi}{2}. \end{aligned}$$

Hence,

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Question: How about the given initial data $F(x)$ is defined on $[-\pi,\pi]$?

We can express $F(x) = f(x) + g(x)$ where f is odd and g is even. Then $f(x)$ and $g(x)$ can be expressed as a sine series and a cosine series respectively. That is,

$$g(x) = \sum_{m=0}^{\infty} A'_m \cos mx.$$

Thus,

$$F(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=1}^{\infty} A'_m \cos mx + \frac{A'_0}{2} \tag{7.1}$$

Remark. (1) The constant $\frac{1}{2}$ in the last term is for making the formula consistant where

$$A'_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \, dx.$$

(2) When $F(x)$ is defined on $[-\pi, \pi]$ and is of the form (7.1), the formulas of the coefficients A_m and A'_m are similar but a slightly different.

$$\begin{aligned} A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin mx\,dx = \frac{1}{2\pi i}\int_{-\pi}^{\pi} F(x)\big(e^{imx} - e^{-imx}\big)\,dx \\ A'_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos mx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\big(e^{imx} + e^{-imx}\big)\,dx. \end{aligned}$$

Remark. Let $f(x)$ be a function defined on $[a, b]$ with $b - a = 2\pi$. Then we can extend $F(x)$ [still called $F(x)$] defined on $\mathbb{R}$ with period 2π. That is, $F(x) = F(x + 2\pi)$. Suppose that

$$F(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=1}^{\infty} A'_m \cos mx + \frac{A'_0}{2}$$

Then we can find the formulas of the coefficients by similar method.

$$\begin{aligned} A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin mx\,dx = \frac{1}{\pi}\int_{a}^{b} F(x)\sin mx\,dx \\ A'_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos mx\,dx = \frac{1}{\pi}\int_{a}^{b} F(x)\cos mx\,dx \end{aligned}$$

❑ Euler Identity

We recall the Euler identity $e^{it} = \cos t + i \sin t$. Suppose that we can express $F(x)$ as the form

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx} \quad \text{where } a_m \in \mathbb{C}.$$

Similarly, since

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 2\pi & \text{if } n = m \end{cases}$$

we have

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-inx}\,dx.$$

The quantity a_n is called the nth **Fourier coefficient** of F.

■ Heuristic Viewpoint[‡]

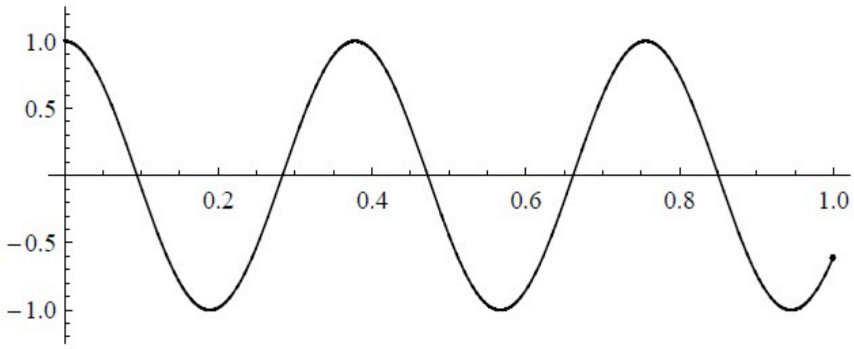
Consider the complex exponential function

$$e_m(x) = e^{2\pi imx} = \cos(2\pi mx) + i\sin(2\pi mx)$$

as a function of x. While x lies in $\mathbb{R}$, the function $e_m(x)$ are complex numbers that lie on the unit circle S^1 in $\mathbb{C}$. If $m > 0$, then as x increases through an interval of length $1/m$, the values $e_m(x)$ moves once around S^1 in the counter-clockwise direction.

[‡]The reference of this part is from Section1.1.2 of Introduction to Harmonic Analysis, Christopher Heil

The function e_m is periodic with period $1/m$ and we therefore say that it has "*frequency m*". In some sense, the function e_m is a "*pure tone*". We can imagine that an ideal vibrating string creates a pressure wave in the air. In general, a real string (wave) is much more complicated than a pure tone with frequency m. The sound created from a musical instrument usually consists of pure tones, overtones and other complications. But let's start with a single pure tone e_m here.

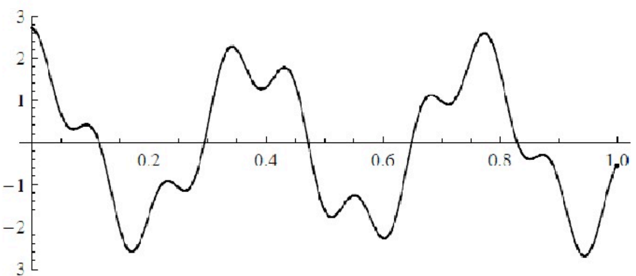


Graph of $\varphi(x) = \cos(2\pi\sqrt{7}x)$.

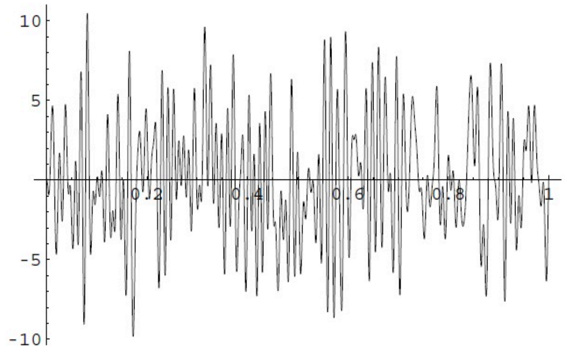
For a fixed m the function $a_m e^{2\pi imx}$ is a pure tone whose "*amplitude*" is the scalar a_m. The larger a_m is, the larger the vibrations of the string and the louder the perceived sound. With several different frequencies $m \in \mathbb{Z}$, the function

$$F(x) = \sum_{m=-N}^{N} a_m e^{2\pi imx}$$

is a superposition of several pure tones.



Graph of $\phi(x) = 2\cos(2\pi 3x) + 0.7\cos(2\pi 9x)$



Graph of 75 superimposed pure tones:

$$\phi(x) = \sum_{m=1}^{75} a_m \cos(2\pi mx)$$

Suppose that any function F can be represented as a series of pure tones $a_m e^{2\pi imx}$ over all possible frequencies $m \in \mathbb{Z}$. By superimposing all the pure tones with the correct amplitudes, we create any sound that we like. Once we have a representation of F in terms of the pure tones, we can act on it. In this sense, we can regard the convolution as a kind of "*filter*".

Question: Given any reasonable function F on $[-\pi, \pi]$, with Fourier coefficients define above, is it true that

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}?$$

■ **Fourier Series on General Intervals**

Let $F(x)$ be defined on $[-L, L]$ with $F(-L) = F(L)$. Suppose that F has the form of Fourier series

$$\begin{aligned} F(x) &= \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right) + \sum_{m=1}^{\infty} A'_m \cos\left(\frac{m\pi x}{L}\right) + \frac{A'_0}{2} \\ &= \sum_{m=-\infty}^{\infty} a_m e^{im\pi x/L} \end{aligned}$$

Then the formulas of the coefficients are

$$\begin{aligned} A_m &= \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx \\ A'_m &= \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx \\ a_m &= \frac{1}{2L}\int_{-L}^{L} F(x)e^{-im\pi x/L}\,dx \end{aligned}$$

Let $F(x)$ be a function on $[a, b]$ with $F(a) = F(b)$ and $b - a = L$. Extend $F(x)$ to a new function [still called $F(x)$] defined on $\mathbb{R}$ and is with period L. Suppose that

$$\begin{aligned} F(x) &= \sum_{m=1}^{\infty} A_m \sin\left(\frac{2\pi mx}{L}\right) + \sum_{m=1}^{\infty} A'_m \cos\left(\frac{2\pi mx}{L}\right) + \frac{A'_0}{2} \\ &= \sum_{m=-\infty}^{\infty} a_m e^{2\pi imx/L}. \end{aligned}$$

Then the formulas of the coefficients are

$$\begin{aligned} A_m &= \frac{2}{L}\int_{a}^{b} F(x)\sin\left(\frac{2\pi mx}{L}\right)dx \\ A'_m &= \frac{2}{L}\int_{a}^{b} F(x)\cos\left(\frac{2\pi mx}{L}\right)dx \\ a_m &= \frac{1}{L}\int_{a}^{b} F(x)e^{-2\pi imx/L}\,dx \end{aligned}$$

Remind that the above discussions are based on some ideal situations of F. For example, the integrability of F, the convergence of Fourier series, etc. We need to discuss them carefully.

7.2 Basic Properties of Fourier Series

In this section, we will rigorously study the convergence of Fourier series. Observe that, for a complex-valued function $f(x)$ defined on $[0, L]$, the Fourier coefficients of f are defined by

$$a_n = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}\,dx, \quad \text{for } n \in \mathbb{Z}.$$

In order to make sure that all those coefficients a_n exist, f needs some suitable integrability conditions. Therefore, for the remainder of this chapter, we assume that all functions are at least Riemann integrable.

■ Periodicity and Functions on the Circle

Definition 7.2.1. A function f is said to be periodic with period p if

$$f(x + p) = f(x)$$

for every x in the domain.

Example 7.2.2. $\sin(x + 2\pi) = \sin x$.

Note. 2π is a period of $\sin nx$, $\cos nx$ and e^{inx} for all $n \in \mathbb{Z}$.

First of all, we consider a 2π-periodic function f defined on $\mathbb{R}$. We can identify f as a function F defined on a circle $\mathbb{T}$ (or S^1) in the complex number plane by

$$f(\theta) = F(e^{i\theta})$$

The integrability, continuity and other smoothness properties of F are determined by those of f. If f is continuous on $\mathbb{R}$, then F is continuous on $\mathbb{T}$.

Moreover, if f is a function defined on $[0, 2\pi]$ for which $f(0) = f(2\pi)$, it can be extended to a 2π-periodic function on $\mathbb{R}$ by and then it can be identified as a function on the circle.

We conclude that two kinds of functions can be regard as functions on the circle. They are "functions on $\mathbb{R}$ with period 2π", and "functions on an interval of length 2π that take one the same value at its endpoints".

❑ Definitions and Some Examples

Definition 7.2.3. Let f be an integrable function defined on $[a, b]$ with $b - a = L$.

(1) The nth "*Fourier coefficient*" of f is defined by

$$\hat{f}(n) = a_n = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}\,dx, \quad n \in \mathbb{Z}. \tag{7.2}$$

(2) The "*Fourier series*" of f is given by

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx/L}$$

and we use the notation

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x/L}.$$

Definition 7.2.4. If f is an integrable function on $[-\pi, \pi]$, then the nth Fourier coefficient of f is

$$\widehat{f}(n) = a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}$$

and the Fourier series of f is

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

Note. If f is a function with period L, the resulting integrals (7.2) are independent of the chosen interval. Thus the Fourier coefficients of a function on the circle are well-defined.

Remark. Let f be integrable on $[0, 2\pi]$ and

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}.$$

Define $g(x) = f(2\pi x)$. Then g is integrable on $[0, 1]$ and

$$g(x) \sim \sum_{n=-\infty}^{\infty} \widehat{g}(n) e^{2\pi i n x}$$

Check that $\widehat{g}(n) = \widehat{f}(n)$.

Example 7.2.5.

(a) $f(x) = x$ on $[-\pi, \pi]$. Then $\widehat{f}(n) = \begin{cases} \dfrac{(-1)^{n+1}}{in} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$

$$f(x) \sim \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{inx} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

(b) $f(x) = \dfrac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$ on $[0, 2\pi]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n + \alpha} \quad \text{whenever } \alpha \notin \mathbb{Z}.$$

The "*trigonometric series*" is a series of the form $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}$ where $c_n \in \mathbb{C}$. Similarly, the "*trigonometric polynomial*" is a finite sum of a trigonometric series, that is, it is of the form $\sum_{n=-M}^{N} c_n e^{2\pi i n x/L}$ for some $M, N > 0$.

Example 7.2.6. If f is a trigonometric polynomial function, that is,

$$f(x) = \sum_{n=1}^{N} s_n \sin nx + \sum_{n=0}^{M} c_n \cos nx,$$

then

$$f(x) \sim \sum_{n=1}^{N} s_n \sin nx + \sum_{n=0}^{M} c_n \cos nx.$$

In other words, the Fourier series of f is itself.

Example 7.2.7. (*Dirichlet kernel*) For $N \in \mathbb{N}$, let $c_n = 1$ for every $n = -N, -N+1, \cdots, -1, 0, 1, \cdots, N-1, N$ and $c_n = 0$ otherwise. The trigonometric polynomial defined on $[-\pi, \pi]$ by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is called the Nth "*Dirichlet kernel* ". Denote $\omega = e^{ix}$. For $x \neq 0$,

$$\sum_{n=0}^{N} \omega^n = \frac{1-\omega^{N+1}}{1-\omega} \quad \text{and} \quad \sum_{n=-N}^{-1} \frac{\omega^{-N}-1}{1-\omega}.$$

Hence,

$$D_N(x) = \sum_{n=-N}^{N} \omega^n = \frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{\omega^{-N-1/2}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}} = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(x/2)} \tag{7.3}$$

For $x = 0$, it is easy to check that $D_N(0) = 2N + 1$. The equation (7.3) is also true by taking limit.

Note that we will see below that $S_N(f)(x)$ can be expressed as the convolution of f and $D_N(x)$ by defining $f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy$.

Example 7.2.8. (*Poisson kernel*) Let $0 \leq r < 1$, the function defined on $[-\pi, \pi]$ by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

is called the "*Poisson kernel* ".

For fixed $0 \leq r < 1$, since the series is absolutely and uniformly convergent in θ, to calculate the Fourier coefficients, we can interchange the order of integration and summation. Moreover, the nth Fourier coefficient equals $r^{|n|}$. Set $\omega = re^{i\theta}$. Then

$$\begin{aligned} P_r(\theta) &= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \bar{\omega}^n \quad \text{(where both series converge absolutely)} \\ &= \frac{1}{1-\omega} + \frac{\bar{\omega}}{1-\bar{\omega}} = \frac{1-\bar{\omega}+(1-\omega)\bar{\omega}}{(1-\omega)(1-\bar{\omega})} \\ &= \frac{1-|\omega|^2}{|1-\omega|^2} = \frac{1-r^2}{1-2r\cos\theta+r^2} \end{aligned}$$

■ Some Questions

The "*trigonometric series*" is a series of the form $\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$ where $c_n \in \mathbb{C}$. Similarly, the "*trigonometric polynomial*" is a finite sum of a trigonometric series, that is, it is of the form $\sum_{n=-M}^{N} c_n e^{2\pi inx/L}$ for some $M, N > 0$. In order to study the convergence of Fourier series, it is natual to consider the limit of its partial sum. But the convergence of the trigonometric polynomials here "$\sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx/L}$" is slightly different the typical forms "$\sum_{n=-M}^{N} \widehat{f}(n)e^{2\pi inx/L}$".

Definition 7.2.9. Let $N \in \mathbb{N}$, then the Nth "*partial sum*" of the Fourier series of f is

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx/L}.$$

Note that the above sum is symmetric since n ranges from $-N$ to N because of the resulting decomposition of the Fourier series as sine and cosine.

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx/L} \\ &= \sum_{n=1}^{N} A_n \sin\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{N} A'_n \cos\left(\frac{2\pi nx}{L}\right) + \frac{A'_0}{2}. \end{aligned}$$

For the convenience, we consider the functions defined on intervals with length 2π. ($[0, 2\pi]$, $[-\pi, \pi]$ or etc).

Question: Does the limit $\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = \lim_{N\to\infty} S_N(f)(x)$ converges and for what values of x the limit converge?

Question: If $S_N(f)$ converges to f, in what sense does $S_n(f)$ converge to f as $N \to \infty$ (pointwise, uniformly, or under a certain norms for instance $\|\cdot\|_{L^p}$)?

Observe that the Fourier coefficients come from an integral $\int f(x)e^{-inx}\,dx$. When f and g have different values only at finitely many points, they will have the same Fourier coefficients. Hence, without any additional assumption for f, it is unreasonable to obtain the convergent result that

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \quad \text{for every } x.$$

Question: Under what conditions of a function is uniquely determined by its Fourier coefficients?

■ Uniqueness of Fourier Series

The question of uniqueness is equivalent to the statement that if a function f has Fourier coefficient $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

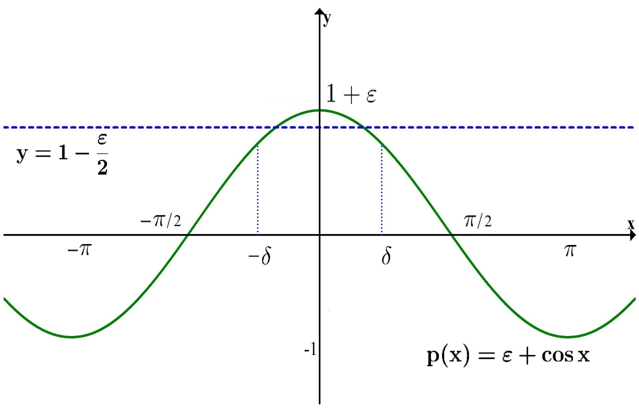
Theorem 7.2.10. *Suppose that f is an integrable function on the circle with $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever f is continuous at the point x_0.*

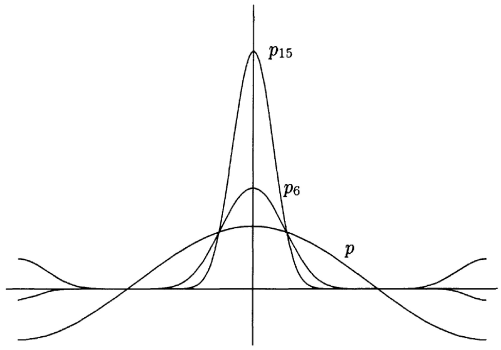
Proof. Firstly, we consider f is real-valued. W.L.O.G, we say that f is defined on $[-\pi, \pi]$ and continuous at $x_0 = 0$. (We will prove, by a contradiction, that $f(0) = 0$ whenever $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$).

The idea is that if $f(0) \neq 0$, we can construct a family of trigonometric polynomials $\{p_k\}$ that "peak" at 0 such that $\int_{-\pi}^{\pi} p_k(x)f(x)\,dx \to \infty$. It is impossible since $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$.

Assume that $f(0) > 0$. Since f is continuous at 0, there exists $0 < \delta < \frac{\pi}{2}$ such that $f(x) > \frac{f(0)}{2}$ for every $x \in [-\delta, \delta]$. Choose a sufficiently small number $\varepsilon > 0$ such that $\left|\varepsilon + \cos x\right| < 1 - \frac{\varepsilon}{2}$ whenever $\delta < |x| \leq \pi$. Denote $p(x) = \varepsilon + \cos x$ and define

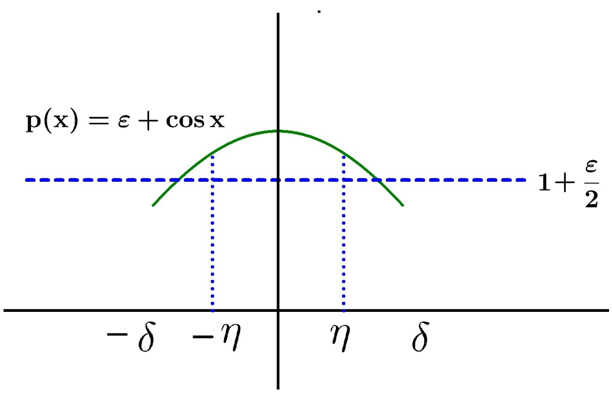
$$p_k(x) = [p(x)]^k.$$





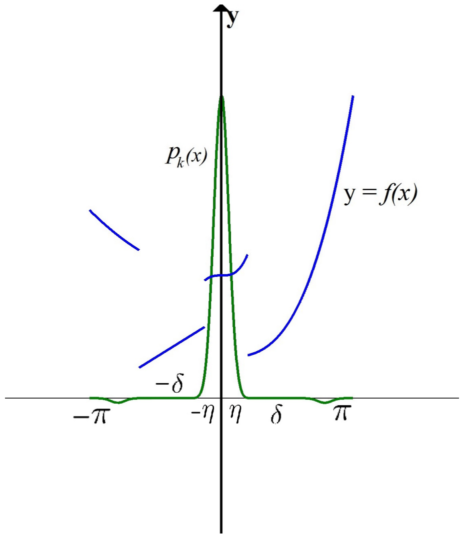
The functions p, p_6, and p_{15} when $\epsilon = 0.1$

Since $\widehat{f}(n) = 0$ for every $n \in \mathbb{Z}$, $\int_{-\pi}^{\pi} f(x)p_k(x)\,dx = 0$ for every $k \in \mathbb{N} \cup \{0\}$. Moreover, f is integrable over $[-\pi, \pi]$. It implies that f is bounded on $[-\pi, \pi]$, say $\left|f(x)\right| \leq B$. Also, we choose $0 < \eta < \delta$ such that $p(x) > 1 + \frac{\varepsilon}{2}$ for every $0 \leq |x| < \eta$.



We have

$$\int_{-\pi}^{\pi} f(x)p_k(x)\,dx = \int_{0\le|x|<\eta} + \int_{\eta\le|x|<\delta} + \int_{\delta\le|x|\le\pi} f(x)p_k(x)\,dx = I + II + III.$$



For $0 \le |x| < \eta$, $f(x) > \dfrac{f(0)}{2}$ and $p_k(x) \ge (1+\frac{\varepsilon}{2})^k$, then

$$I \ge 2\eta \cdot \frac{f(0)}{2} \cdot (1+\frac{\varepsilon}{2})^k \to \infty \quad as \quad k \to \infty$$

For $\eta \le |x| < \delta < \dfrac{\pi}{2}$, $p(x) \ge 0$ and $f(x) > \dfrac{f(0)}{2} > 0$, then

$$II \ge 0.$$

For $\delta \le |x| \le \pi$, $\left|p_k(x)\right| \le (1-\frac{\varepsilon}{2})^k$, then

$$III \le 2\pi \cdot B \cdot (1-\frac{\varepsilon}{2})^k \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, we can choose k sufficiently large such that

$$\int_{-\pi}^{\pi} f(x)p_k(x)\,dx > 0 \quad \text{(Contradiction!)}.$$

Thus, $f(0) = 0$.

Generally, suppose that f is complex-valued, say $f(x) = u(x) + iv(x)$. Define $\bar{f}(x) = \overline{f(x)}$. Then $u(x) = \dfrac{f(x)+\bar{f}(x)}{2}$ and $v(x) = \dfrac{f(x)-\bar{f}(x)}{2}$. Hence u and v are integrable over $[-\pi,\pi]$ and continuous at 0. Since $\widehat{\bar{f}}(n) = \overline{\widehat{f}(-n)}$, we have $\widehat{u}(n) = \widehat{v}(n) = 0$ for all $n \in \mathbb{Z}$. Therefore, $u(0) = v(0) = 0$. □

Corollary 7.2.11. *If f is continuous on the circle and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x) \equiv 0$ on the circle.*

Corollary 7.2.12. *Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, that is $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$. Then*

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \quad \textit{uniformly}.$$

Proof. Since $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$, then series

$$g(x) := \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}$$

converges uniformly. Hence, g is continuous on the circle and the Fourier coefficients $\widehat{g}(n) = \widehat{f}(n)$ for all $n \in \mathbb{Z}$.

On the other hand, since $f - g$ is continuous on the circle and $\widehat{(f-g)}(n) = 0$ for all $n \in \mathbb{Z}$. Thus, $f \equiv g$ on the circle. Then

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} = \lim_{N\to\infty} S_N(f)(x).$$

□

Question: In what conditions of f, the Fourier series of f converges absolutely?

Corollary 7.2.13. *Suppose that f is a twice continuously differentiable function on the circle. Then*

$$\widehat{f}(n) = O\Big(\frac{1}{|n|^2}\Big) \quad as \quad |n| \to \infty$$

Hence, the Fourier series of f converges absolutely and uniformly to f.

Proof. By the integration by parts twice, for $n \neq 0$,

$$\begin{aligned} 2\pi\widehat{f}(n) &= \int_0^{2\pi} f(x)e^{-inx}\,dx \\ &= \underbrace{\Big[f(x)\cdot\frac{e^{-inx}}{-in}\Big]_0^{2\pi}}_{=0} + \frac{1}{in}\int_0^{2\pi} f'(x)e^{-inx}\,dx \\ &= \underbrace{\frac{1}{in}\Big[f'(x)\cdot\frac{e^{-inx}}{-in}\Big]_0^{2\pi}}_{=0} + \frac{1}{(in)^2}\int_0^{2\pi} f''(x)e^{-inx}\,dx \end{aligned}$$

Since f is twice continuously differentiable on the circle, $f''(x)$ is bounded, say $|f''(x)| \leq B$ for all $x \in \mathbb{T}$. Then

$$2\pi|n|^2|\widehat{f}(n)| \leq \int_0^{2\pi} |f''(x)|\,dx \leq 2\pi B.$$

Thus, $|\widehat{f}(n)| \leq \dfrac{B}{|n|^2}$. Moreover, since $\sum \dfrac{1}{n^2}$ converges, the proof is complete. □

Remark.

(1) Heuristically, the index "n" represents the frequency and $\widehat{f}(n)$ reflects the amplitude of nth harmonic with frequency n when regarding f as a superposition of infinite standing waves with different frequencies. Hence, the larger frequencey will be corresponding to the size (weight) of derivatives of f.

(2) More rigorously, we can compute that

$$\widehat{f'}(n) = in\widehat{f}(n), \quad \text{for all } n \in \mathbb{Z}.$$

Thus if f is differentiable and $f \sim \sum a_n e^{inx}$, then $f' \sim \sum a_n in e^{inx}$. Also, if f is twice continuously differentiable, then $f'' \sim \sum a_n (in)^2 e^{inx}$, and so on. Further smoothness conditions on f imply better decay of the Fourier coefficients.

(3) Similar as the corollary, to make the Fourier series of f converges absolutely and uniformly to f, we only need

$$\widehat{f}(n) = O\Big(\frac{1}{|n|^{\alpha}}\Big) \quad \text{as} \quad |n| \to \infty \tag{7.4}$$

for $\alpha > 1/2$. If f satisfies a "*Hölder condition*" of order α, with $\alpha > 1/2$, that is

$$\sup_x |f(x+t) - f(x)| \le A|t|^{\alpha} \quad \text{for all } t,$$

we can obtain (7.4).

7.3 Convolutions of periodic functions and good kernels

Recall that, for given two 2π-periodic integrable functions f and g on $\mathbb{R}$, the convolution of f and g on $[-\pi, \pi]$ is defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy.$$

■ **Properties of Convolution**

Proposition 7.3.1. *Suppose that f, g and h are 2π- periodic integrable functions. Then*

(1) $f * (g + h) = f * g + f * h$.

(2) $(cf) * g = c(f * g) = f * (cg)$ *for every* $c \in \mathbb{C}$.

(3) $f * g = g * f$.

(4) $(f * g) * h = (f * g) * h$.

(5) $f * g$ *is continuous.*

(6) $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.

Proof. The proofs of (1)-(5) are left to the readers. We will prove part(6) here.

$$\begin{aligned}
\widehat{f * g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f * g)(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} f(y)g(x-y)\,dy\Big)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-y)e^{-in(x-y)}\,dx\Big)\,dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx\Big)\,dy \\
&= \widehat{f}(n)\widehat{g}(n).
\end{aligned}$$

□

Remark. Property (5) exhibits that the convolution of $f * g$ is "more regular" than f or g.

Note. One of our goal is to understand whether a function f can be expressed as its Fourier series. That is, $\lim_{N\to\infty} S_N(f)(x) = f(x)$ for every x? Consider the partial sum of the Fourier series of f

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} \\ &= \sum_{n=-N}^{N} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\Big)e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\Big(\sum_{n=-N}^{N} e^{in(x-y)}\Big)\,dy \\ &= (f * D_N)(x) \end{aligned}$$

where D_N is the Nth Dirichlet kernel given by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$

Hence the problem of understanding $S_N(f)$ reduces to the understanding of the convolution $f * D_N$.

❑ Good kernels

In Section3.10 we can regard the convolution $f * g$ as a "weighted average" of f when $\int g(x)\,dx = 1$. Moreover, if g is a highly peaked functoin and is concentrated at 0, the value of $(f * g)(x)$ is close to $f(x)$ if f is continuous there. The same phenomenon also occurs in the proof of Theorem7.2.10. It motivates us to study the "kernels" of operators and discuss the characteristic properties of such functions.

Definition 7.3.2. Let $\{K_n(x)\}_{n=1}^{\infty}$ be a family of functions defined on the circle. This family is called a family of "*good kernels*" if it satisfies the following properties:

(a) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

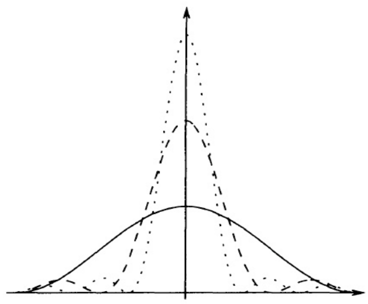
$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta\leq|x|\leq\pi} |K_n(x)|\,dx \to 0, \quad \text{as } n \to \infty.$$

Note.

Property (a) says that K_n assigns unit mass to the whole circle $[-\pi, \pi]$ and K_n is interpreted as weight distributions on the circle. Property (c) exhibits that the mass concentrates near the origin as n becomes large.



Theorem 7.3.3. *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels and f be an integrable function on the circle. Then*

$$\lim_{n\to\infty}(f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, then above limit is uniform.

Proof. Since f is continuous at x, for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon \tag{7.5}$$

as $|y| < \delta$. Consider

$$\begin{aligned}
\left|(f * K_n)(x) - f(x)\right| &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\big[f(x-y) - f(x)\big]\,dy \quad \text{(by condition (a))}\\
&\le \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\big|f(x-y) - f(x)\big|\,dy\\
&\qquad + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)||f(x-y) - f(x)|\,dy\\
&= I + II.
\end{aligned}$$

By the condition (b) and (7.5), $I \le \dfrac{M\varepsilon}{2\pi}$.

Since f is integrable on the circle, it is bounded, say $|f(x)| \le B$ on the circle. From condition (c),

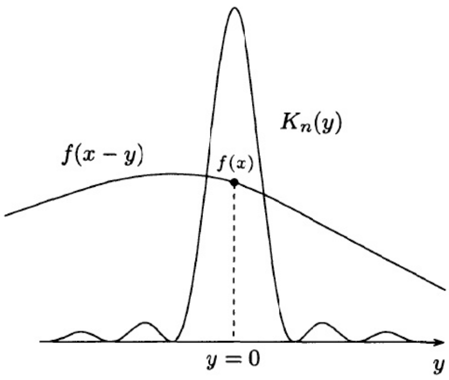
$$II \le \frac{2B}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,dy \to 0 \quad \text{as } n \to \infty.$$

Hence, as n sufficiently large,

$$|(f * K_n)(x) - f(x)| \le C\varepsilon.$$

We have

$$\lim_{n\to\infty}(f * K_n)(x) = f(x).$$



Moreover, if f is continuous everywhere, then f is uniformly continuous on the circle. For the given $\varepsilon > 0$, there exists $\delta > 0$ (which is independent of x) such that

$$|f(x-y) - f(x)| < \varepsilon$$

for every x on the circle. Hence, $f * K_n(x)$ converges to $f(x)$ everywhere and this convergence is independent of x. That is, $f * K_n \to f$ uniformly. □

Remark.

(i) Heuristically, the weighted distribution K_n concentrates its mass at $y = 0$ as n becomes large. Therefore, the value $f(x)$ is assigned the full mass as $n \to \infty$. The convolution

$$(f * K_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) K_n(y)\, dy$$

is the average of $f(x-y)$, where the weights are given by $K_n(y)$.

(ii) The family $\{K_n\}$ is refered to as an **approximation to the identity**.

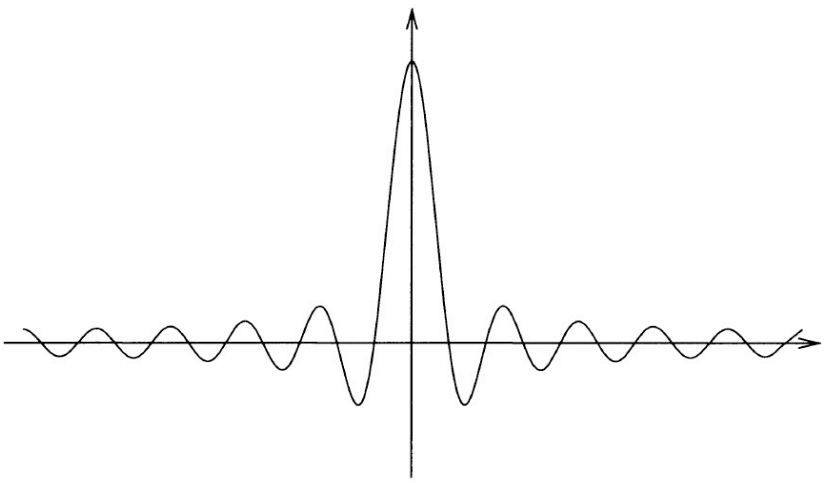
■ <u>Dirichlet Kernel</u>

Question: Is the family of Dirichlet kernels $\left\{D_N(x) = \sum_{n=-N}^{N} e^{inx}\right\}_{N=1}^{\infty}$ a family of good kernels?

It is easy to check that $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)\, dx = 1$ for all $N \geq 1$. Thus, condition (a) holds. Unfortunately, the absolute integral

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq c \log N, \quad \text{as } N \to \infty.$$

Then the condition (b) does not hold. This observation suggests that the pointwise convergence of Fourier series may fail at points of continuity. In fact, the function $D_N(x)$ oscillates very rapidly as N gets large.



The Dirichlet kernel for large N

7.4 Fejér kernel and Poisson kernel

❑ Fejér kernel

Definition 7.4.1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of numbers and $s_n = \sum_{k=0}^{n-1} a_k$ be the nth parital sum of $\{a_n\}$.

(1) The average of the first N partial sums

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N} = \frac{1}{N}\sum_{n=0}^{N-1} s_n$$

is called the Nth "*Cesàro mean*" of the sequence $\{s_n\}$ or the Nth "*Cesàro sum* of the series $\sum_{n=1}^{\infty} a_n$.

(2) If σ_N converges to σ as N tends to infinity, we say that the series $\sum a_n$ is "*Cesàro summable*" to σ.

Exercise.

(1) Let $a_n = (-1)^n$. Then $\sigma_N = \frac{1}{2} + \frac{1 + (-1)^{N-1}}{4N}$ and σ_N converges to $\frac{1}{2}$.

(2) If $\{a_n\}$ is summable to L (that is s_n converges to L), then σ_N converges to L.

(3) If s_n diverges to $\pm\infty$, then σ_N diverges to $\pm\infty$.

Note. The Dirichlet kernels fail to belong to the family of good kernels. But their averages are very well behaved functons, in the sense that they indeed form a family of good kernels.

Definition 7.4.2. Let $D_n(x)$ be the family of Dirichlet kernel. We call the function

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}$$

the Nth "*Fejér kernel*".

Consider the Cesàro mean of the Fourier series

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f) + \cdots + S_{N-1}(f)(x)}{N} \\
&= \frac{(f * D_0)(x) + \cdots (f * D_{N-1})(x)}{N} \\
&= \Big(f * \frac{D_0 + \cdots + D_{N-1}}{N}\Big)(x) \\
&= (f * F_N)(x).
\end{aligned}$$

Lemma 7.4.3. *The Fejér kernel*

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)} \tag{7.6}$$

and it is a good kernel.

Proof. Since $D_N(x) = \dfrac{\omega^{-N} - \omega^{N+1}}{1-\omega}$ with $\omega = e^{ix}$, the equality (7.6) is obtained by direct computation.

Moreover, since $F_N \geq 0$ from (7.6) and $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_n(x)\,dx = 1$ for every $n \in \mathbb{N}$, the average of partial sum of $\{D_n\}_{n=0}^{\infty}$ is also equal to 1. That is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)\,dx = 1.$$

The conditions (a) and (b) of good kernels hold. For every $\delta > 0$, there exists $C_\delta > 0$ such that $\sin^2(x/2) \geq c_\delta$ for every $|x| > \delta$. Hence, $F_N(x) \leq 1/(Nc_\delta)$ and

$$\int_{\delta \leq |x| \leq \pi} |F_N(x)|\,dx \to 0 \quad \text{as } N \to \infty.$$

This implies that the condition (c) of good kernel holds. □

Theorem 7.4.4. *If f is integrable on the circle, then the Fourier series of f is Cesàro summable to f at every point of continuity of f. That is,*

$$\sigma_N(f)(x) \to f(x) \quad \text{as } N \to \infty$$

for every x where f is continuous.
Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesàro summable to f.

Corollary 7.4.5. *If f is integrable on the circle and $\widehat{f}(n) = 0$ for all n, then $f = 0$ at all points of continuity of f.*

Proof. Since $S_N(f) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = 0$ for every $N \in \mathbb{N}$, the Casàro mean of $\{S_n\}$ is equal to 0 and hence the Nth Fejér kernel $F_N(x) \equiv 0$ for every N. Then

$$0 = f * F_N(x) \to f(x)$$

at every continuity of f. □

Corollary 7.4.6. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials. That is, if f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $\varepsilon > 0$, then there exists a trigonometric polynomial P such that*

$$|f(x) - P(x)| < \varepsilon \quad \textit{for all} \ -\pi \le x \le \pi.$$

Proof. The corollary is followed by the theorem since the Cesàro means are trigonometric polynomials.

□

❑ Poisson kernel

Definition 7.4.7. A series of complex number $\sum_{k=0}^{\infty} c_k$ is said to be "*Abel summable*" to s if for every $0 \le r < 1$, there series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantities $A(r)$ are called the "*Abel means*" of the series.

Remark. If $\sum_{k=0}^{\infty} c_k$ is Cesàro summable to s, then it is also Abel summable to s. But the converse is not true. For example, $c_k = (-1)^k(k+1)$. Then

$$A(r) = \sum_{k=0}^{\infty} (-1)^k(k+1)r^k = \frac{1}{(1+r)^2}.$$

The series is Abel summable to $\lim_{r \to 1} A(r) = 1/4$ but it is not Cesàro summable.

Definition 7.4.8. Let $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Define

$$A_r(f)(x) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{inx}.$$

Remark. Since f is integrable (that is, $\int_{-\pi}^{\pi} |f(x)|\, dx < \infty$),

$$|a_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx < \infty.$$

The uniform boundedness of $|a_n|$ implies that $A_r(f)$ converges absolutely and uniformly for each $0 \le r < 1$.

Definition 7.4.9. We define the "*Poisson kernel*" by

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|r|} e^{inx}.$$

Note. The Abel mean of f is equal to the convolution $(f * P_r)(x)$. In fact,

$$\begin{aligned} A_r(f)(x) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{inx} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \Big) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \Big(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(y-x)} \Big)\, dy \\ &= (f * P_r)(x). \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniorm convergence of the series.

Lemma 7.4.10. *If* $0 \le r < 1$*, then*

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{7.7}$$

The poisson kernel is a good kernel, as r tends to 1 *from below.*

Proof. The identity is obtained by direct computation by setting $\omega = e^{ix}$. Since $P_r(x)$ is positive and evaluating the integral term by term, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x)\, dx = 1.$$

The condtions (a) and (b) of good kernel hold. Moreover, for $1/2 \le r \le 1$ and $\delta \le |x| \le \pi$,

$$1 - 2r\cos x + r^2 = (1-r)^2 + 2r(1 - \cos x) \ge c_\delta > 0$$

where c_δ could be given by $1 - \cos\delta$. Then $P_r(x) \le \dfrac{(1-r^2)}{c_\delta}$ when $\delta \le |x| \le \pi$. Then

$$\int_{\delta \le |x| \le \pi} |P_r(x)|\, dx \le \frac{\pi(1-r^2)}{c_\delta} \to 0 \quad \text{as } r \to 1^-.$$

The condition (c) of good kernel holds.

□

Theorem 7.4.11. *The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f.*

7.5 Convergence of Fourier Series

In the present section, we will discuss the convergence of Fourier series in three different senses, mean-square, pointwise and uniform convergence. The mean-square convergence reflects the global bahaviors of the partial sum $S_N(f)$. The pointwise and uniforn convergence reveal the local behaviors of $S_N(f)$. We want to find the sufficient conditions of these convergence.

Recall that a Hilbert space is a complete inner product space.

Example 7.5.1.

(1) Let $\ell^2(\mathbb{Z},\mathbb{C}) = \left\{(\cdots, a_{-1}, a_0, a_1, \cdots) \,\middle|\, a_n \in \mathbb{C} \text{ with } \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\right\}$. Define

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}$$

for $\mathbf{a} = (\cdots, a_{-1}, a_0, a_1, \cdots)$ and $\mathbf{b} = (\cdots, b_{-1}, b_0, b_1, \cdots)$. Then $\ell^2(\mathbb{Z},\mathbb{C})$ is a Hilbert space.

(2) $\mathcal{R} = \left\{f : [0, 2\pi] \to \mathbb{C} \,\middle|\, f \text{ is a Riemann integrable function on } [0, 2\pi]\right\}$ with

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

$\mathcal{R}$ is not a Hilbert space.

Let

$$f_n(x) = \begin{cases} x^{-1/4} & \text{if } x \in [\frac{1}{n}, \pi] \\ 0 & \text{otherwise} \end{cases}$$

Then f_n is a Cauchy sequenc of $\mathcal{R}$. For any bounded function $f \in \mathcal{R}$,

$$\lim_{n\to\infty} \|f_n - g\| \neq 0.$$

Hence, $\mathcal{R}$ is not complete.

Before discussing the convegence of Fourier series, we review some properties of inner product spaces and Hilbert spaces.

❑ Orthonormal Sequence

Definition 7.5.2. Let X be a vector space with an inner product $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ be the incuced norm on X which is defined by

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle \quad \text{for every } \mathbf{x} \in X.$$

We say that the two vectors $\mathbf{x}, \mathbf{y} \in X$ are "*orthogonal*" if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.

■ Some Properties

(1) (**Pythagorean theorem**) If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

(2) (**Cauchy-Schwarz inequality**) For $\mathbf{x}, \mathbf{y} \in X$,

$$\left|\langle \mathbf{x}, \mathbf{y}\rangle\right| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

(3) (**Triangle inequaltiy**) For $\mathbf{x}, \mathbf{y} \in X$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Definition 7.5.3. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{C}$. We say that $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is a sequence of orthonormal vectors if

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Remark. Let $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ be a sequence of orthonormal vectors in a Hilbert space X. The closed span

$$M = \overline{span\{\mathbf{e}_n\}}$$

is a closed subspace of X.

Theorem 7.5.4. *Let X be a Hilbert space and $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in X. Then the following statements hold.*

(a) **Bessel's Inequality:**

$$\sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2 \leq \|\mathbf{x}\|^2$$

for every $\mathbf{x} \in X$.

(b) If the series $\mathbf{x} = \sum_{n=1}^{\infty} c_n \mathbf{e}_n$ converges, then $c_n = \langle \mathbf{x}, \mathbf{e}_n \rangle$ for each $n \in \mathbb{N}$.

(c) The following equivalence holds:

$$\sum_{n=1}^{\infty} c_n \mathbf{e}_n \text{ converges} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

Furthermore, in this case the series $\sum_{n=1}^{\infty} c_n \mathbf{e}_n$ converges unconditionally, i.e., it converges regardless of the ordering of the index set.

(d) If $\mathbf{x} \in X$, then

$$\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$$

is the orthogonal projection of $\mathbf{x}$ onto $M := \overline{span\{\mathbf{e}_n\}}$, and $\|\mathbf{p}\|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2$.

(e) If $\mathbf{x} \in X$*, then the following three statements are equivalent*

(i) $\mathbf{x} \in M := \overline{span\{\mathbf{e}_n\}}$.

(ii) $\mathbf{x} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$.

(iii) $\|\mathbf{x}\|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2$.

Proof. (a) Choose $\mathbf{x} \in X$. For each $N \in \mathbb{N}$ define

$$\mathbf{p}_N = \sum_{n=1}^{N} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n \quad \text{and} \quad \mathbf{q}_N = \mathbf{x} - \mathbf{p}_N.$$

Since the $\mathbf{e}_n$ are orthonormal, the Pythagorean Theorem implies that

$$\|\mathbf{p}_N\|^2 = \sum_{n=1}^{N} \|\langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n\|^2 = \sum_{n=1}^{N} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2.$$

Also,

$$\langle \mathbf{p}_N, \mathbf{q}_N \rangle = \langle \mathbf{p}_N, \mathbf{x} \rangle - \langle \mathbf{p}_N, \mathbf{p}_N \rangle = \sum_{n=1}^{N} \langle \mathbf{x}, \mathbf{e}_n \rangle \langle \mathbf{e}_n, \mathbf{x} \rangle - \|\mathbf{p}_N\|^2 = 0.$$

Then the vectors $\mathbf{p}_N$ and $\mathbf{q}_N$ are orthogonal. By the Pythagorean Theorem again,

$$\sum_{n=1}^{N} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2 = \|\mathbf{p}_N\|^2 \le \|\mathbf{p}_N\|^2 + \|\mathbf{q}_N\|^2 = \|\mathbf{p}_N + \mathbf{q}_N\|^2 = \|\mathbf{x}\|^2.$$

Let $N \to \infty$, we obtain Bessel's Inequality.

(b) If $\mathbf{x} = \sum_{n=1}^{\infty} c_n \mathbf{e}_n$ converges, for each fixed m, we have

$$\langle \mathbf{x}, \mathbf{e}_m \rangle = \left\langle \sum_{n=1}^{\infty} c_n \mathbf{e}_n, \mathbf{e}_m \right\rangle = \sum_{n=1}^{\infty} c_n \langle \mathbf{e}_n, \mathbf{e}_m \rangle = c_m.$$

(Notice that the second equality is valid since the sequence is convergent.)

(c) ($\Longrightarrow$) By part(b), $c_n = \langle \mathbf{x}, \mathbf{e}_n \rangle$ since $\mathbf{x} = \sum_{n=1}^{\infty} c_n \mathbf{e}_n$. Thus, by Bessel's inequality,

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2 \le \|\mathbf{x}\|^2.$$

($\Longleftarrow$) Suppose that $\sum_{n=1}^{\infty} |c_n|^2 < \infty$. Set

$$\mathbf{s}_n = \sum_{n=1}^{N} c_n \mathbf{e}_n \quad \text{and} \quad t_N = \sum_{n=1}^{N} |c_n|^2.$$

To prove that $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ is a convergent sequence in X. If $M < N$, then

$$\begin{aligned} \|\mathbf{s}_N - \mathbf{s}_M\|^2 &= \Big\| \sum_{n=M+1}^{N} c_n \mathbf{e}_n \Big\|^2 \\ &= \sum_{n=M+1}^{N} \|c_n \mathbf{e}_n\|^2 \quad \text{(Pythagorean Theorem)} \\ &= \sum_{n=M+1}^{N} |c_n|^2 = |t_N - t_M.| \end{aligned}$$

Since $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, the sequence $\{t_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a Hilbert space, the sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ converges and so does $\sum_{n=1}^{\infty} c_n \mathbf{e}_n$.

Furthermore, since $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, the sequence $\{|c_n|^2\}_{n\in\mathbb{N}}$ is absolutely summable and the summation does not change if reordering of the series. Thus, $\sum_{n=1}^{\infty} c_n \mathbf{e}_n$ converges unconditionally.

(d) By Bessel's inequality and part(c), the series $\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$ converges. For fixed k,

$$\begin{aligned} \langle \mathbf{x} - \mathbf{p}, \mathbf{e}_k \rangle &= \langle \mathbf{x}, \mathbf{e}_k \rangle - \Big\langle \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n, \mathbf{e}_k \Big\rangle \\ \text{(Convergence)} \longrightarrow \quad &= \langle \mathbf{x}, \mathbf{e}_k \rangle - \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \langle \mathbf{e}_n, \mathbf{e}_k \rangle \\ &= \langle \mathbf{x}, \mathbf{e}_k \rangle - \langle \mathbf{x}, \mathbf{e}_k \rangle = 0 \end{aligned}$$

The vector $\mathbf{x} - \mathbf{p}$ is orthogonal to each vector $\mathbf{e}_k$ and thus it is orthogonal to every vector in M. We have that $\mathbf{p} \in M$ and $\mathbf{x} - \mathbf{p} \in M^{\perp}$. This implies that $\mathbf{p}$ is the orthogonal projection of $\mathbf{x}$ onto M.

(e) By part(d), $\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e} \rangle \mathbf{e}_n$ is the orthogonal projection of $\mathbf{x}$ onto M and

$$\|\mathbf{p}\|^2 = \langle \mathbf{p}, \mathbf{p} \rangle = \sum_{n=1}^{\infty} \big|\langle \mathbf{x}, \mathbf{e}_n \rangle\big|^2.$$

"(i) ⇒ (ii)" If $\mathbf{x} \in M$, the orthogonal projection of $\mathbf{x}$ onto M is $\mathbf{x}$ itself. Thus, $\mathbf{x} = \mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$.

"(ii) ⇒ (iii)" If $\mathbf{x} = \mathbf{p}$, then $\|\mathbf{x}\|^2 = \|\mathbf{p}\|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2$.

"(iii) ⇒ (i)" Suppose $||\mathbf{x}||^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n\rangle|^2$. Then since $\mathbf{x} - \mathbf{p} \perp \mathbf{p}$,

$$\begin{aligned} ||\mathbf{x}||^2 &= ||(\mathbf{x} - \mathbf{p}) + \mathbf{p}||^2 = ||\mathbf{x} - \mathbf{p}||^2 + ||\mathbf{p}||^2 \\ &= ||\mathbf{x} - \mathbf{p}||^2 + \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{e}_n\rangle|^2 = ||\mathbf{x} - \mathbf{p}||^2 + ||\mathbf{x}||^2. \end{aligned}$$

Hence $||\mathbf{x} - \mathbf{p}|| = 0$ and $\mathbf{x} = \mathbf{p} \in M$.

□

Remark. We say that the sequence $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is "*complete*" in X if

$$\overline{span\{\mathbf{e}_n\}} = X.$$

7.5.1 Mean-Square Convergence

Consider the space $\mathcal{R}$ of integrable functions on the circle with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\, dx$$

and the induced norm

$$||f||^2 = \langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx.$$

Note. The norm $||\cdot||$ is equivalent to $||\cdot||_{L^2}$. In fact,

$$2\pi||\cdot||^2 = ||\cdot||^2_{L^2([0,2\pi])}.$$

We will prove that $||S_N(f) - f|| \to 0$ as N tends to infinity. It also implies $S_N(f)$ converges to f in L^2 norm.

Set $\mathbf{e}_n(x) = e^{inx}$. Then $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ is an orthonormal sequence. Let

$$a_n = \langle f, \mathbf{e}_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx = \widehat{f}(n)$$

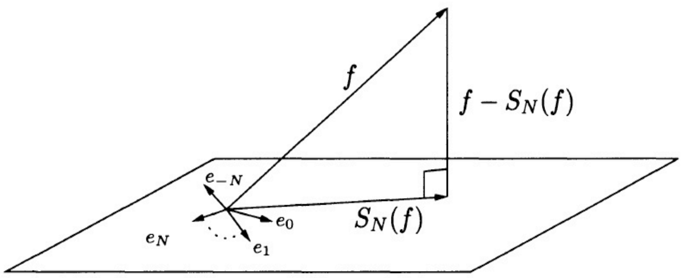
be the Fourier coefficient of f. Then

$$S_N(f)(x) = \sum_{|n|\le N} a_n\mathbf{e}_n.$$

Lemma 7.5.5. *For every $N \in \mathbb{N}$,*

$$\Big(f - \sum_{|n|\le N} a_n\mathbf{e}_n\Big) \perp \sum_{|n|\le N} b_n\mathbf{e}_n$$

for any $b_n \in \mathbb{C}$.



The best approximation lemma

Proof. For every $|n| \le N$,

$$\begin{aligned}\langle f - \sum_{|m|\le N} a_m \mathbf{e}_m, \mathbf{e}_n\rangle &= \langle f, \mathbf{e}_n\rangle - \sum_{|m|\le N} a_m \langle \mathbf{e}_m, \mathbf{e}_n\rangle \\ &= a_n - a_n = 0.\end{aligned}$$

By the linear combination, we have $\left(f - \sum_{|n|\le N} a_n \mathbf{e}_n\right) \perp \sum_{|n|\le N} b_n \mathbf{e}_n$. □

■ **Bessel's Inequality**

By Lemma7.5.5, we write $f = (f - \sum_{|n|\le N} a_n \mathbf{e}_n) + \sum_{|n|\le N} a_n \mathbf{e}_n$ and

$$\begin{aligned}||f||^2 &= ||f - \sum_{|n|\le N} a_n \mathbf{e}_n||^2 + ||\sum_{|n|\le N} a_n \mathbf{e}_n||^2 \quad \text{(Pythagorean Theorem)} \\ &= ||f - \sum_{|n|\le N} a_n \mathbf{e}_n||^2 + \sum_{|n|\le N} |a_n|^2 ||\mathbf{e}_n||^2 \\ &= ||f - \sum_{|n|\le N} a_n \mathbf{e}_n||^2 + \sum_{|n|\le N} |a_n|^2 \\ &= ||f - S_N(f)||^2 + \sum_{|n|\le N} |a_n|^2.\end{aligned}$$

Hence, for every $N \in \mathbb{N}$, $\sum_{|n|\le N} |a_n|^2 \le ||f||^2$. Letting $N \to \infty$, we have the Bessel's inequality

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \le ||f||^2.$$

Remark. Suppose that $\{\mathbf{u}_n\}$ is any orthonormal sequence and $b_n = \langle f, \mathbf{u}_n\rangle$ for every n. We still have a corresponding Bessel's inequality,

$$\sum |b_n|^2 \le ||f||^2.$$

Lemma 7.5.6. *(Best approximation) If f is integrable on the circle with Fourier coefficients a_n, then*

$$||f - S_N(f)|| \le ||f - \sum_{|n|\le N} c_n \mathbf{e}_n|| \tag{7.8}$$

for any $c_n \in \mathbb{C}$. Moreover, the equality holds precisely when $c_n = a_n$ for all $|n| \le N$.

Proof. Let $b_n = a_n - c_n$. Then

$$f - \sum_{|n|\le N} c_n \mathbf{e}_n = f - S_N(f) + \sum_{|n|\le N} b_n \mathbf{e}_n.$$

By Pythagorean theorem, since $\left(f - S_N(f)\right) \perp \sum_{|n|\le N} b_n \mathbf{e}_n$,

$$\|f - \sum_{|n|\le N} c_n \mathbf{e}_n\|^2 = \|f - S_N(f)\|^2 + \sum_{|n|\le N} |b_n|^2.$$

Thus, the inequality (7.8) is proved. □

Theorem 7.5.7. *If f is Riemann integrable on the circle, then*

$$\|S_N(f) - f\| \to 0 \quad as \quad N \to \infty.$$

Proof.

Step1: To show that the theorem is ture if f is (2π-periodic) continuous on the circle. For given $\varepsilon > 0$, by Corollary7.4.6, there exists a trigonometric polynomial P with degree M such that

$$\|f - P\|_{L^\infty\left([0,2\pi]\right)} < \varepsilon.$$

Therefore,

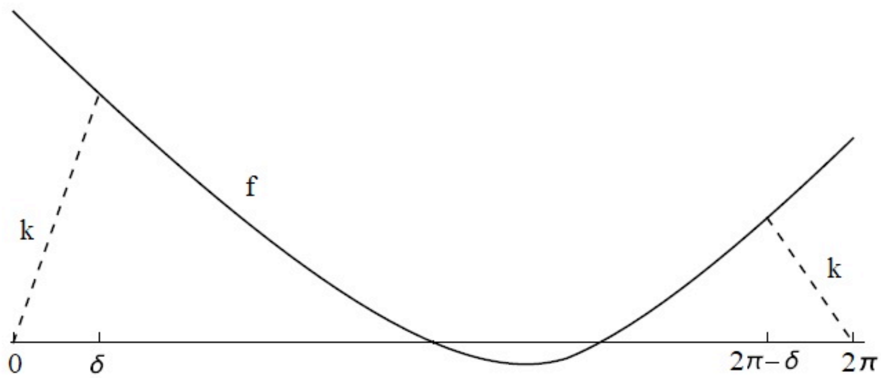
$$\frac{1}{2\pi}\int_0^{2\pi} |f - P|^2\, dx \le \frac{1}{2\pi} \cdot 2\pi\varepsilon^2 = \varepsilon^2.$$

Then $\|f - P\| < \varepsilon$. By the best approximation,

$$\|f - S_M(f)\| \le \|f - P\| < \varepsilon.$$

Step2: If f is a continuous function (but possibly $f(0) \ne f(2\pi)$), we define

$$k(x) = \begin{cases} 0, & x = 0 \\ \text{linear}, & 0 < x < \delta \\ f(x), & \delta < x < 2\pi - \delta \\ \text{linear}, & 2\pi - \delta \le x < 2\pi \\ 0, & x = 2\pi \end{cases}$$



The function k (dashed) is close in L^2-norm to f (solid), and also satisfies $k(0) = k(2\pi)$.

Then k is continuous on $[0, 2\pi]$ with $k(0) = k(2\pi)$ and

$$\|f - k\| < \varepsilon$$

if δ is sufficiently small. Also, $f - k$ is integrable on the circle. By the Bessel's inequality,

$$\|S_N(f) - S_N(k)\| = \|S_N(f - k)\| \leq \|f - k\| < \varepsilon$$

for every $N \in \mathbb{N}$.
Step3: If f is integrable on the circle, by using the method of mollifiers, we can choose a continuous function g on $[0, 2\pi]$ such that

$$\|f - g\| < \varepsilon$$

and hence $\|S_N(f) - S_N(g)\| = \|S_N(f - g)\| \leq \|f - g\| < \varepsilon$. Then

$$\begin{aligned} \|f - S_N(f)\| &\leq \|f - g\| + \|g - S_N(g)\| + \|S_N(g) - S_N(f)\| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

as N is sufficiently large. □

Corollary 7.5.8. *(***Parseval's Identity***) Let f be an integrable function on the circle. If a_n is the nth Fourier coefficients of f, then*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2.$$

Proof. The identity is clear since

$$\begin{aligned} \|f\|^2 &= \|f - S_N(f)\|^2 + \|S_N(f)\|^2 \quad \text{(Pythagorean Theorem)} \\ &= \|f - S_N(f)\|^2 + \sum_{n=-N}^{N} |a_n|^2. \end{aligned}$$

Let $N \to \infty$ and we obtain $\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2$. □

Theorem 7.5.9. *(***Riemann-Lebesgue lemma***) If f is integrable on the circle, then $\widehat{f}(n) \to 0$ as $|n| \to 0$.*

Proof. Since f is integrable on the circle, f is bounded and this implies that $\|f\|^2 < \infty$. By Bessel's identity,

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)^2| = \|f\|^2 < \infty.$$

Then $\widehat{f}(n) \to 0$ as $|n| \to \infty$.

□

Note. An equivalent result of this theorem is that if f is integrable on $[0, 2\pi]$, then

$$\int_0^{2\pi} f(x)\sin(Nx)\,dx \to 0 \quad \text{as } N \to \infty$$

and

$$\int_0^{2\pi} f(x)\cos(Nx)\,dx \to 0 \quad \text{as } N \to \infty$$

Lemma 7.5.10. *Suppose F and G are integrable on the circle with*

$$F \sim \sum a_n e^{inx} \quad \text{and} \quad G \sim \sum b_n e^{inx}.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} F(x)\overline{G(x)}\,dx = \sum_{n=-\infty}^{\infty} a_n\overline{b_n}.$$

Proof. Since

$$\langle F, G\rangle = \frac{1}{4}\left[\|F+G\|^2 - \|F-G\|^2 + i\left(\|F+iG\|^2 - \|F-iG\|^2\right)\right]$$

by Parseval's identity

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} F(x)\overline{G(x)}\,dx &= \langle F, G\rangle \\
&= \frac{1}{4}\left[\|F+G\|^2 - \|F-G\|^2 + i\left(\|F+iG\|^2 - \|F-iG\|^2\right)\right] \\
&= \frac{1}{4}\sum_{n=-\infty}^{\infty}\left[|a_n+b_n|^2 - |a_n-b_n|^2 + i\left(|a_n+ib_n|^2 - |a_n-ib_n|^2\right)\right] \\
&= \sum_{n=-\infty}^{\infty} a_n\overline{b_n}.
\end{aligned}$$

□

7.5.2 Pointwise Convergence

The mean-square convergence theorem does not guarantee that the Fourier series converges for any x. In order to obtain the pointwise convergence of Fourier series, the function may have good local behaviors near x_0.

Observe that

$$\begin{aligned}
S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)D_N(y)\,dy - f(x_0) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x_0-y) - f(x_0)\right]D_N(y)\,dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x_0-y) - f(x_0)\right]\frac{\sin\left((N+\frac{1}{2})y\right)}{\sin(\frac{y}{2})}\,dy
\end{aligned}$$

We expect the integral decays to 0 as N tends to infinity. However, the denominator $\sin(\frac{y}{2})$ become small as $|y|$ tends to 0. Hence, we hope to obtain a better control of $\dfrac{f(x_0 - y) - f(x_0)}{\sin(\frac{y}{2})}$ that will give the pointwise convergence.

Theorem 7.5.11. *Let f be an integrable function on the circle which is differentiable at a point x_0. Then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.*

Proof. Define

$$F(y) = \begin{cases} \dfrac{f(x_0 - y) - f(x_0)}{y} & \text{if } y \neq 0 \text{ and } |y| < \pi \\ -f'(x_0) & \text{if } y = 0 \end{cases}$$

Since f is differentiable at x_0, there exists $\delta > 0$ such that F is bounded for $|y| \le \delta$. Moreover, F is integrable on $[-\pi, -\delta] \cup [\delta, \pi]$ because f is integrable on the circle. Then F is integrable on the circle.

On the other hand, since $\dfrac{y}{\sin(y/2)}$ is continuous on $[-\pi, \pi]\backslash\{0\}$, the functions

$$F(y) \cdot \frac{y}{\sin(y/2)} \cos(y/2) \quad \text{and} \quad F(y)y$$

are Riemann integrable on $[-\pi, \pi]$. Also,

$$\sin\big((N + 1/2)y\big) = \sin(Ny)\cos(y/2) + \cos(Ny)\sin(y/2).$$

Then

$$\begin{aligned} S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - y)D_N(y)\,dy - f(x_0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[f(x_0 - y) - f(x_0)\big] D_N(y)\,dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[f(x_0 - y) - f(x_0)\big] \frac{\sin\big((N + \frac{1}{2})y\big)}{\sin(\frac{y}{2})}\,dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(F(y) \cdot \frac{y}{\sin(y/2)}\cos(y/2)\Big)\sin(Ny)\,dy \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} F(y)y\cos(Ny)\,dy. \end{aligned}$$

By Riemann-Lebesgue lemma, the above two integrals converge to 0 as $N \to 0$ and the theorem is proved. □

Remark. According to the above analysis, we need to control the term $\dfrac{f(x_0 - y) - f(x_0)}{\sin(y/2)}$ as $|y|$ is small. In fact, the conclusion of the theorem still holds if we assume that f satisfies a "*Lipschitz condition*" at x_0; that is,

$$|f(x) - f(x_0)| \le M|x - x_0|$$

for some $M \ge 0$ and all x.

Theorem 7.5.12. *Suppose f and g are two integrable functions defined on the circle, and for some x_0 there exists an open interval I containing x_0 such that*

$$f(x) = g(x) \quad \text{for all } x \in I.$$

Then $S_N(f)(x_0) - S_N(g)(x_0) \to 0$ as $N \to \infty$.

Proof. Since the function $f - g$ is 0 in I, it is differentiable at x_0. Therefore, by Theorem7.5.11,

$$S_N(f)(x_0) - S_N(g)(x_0) = S_N(f - g)(x_0) \to (f - g)(x_0) = 0.$$

□

■ Piecewise Continuous Functions

If f is a piecewise continuous function on the circle, then it is bounded and integrable on the circle. Denote

$$f(x-) = \lim_{h\to 0^+} f(x-h) \quad \text{and} \quad f(x+) = \lim_{h\to 0^+} f(x+h).$$

Let $\overline{f(x)}$ be the average value

$$\overline{f(x)} = \frac{1}{2}[f(x+) + f(x-)].$$

Note that if f is continuous at x, then $f(x) = f(x+) = f(x-) = \overline{f(x)}$.

Definition 7.5.13. A piecewise continuous function f is said to be "*one-sided differentiable*" at x if the two limits

$$\lim_{h\to 0^+} \frac{f(x-) - f(x-h)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(x+h) - f(x+)}{h}$$

both exist.

Example 7.5.14. The function $f(x) = |x|$ is one-sided differentiable at $x = 0$ since

$$\lim_{h\to 0^+} \frac{|0| - |-h|}{h} = -1 \quad \text{and} \quad \lim_{h\to 0^+} \frac{|h| - |0|}{h} = 1.$$

Theorem 7.5.15. *Let f be a piecewise continuous function on $[-\pi, \pi]$ such that its 2π-periodic extension is one-sided differentiable for all $x \in \mathbb{R}$. Then $S_N(f)$ converges pointwise to $\overline{f(x)}$ for all $x \in \mathbb{R}$.*

Proof. Since $D_N(y)$ is an even function, then

$$\frac{1}{2\pi}\int_{-\pi}^{0} D_N(y)\,dy = \frac{1}{2\pi}\int_{0}^{\pi} D_N(y)\,dy = \frac{1}{2}.$$

We have

$$\overline{f(x)} = \frac{1}{2\pi}\Big[\int_{-\pi}^{0} D_N(y)f(x+)\,dy + \int_{0}^{\pi} D_N(y)f(x-)\,dy\Big].$$

$$\begin{aligned}
S_N(f)(x) - \overline{f(x)} &= \frac{1}{2\pi}\Big[\int_{-\pi}^{0} D_N(y)\Big(f(x-y) - f(x+)\Big)\,dy \\
&\quad + \int_{0}^{\pi} D_N(y)\Big(f(x-y) - f(x-)\Big)\,dy\Big] \\
&= \frac{1}{2\pi}\Big[\int_{\pi}^{0} D_N(-y)\Big(f(x+y) - f(x+)\Big)\,(-dy) \\
&\quad + \int_{0}^{-\pi} D_N(-y)\Big(f(x+y) - f(x-)\Big)\,(-dy)\Big] \quad \text{(let } y \to -y) \\
&= \frac{1}{2\pi}\Big[\int_{0}^{\pi} D_N(y)\Big(f(x+y) - f(x+)\Big)\,dy \\
&\quad + \int_{-\pi}^{0} D_N(y)\Big(f(x+y) - f(x-)\Big)\,dy\Big] \quad (D_N \text{ is even .}) \\
&= \frac{1}{2\pi}\Big[\int_{0}^{\pi} \frac{f(x+y) - f(x+)}{\sin(y/2)} \cdot \sin\big((N+1/2)y\big)\,dy \\
&\quad + \int_{-\pi}^{0} \frac{f(x+y) - f(x-)}{\sin(y/2)} \cdot \sin\big((N+1/2)y\big)\,dy\Big] \\
&= \frac{1}{\pi}\Big[\int_{0}^{2\pi} \frac{f(x+2z) - f(x+)}{\sin z} \cdot \sin\big((2N+1)z\big)\,dz \\
&\quad + \int_{-2\pi}^{0} \frac{f(x+2z) - f(x-)}{\sin z} \cdot \sin\big((2N+1)z\big)\,dz\Big] \quad \text{(let } y = 2z) \\
&= I + II
\end{aligned}$$

By the similar argument as the one of Theorem7.5.11, since f is one-sided differentiable, the functions

$$\frac{f(x+2z) - f(x+)}{\sin z} \quad \text{and} \quad \frac{f(x+2z) - f(x-)}{\sin z}$$

are integrable on $[0, 2\pi]$ and $[-2\pi, 0]$ respectively. From Riemann-Lebesgue lemm, both I and II converge to 0 as N tends to infinity. The theorem is proved.

□

Example 7.5.16. Let $f(x) = |x|$ be defined on $[-\pi, \pi]$. Then the Fourier coefficients of f are

$$\widehat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0 \\ \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0 \end{cases}$$

Then the Fourier series

$$|x| \sim \frac{\pi}{2} + \sum_{|n|=1}^{\infty} \frac{-1 + (-1)^n}{\pi n^2} e^{inx} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,\ odd}^{\infty} \frac{\cos(nx)}{n^2}.$$

Since f is continuous on $[-\pi, \pi]$ and one-sided differentiable, f can be expressed as its Fourier series. That is

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,odd}^{\infty} \frac{\cos(nx)}{n^2}.$$

Taking $x = 0$, we have

$$\sum_{n=1,\ odd}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

7.5.3 Uniform Convergence

In the present subsection, we want to find the sufficient condition for the uniform convergence of Fourier series. Corollary7.2.13 says that the twice continuous differentiability of f will give rise to the uniform convergence. Besides, since uniform convergence automatically implies pointwise convergence, we naturally expect the sufficient conditions for uniform convergence are strong than the hypotheses in Theorem7.5.11.

The following theorem will apply Corollary7.2.11 and give a better hypothesis than the ones of Corollary7.2.13.

Theorem 7.5.17. *Let f be a function defined on $[-\pi, \pi]$ such that its periodic extension is continuous (i.e $f(-\pi) = f(\pi)$) and let f' be piecewise continuous. Then $S_N(f)$ converges uniformly to f on $[-\pi, \pi]$.*

Proof. By Corollary7.2.11, it suffices to show that $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$. Since f' is piecewise continuous, it is integrable on $[-\pi, \pi]$ and hence its Fourier coefficients are well-defined and

$$\widehat{f'}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx.$$

Moreover, from Bessel's inequality,

$$\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2 \le \|f'\|^2 < \infty.$$

On the other hand, for every $n \in \mathbb{Z}$,

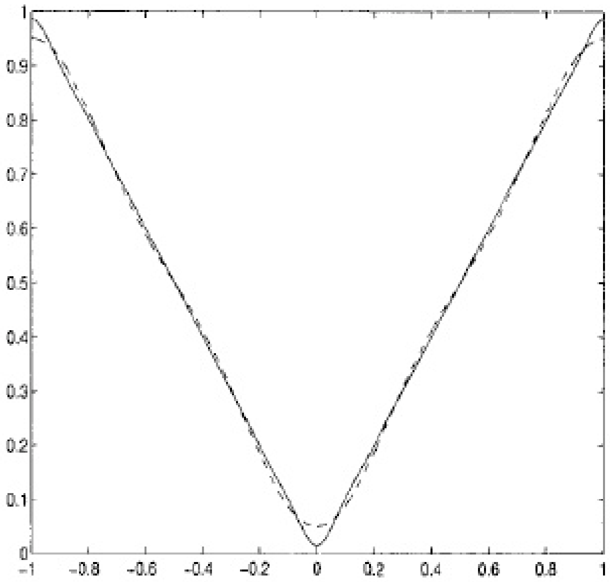
$$\begin{aligned}
\widehat{f'}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx \\
&= \frac{1}{2\pi} \Big[f(x) e^{-inx} \Big|_{-\pi}^{\pi} + in \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \Big] \\
&= 0 + \frac{in}{2\pi} \int_{\pi}^{\pi} f(x) e^{-inx}\, dx \quad (\text{since } f(-\pi) = f(\pi)) \\
&= (in)\widehat{f}(n).
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| &= |\widehat{f}(0)| + \sum_{|n|=1}^{\infty} \frac{|\widehat{f'}(n)|}{|n|} \\
&\le |\widehat{f}(0)| + \Big(\sum_{|n|=1}^{\infty} \frac{1}{n^2} \Big)^{1/2} \Big(\sum_{|n|=1}^{\infty} |\widehat{f'}(n)|^2 \Big)^{1/2} \\
&< \infty.
\end{aligned}$$

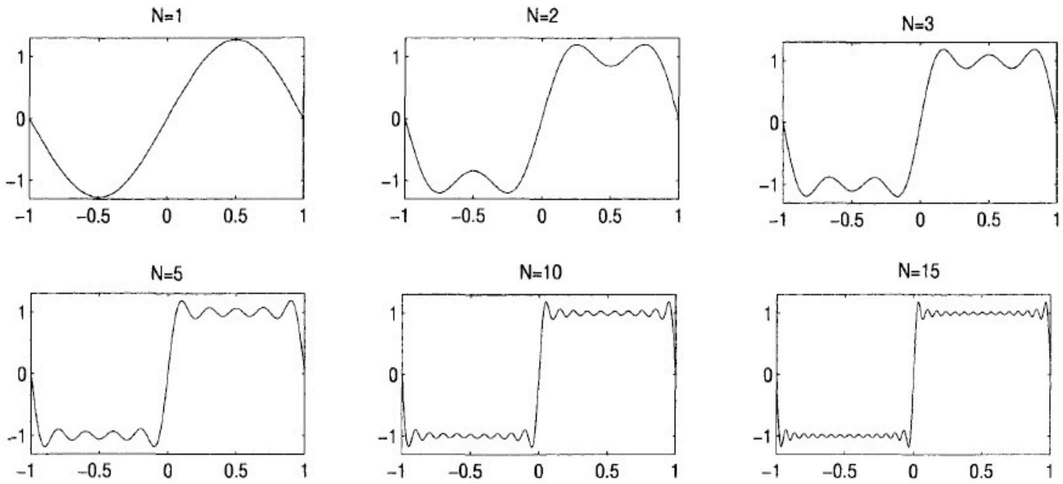
By Corollary7.2.11, $S_N(f)$ converges to f uniformly. □

Example 7.5.18. Let $f(x) = |x|$ be defined on $[-\pi, \pi]$ and the 2π periodic extension of f and $f'(x) = sign(x)$ is piecewise continuous. Therefore, $S_N(f)$ converges to f uniformly.



$S_2(f)$ *(dashed) and* $S_7(f)$ *when* $f(x) = |x|$.

Example 7.5.19. Let $f(x) = sign(x)$. Since f is not continuous, we cannot conclude that $S_N(f)$ converges to f uniformly on $[-\pi, \pi]$. If fact, it is impossible that $S_N(f)$ convergs to f uniformly since the limit function of uniform convergence of continuous functions should be continuous.



S_{2N-1} for different values of N when f(x) = sign(x)

7.6 Smoothness and Decay of Fourier Coefficients

From the proofs of Corollary7.2.13 and Theorem7.5.17, we have an insight that the smoother f is the faster the Fourier coefficients will converge to zero. The rate at which the Fourier coefficients tend to zero will be measured by checking if

$$\sum_{n=-\infty}^{\infty} n^{2m}|\widehat{f}(n)|^2 < \infty$$

for positive integers m.

Let C_p^m denote the set of functions on $\mathbb{R}$ such that $f,\ f',\ \cdots,\ f^{(m)}$ are all continuous and 2π periodic. Hence, if $f \in C_p^m$, then

$$f^{(j)}(-\pi) = f^{(j)}(\pi) \quad \text{for } j = 0, 1, \cdots, m.$$

Theorem 7.6.1. *Let $m \geq 1$ be an integer. Assume that $f \in C_p^{m-1}$ and $f^{(m)}$ is piecewise continuous. Then*

$$\sum_{n=-\infty}^{\infty} n^{2m}|\widehat{f}(n)|^2 = \|f^{(m)}\|^2.$$

Proof. Assume that $m = 1$. Then f is continuous on the circle and f' is piecewise continuous on $[-\pi, \pi]$. Hence, f' is integrable on $[-\pi, \pi]$ and

$$\widehat{f'}(n) = in\widehat{f}(n) \quad \text{for all } n \in \mathbb{Z}.$$

By Parseval's inequality,

$$\sum_{n=-\infty}^{\infty} n^2|\widehat{f}(n)|^2 = \|f'\|^2.$$

Assume that the theorem holds for m. Let $f \in C_p^m$ with $f^{(m+1)}$ piecewise continuous, then $f' \in C_p^{m-1}$ with $\frac{d^m}{dx^m}f' = f^{(m+1)}$ piecewise continuous. Then

$$\sum_{n=-\infty}^{\infty} n^{2(m+1)}|\widehat{f}(n)|^2 = \sum_{n=-\infty}^{\infty} n^{2m}\left|(in)\widehat{f}(n)\right|^2 = \sum_{n=-\infty}^{\infty} n^{2m}\left|\widehat{f'}(n)\right|^2 = \|f^{(m+1)}\|^2.$$

The theorem is proved by induction on m. □

Example 7.6.2. In Example7.5.16, we consider the function $f(x) = |x|$ on $[-\pi, \pi]$. The Fourier coefficients are

$$\widehat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0 \\ \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0 \end{cases}$$

Hence,

$$\sum_{n=-\infty}^{\infty} n^2|\widehat{f}(n)|^2 = 2\sum_{n=1,\ odd}^{\infty} n^2\frac{4}{\pi^2 n^4} = \frac{8}{\pi^2}\sum_{n=1,\ odd}^{\infty}\frac{1}{n^2}.$$

It is easy to check that $f \in C_p^0$ and $f'(x) = sign(x)$ is piecewise continuous. Also, we can compute that $\|f'\|^2 = 1$. This also implies that

$$\sum_{n=1,\ odd}\frac{1}{n^2} = \frac{\pi^2}{8}.$$

7.7 Applications

In the present section, we will use the Fourier series to solve an PDE problem.

■ Heat Equation

We consider the heat equation on the domain $(0, 1)$ satisfying

$$u_t(t,x) - u_{xx}(t,x) = 0 \qquad x \in [0,1],\ t \geq 0 \tag{7.9}$$
$$u(t,0) = u(t,1) = 0 \qquad t \geq 0 \tag{7.10}$$
$$u(0,x) = f(x) \in C^2([0,1]) \quad 0 \leq x \leq 1 \tag{7.11}$$

We want to look for special solutions of the form

$$u(t,x) = A(t)B(x).$$

The heat equation implies that

$$A'(t)B(x) - A(t)B''(x) = 0.$$

Hence,

$$\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} = \lambda.$$

The number λ is a constant since it is independent of both x and t. Then we have

$$A(t) = e^{\lambda t} \quad \text{and} \quad B(x) = b_1 e^{\sqrt{\lambda}x} + b_2 e^{-\sqrt{\lambda}x}.$$

From the boundary condition(7.10), we have $B(0) = B(1) = 0$. Then $B(x)$ is a 1-periodic function and hence $\lambda < 0$ and $\sqrt{|\lambda|}$ is an integer multiple of 2π. Set $\lambda = -4\pi^2 n^2$ for $n \in \mathbb{N}$. Let

$$A_n(t) = e^{-4\pi^2 n^2 t} \quad \text{and} \quad B_n(x) = b_{1n} e^{2\pi i n x} + b_{2n} e^{-2\pi i n x}.$$

The for every $n \in \mathbb{N}$, the function

$$u_n(t,x) = A_n(t)B_n(x) = e^{-4\pi^2 n^2 t}\left(b_{1n} e^{2\pi i n x} + b_{2n} e^{-2\pi i n x}\right), \quad b_{1n}, b_{2n} \in \mathbb{C}$$

satisfies (7.9) and (7.10). Since the heat equation is linear, the linear combination

$$u(t,x) = \sum_{n=-\infty}^{\infty} A_n(t)B_n(x) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

also solves (7.9) and (7.10). To determine whether $u(t,x)$ satisfies (7.11), setting $t = 0$ and

$$f(x) = u(0,x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

where $a_n = \widehat{f}(n) = \displaystyle\int_0^1 f(x)e^{-2\pi inx}\,dx$ are the Fourier coefficients of f.
Since f is a twice continuously differentiable function, the Fourier coefficients $a_n's$ are bounded. Also, for every $t > 0$, $e^{-4\pi^2 n^2 t}$ decays repidly as n tends to infinity. Hence the series

$$u(t,x) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx}$$

converges for every $t > 0$. Thus, the above series solves (7.9), (7.10) and (7.11). In fact, $u \in C^2$.

Question: Does $u(t,x)$ converge to $f(x)$ as t tends to 0?
That is,

$$\begin{aligned}
\lim_{t\to 0} u(t,x) &= \lim_{t\to 0}\lim_{N\to\infty} \sum_{n=-N}^{N} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx} \\
&\overset{??}{=} \lim_{N\to\infty}\lim_{t\to 0} \sum_{n=-N}^{N} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx} \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} a_n e^{2\pi inx} \\
&= f(x).
\end{aligned}$$

Since f is twice continuously differentiable, $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| = \sum_{n\in\mathbb{Z}} |a_n| < \infty$. For given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\sum_{|n|\geq N_0} |a_n| < \dfrac{\varepsilon}{3}$. We have

$$\left| f(x) - \sum_{|n|<N_0} a_n e^{2\pi inx} \right| < \frac{\varepsilon}{3}$$

for every $x \in [0,1]$. Choose $\delta > 0$ such that $0 < t < \delta$, then

$$\left| \sum_{|n|<N_0} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx} - \sum_{|n|<N_0} a_n e^{2\pi inx} \right| < \frac{\varepsilon}{3}$$

for every $x \in [0,1]$. Then for $0 < t < \delta$,

$$\begin{aligned}
|f(x) - u(t,x)| &\leq \left| f(x) - \sum_{|n|<N_0} a_n e^{2\pi inx} \right| + \left| \sum_{|n|<N_0} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx} - \sum_{|n|<N_0} a_n e^{2\pi inx} \right| \\
&\quad + \left| \sum_{|n|\geq N_0} a_n e^{-4\pi^2 n^2 t} e^{2\pi inx} \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore, $u(t,x)$ converges to $f(x)$ uniformly on $[0,1]$ as t tends to 0.

Question: Is the solution of (7.10) and (7.11) unique?

Suppose that u_1 and u_2 are solutions of (7.10) and (7.11). Let $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} v_t(t,x) - v_{xx}(t,x) &= 0 \quad && x \in [0,1],\ t \geq 0 \\ v(t,0) = v(t,1) &= 0 && t \geq 0 \\ v(0,x) &= 0 && 0 \leq x \leq 1 \end{aligned}$$

Define $w(t,x) = e^{-t}v(t,x)$. Then

$$\begin{aligned} w_t(t,x) - w_{xx}(t,x) + w(t,x) &= 0 \quad && x \in [0,1],\ t \geq 0 \\ w(t,0) = w(t,1) &= 0 && t \geq 0 \\ w(0,x) &= 0 && 0 \leq x \leq 1 \end{aligned}$$

Claim: $w(t,x) \leq 0$ for $t \geq 0$ and $0 \leq x \leq 1$.

Suppose the contrary, there exists $t_0 > 0$ and $0 < x_0 < 1$ such that $w(t_0,x_0) > 0$. Since $w(t_0,x)$ is continuous on $\{t_0\} \times [0,1]$, we may assume that x_0 such that $w(t_0,x_0) = \max_{0 \leq x \leq 1} w(t_0,x)$. Then

$$w_{xx}(t_0,x_0) \leq 0.$$

Therefore, $w_t(t_0,x_0) \leq -w(t_0,x_0) < 0$. We have

$$\max_{0 \leq x \leq 1} w(t,x) > 0 \quad \text{for all } 0 \leq t \leq t_0.$$

We can repeat the above argument on $[0,t_0] \times [0,1]$ until the process goes back to the initial time $t = 0$. It will implies that $\max_{0 \leq x \leq 1} w(0,x) > 0$ and obtain a contradiction.

The claim $w(t,x) \leq 0$ shows that $v(t,x) \leq 0$. On the other hand, the same argument also holds with v replaced by $-v$. We will obtain that $v(t,x) \geq 0$ and hence $v(t,x) \equiv 0$. This proves that the solution of (7.10) and (7.11) is unique.

Homework

Homework 1

1. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed vector space and $A \subseteq M$. Check that $\big(C_b(A; V), \|\cdot\|_\infty\big)$ is a normed vector space.

2. Let (M, d) be a metric space and $A \subseteq M$. Define $\mathbb{R}^A := \big\{f : A \to \mathbb{R}\big\}$ = the set of all real-valued functions defined on $\mathbb{R}$. Prove that $\Big(C_b\big(A; \mathbb{R}\big), \|\cdot\|_\infty\Big)$ is closed in $\mathbb{R}^A$.

3. Let $U = \Big\{f \in C\big((0,1); \mathbb{R}\big) \;\Big|\; f(x) > 0 \text{ for every } x \in (0,1)\Big\}$. Determine whether U is relatively open in $\big(C_b((0,1); \mathbb{R}), \|\cdot\|_\infty\big)$.

4. Let $f_n(x) = \displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for $n = 1, 2, \cdots$ and $f(x) = \sin x$.

   (a) Determine whether $\{f_n\}_{n=1}^\infty$ converges to f in $\Big(C_b\big([0,1]; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

   (b) Determine whether $\{f_n\}_{n=1}^\infty$ converges to f in $\Big(C_b(\mathbb{R}; \mathbb{R}), \|\cdot\|_\infty\Big)$.

   (c) Prove that the set $\big\{f, f_1, f_2, f_3, \cdots\big\}$ is compact in $\Big(C_b([0,1]; \mathbb{R}), \|\cdot\|_\infty\Big)$.

5. Let $f_n(x) = \begin{cases} 0, & x \in (-\infty, n-1) \\ x - (n-1), & x \in [n-1, n] \\ (n+1) - x, & x \in [n, n+1] \\ 0, & x \in (n+1, \infty) \end{cases}$ for $n = 1, 2, \cdots$ and let $\mathbf{0}$ be the zero element in $\Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

   (a) Prove that every f_n is in the unit ball $\overline{B(\mathbf{0}, 1)} \subset \Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

   (b) Prove that the sequence $\{f_n\}_{n=1}^\infty$ does not contain a convergent subsequence in $\Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

   (c) Prove that the set $\{f_1, f_2, f_3, \cdots\}$ is closed in $\Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

   (d) Prove that $\overline{B(\mathbf{0}, 1)}$ is closed and bounded in $\Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$, but is not compact in $\Big(C_b\big(\mathbb{R}; \mathbb{R}\big), \|\cdot\|_\infty\Big)$.

6. Let $f \in C(\mathbb{R}^n; \mathbb{R})$. Prove that $\big(f \circ \phi\big)(x) = f\big(\phi\big)(x) \in C_b\big([a,b]; \mathbb{R}\big)$ for every $\phi \in C_b\big([a,b]; \mathbb{R}^n\big)$.

7. Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$ on $[0,1]$. Show that $\mathcal{F} = \{f_n \,|n \in \mathbb{N}\}$ is not equicontinuous.

Lecture Note :(Page 200)

8. Problem 5.13
9. Problem 5.14(1)(3)

Homework 2

Part I:

1. Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a Banach space and $A \subseteq M$ be a countable subset. Suppose that $\{f_k\}_{k=1}^{\infty}$ be pointwise compact on A. Prove that $\{f_k\}_{k=1}^{\infty}$ contains a subsequence which converges (pointwise) on A.
2. Let (M, d) be a metric space and $A \subseteq M$, $(V, \|\cdot\|)$ be a normed space and $K \subseteq M$ be a compact subset. Prove "directly" (without using the "contradiction argument'' 反証法) that if B is precompact in $\big(C(K; V), \|\cdot\|_\infty\big)$, then B is equicontinuous.
3. Fix $N \in \mathbb{N}$. Let $\mathcal{F} = \Big\{P(x) \;\Big|\; P(x) = \sum_{k=0}^{N} a_k x^k, \text{ where } -1 \le a_0, a_1, \cdots, a_N \le 1\Big\}$ be the collection of all polynomials of degree $\le N$ with coefficients in $[-1, 1]$. Prove that $\mathcal{F}$ is equicontinuous on any bounded set in $\mathbb{R}$.
4. Suppose that $\{f_n\}_{n=1}^{\infty}$ be a sequence of twice differentiable functions on $[0, 1]$ such that $f_n(0) = 0$, $|f_n'(0)| < 1$ and $|f_n''(x)| \le M$ for all $x \in [0, 1]$ and every $n \in \mathbb{N}$.
   (a) Suppose that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0, 1]$, then it also converges uniformly on $[0, 1]$.
   (b) Prove that even $\{f_n\}_{n=1}^{\infty}$ itself does not converge pointwise on [0,1], it still contains a uniformly convergent subsequence on $[0, 1]$.

<u>Lecture Note :(Page 200)</u>

5. Problem 5.15 (1)(3)
6. Problem 5.16
7. Problem 5.20 (in this problem $A = K$)

Part II:

1. Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \cdots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?
2. Let $a < b < c$ and $B \subseteq C\big([a, c]; \mathbb{R}\big)$. Suppose that B is equicontinuous on $[a, b]$ and on $(b, c]$ respectively. Determine whether B is equicontinuous on $[a, c]$.
3. Let $B \subseteq C^1\big([a, b]; \mathbb{R}\big)$.
   (a) If there exists $M > 0$ such that $|f'(x)| < M$ for every $f \in B$ and $x \in (a, b)$, prove that B is equicontinuous on $[a, b]$.
   (b) Determine whether the converse of (a) still holds.

<u>Lecture Note:(Page 201)</u>

4. Problem 5.18(2)

Homework 3

Part I:

1. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are integrable on $[a, b]$, and put
$$F_n(x) = \int_a^x f_n(t)\, dt$$
for $x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

2. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \dfrac{1}{(x-n)^2+1}$.
   (a) Prove that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and converges to 0 pointwise.
   (b) Prove that there exists no subsequence of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly.
   (c) Which hypothesis of Arzelà-Ascoli theorem is not satisfied and show your assertion.

3. Check that each of the following families of real-valued functions defined on the given set is an algebra.
   (a) The collection of simple functions defined on $[a, b]$.
   (b) $\mathcal{P}\big(K\big)$ denote the collection of polynomials defined on $K \subseteq \mathbb{R}^n$.
   (c) $\mathcal{P}_{\text{even}}\big([a, b]\big)$ in Example 5.84

4. Prove that $\Big(C_b\big([0, 1]; \mathbb{R}\big), \|\cdot\|_\infty\Big)$ is separable. (That is, $C_b\big([0, 1]; \mathbb{R}\big)$ contains a countable dense subset.)

5. Let $a > 0$. Prove that there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that $P_n(0) = 0$ and $P_n(x) \to |x|$ uniformly on $[-a, a]$.

Lecture Note :(Page 202)

6. Problem 5.23 (Hint: (1) Use the Weierstrass Theorem to show that $\int_0^1 f^2(t)\, dt = 0$.)

7. Problem 5.26

Part II:

1. For every $n \in \mathbb{N}$, define $Q_n(x) = c_n(1-x^2)^n$ where c_n is chosen so that $\displaystyle\int_{-1}^{1} Q_n(x)dx = 1$.
   (a) Show that $c_n < \sqrt{n}$ by proving that
$$\int_{-1}^{1} (1-x^2)^n\, dx > \frac{1}{\sqrt{n}} \quad \text{for every } n = 1, 2 \cdots .$$

(b) Prove that for every $0 < \delta < 1$ and $\delta \le |x| \le 1$,

$$Q_n(x) \le \sqrt{n}(1-\delta^2)^n \quad \text{(Hint: show that } (1-x^2)^n \ge 1 - nx^2\text{)}$$

and prove that for any given $0 < \delta < 1$, $Q_n(x) \to 0$ uniformly on $\left\{x \mid \delta \le |x| \le 1\right\}$.

(c) Let f be a continuous function on $[0,1]$ with $f(0) = f(1) = 0$. We extend f outside $[0,1]$ such that $f(x) = 0$ for $x \notin [0,1]$ and still called f. Define $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,dt$. Prove that $P_n(x)$ is a polynomial in x on $[0,1]$.

(d) Prove that $P_n \to f$ uniformly on $[0,1]$. [Hint: Consider

$$P_n(x) - f(x) = \int_{-1}^{1} \left[f(x+t) - f(x)\right] Q_n(t)\,dt = \int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{1} \cdots dt \le \cdots$$

and choosing suitably small δ and sufficiently large n to estimate $|P_n(x) - f(x)| \le \varepsilon$.].

2. Let $I_j = [a_j, b_j]$ be disjoint intervals in $\mathbb{R}$ for $j = 1, \cdots k$ and $I = \bigcup_{j=1}^{k} I_j$. Let f be a continuous function defined on I. Prove that there exists a sequnce of polynomials $\{P_n\}_{n=1}^{\infty}$ such that $P_n \to f$ uniformly on I.
(Note: until now, we only know the Weierstruass theorem holds on any single interval $[a,b]$.)

3. Suppose that f is an integrable function on $[a,b]$. Given $\varepsilon > 0$.

(a) Prove that there exists a simple function g on $[a,b]$ such that

$$\int_a^b \left|f(x) - g(x)\right| dx < \varepsilon.$$

(b) Prove that there exists a continuous function h on $[a,b]$ such that

$$\int_a^b \left|f(x) - h(x)\right| dx < \varepsilon.$$

4. Let f be an integrable function on $[a,b]$. Prove that there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that

$$\int_a^b |f(x) - P_n(x)|\,dx \to 0 \quad \text{as } n \to \infty.$$

(Be careful, f may not be continuous.) (Hint: use Problem 2)

Homework 4

Part I:

1. For a given set $A \subseteq \mathbb{R}$, let $\mathcal{P}(A)$ be the collection of all polynomials defined on A and let X be a given collection of functions defined on A. Determine whether $\mathcal{P}(A)$ is dense in $(X, \|\cdot\|_\infty)$.

   (a) $A = \mathbb{R}$ and $X = C_b(\mathbb{R};\mathbb{R})$.

   (b) $A = \mathbb{R}$ and $X = \left\{ f \in C(\mathbb{R};\mathbb{R}) \,\middle|\, \lim\limits_{|x|\to\infty} f(x) = 0 \right\}$.

   (c) $A = (0,1)$ and $X = C\big((0,1);\mathbb{R}\big)$.

   (d) $A = (0,1)$ and $X = C_b\big((0,1);\mathbb{R}\big)$.

2. Let $I = [a,b]$ and $\mathcal{A}$ be the subset of $C(I;\mathbb{R})$ consisting of all piecewise linear (continuous) functions. Determine whether $\mathcal{A}$ is dense in $C(I;\mathbb{R})$.

3. Let $\mathbf{f} = (f_1, \cdots, f_n) : [a,b] \to \mathbb{R}^n$. We say that $\mathbf{f}$ is integrable on $[a,b]$ if each f_i is integrable on $[a,b]$ and define

$$\int_a^x \mathbf{f}(t)\,dt := \Big(\int_a^x f_1(t)\,dt, \cdots, \int_a^x f_n(t)\,dt \Big) = \mathbf{F}(x)$$

   for $x \in [a,b]$. Let $X := \left\{ \mathbf{f} : [a,b] \to \mathbb{R}^n \,\middle|\, \mathbf{f} \text{ is integrable on } [a,b] \right\}$. Define a map Φ on X by

$$\Phi(\mathbf{f}) = \mathbf{F}.$$

   Prove that Φ maps from $\big(C([a,b];\mathbb{R}^n), \|\cdot\|_\infty\big)$ to itself.

4. Let $f(x) = 1 + x^{1/3}$.

   (a) Show that f is a contraction mapping on $[1,8]$.

   (b) By the Contraction Mapping Theorem, there exists a fixed point $a \in [1,8]$ for f. Set $x_1 = 1$ and $x_{n+1} = f(x_n)$. Find a number $N \in \mathbb{N}$ such that for every $n \geq N$,

$$|x_n - a| \leq \frac{1}{10000}.$$

<u>Lecture Note :(Page 202)</u>

5. Problem 5.28

6. Problem 5.29

7. Problem 5.30

Part II:

1. Show that the Stone-Weierstrass Theorem fails to hold if the set K (domain of continuous functions) is not compact.

2. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a 2×2 matrix. Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   Prove that Φ is a contraction mapping on $\mathbb{R}^2$ if and only if all eigenvalues of A are between -1 and 1 (that is,$-1 < \lambda_1, \lambda_2 < 1$).

Lecture Note:(Page 203)

3. Problem 5.24

4. Problem 5.25

Homework 5

Part I:

1. Let $f(x) = x^5 - 5x - 2$. Then there exists a zero, say x_0, of $f(x)$ in $(1,2)$. Find a subinterval I of x_0 contained in $(1,2)$ such that the map $\phi(x) = x - \dfrac{f(x)}{f'(x)}$ is a contraction mapping on I.

2. Let $(\mathbb{R}^n, \|\cdot\|_{\mathbb{R}^n})$, $(\mathbb{R}^m, \|\cdot\|_{\mathbb{R}^m})$ and $(\mathbb{R}^k, \|\cdot\|_{\mathbb{R}^k})$ be normed spaces.

   (a) Let $L \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^k)$. Write the condition (definition) if L is bounded.

   (b) Let $T \in \mathcal{L}\Big(\mathbb{R}^n; \mathcal{B}(\mathbb{R}^m; \mathbb{R}^k)\Big)$. Write the condition (definition), if T is bounded.

   (c) Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ be a 2×2 matrix with real-valued entries and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.
   Define
   $$A\mathbf{x} := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix}.$$
   Prove that $A \in \mathcal{B}(\mathbb{R}^2; \mathbb{R}^2)$ and $\|A\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R}^2)} = \max(|a|, |b|)$.

   (d) Let A be defined as above and define a map T on $\mathbb{R}^3$ by
   $$T(r, s, t) = (r + s + t)A.$$
   Prove that $T \in \mathcal{B}\Big(\mathbb{R}^3; \mathcal{B}(\mathbb{R}^2; \mathbb{R}^2)\Big)$ and find $\|T\|_{\mathcal{B}\left(\mathbb{R}^3;\mathcal{B}(\mathbb{R}^2;\mathbb{R}^2)\right)}$.

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $L \in \mathcal{B}(X; Y)$. Prove that
   $$\|L\|_{\mathcal{B}(X;Y)} = \sup_{\|x\|_X=1} \|Lx\|_Y = \sup_{\|x\|_X\le 1} \|Lx\|_Y = \sup_{x\neq 0} \frac{\|Lx\|_Y}{\|x\|_X} = \inf\Big\{M > 0 \;\Big|\; \|Lx\|_Y \le M\|x\|_X\Big\}.$$

4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Prove that $\big(\mathcal{B}(X; Y), \|\cdot\|_{\mathcal{B}(X;Y)}\big)$ is a normed space.

<u>Lecture Note :(Page 205, 261)</u>

5. Problem 5.33

6. Problem 5.35

7. Problem 6.2

Part II:

1. Let X be a finite dimensional vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Prove that the identity map $id_X : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is a bounded linear map.

2. Let $K(x, y) : [0, 1]\times[0, 1] \to \mathbb{R}$ be a continuous function. Denote $X = \big(C([0, 1]; \mathbb{R}), \|\cdot\|_\infty\big)$. Define a map Φ on X by

$$[\Phi(f)](x) = \int_0^1 K(x, y)f(y)\, dy$$

for every $f \in X$.

   (a) Prove that $\Phi \in \mathcal{B}(X; X)$

   (b) Assume that $K(x, y) \geq 0$. Find $\|\Phi\|_{\mathcal{B}(X;X)}$.

Lecture Note:(Page 205, 261)

3. Problem 5.32

4. Problem 6.3

Homework 6

Part I:

1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces, and $L \in \mathcal{B}(X, Y)$, $K \in \mathcal{B}(Y, Z)$. Prove that $K \circ L \in \mathcal{B}(X, Z)$ and
$$\|K \circ L\|_{\mathcal{B}(X,Z)} \leq \|K\|_{\mathcal{B}(Y,Z)} \|L\|_{\mathcal{B}(X,Y)}.$$

2. Let $n, m \in \mathbb{N}$ and $A \in \mathcal{M}_{m\times n}(\mathbb{R})$.
   (a) If $n \geq m$ and $rank(A) = m$, prove that A is a surjective mapping.
   (b) If $n \leq m$ and $rank(A) = n$, prove that A is a injective mapping.
   (c) Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Prove that A is invertible if and only if $\det A \neq 0$.
   (d) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $\delta > 0$ such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible whenever $|a - 1| < \delta, |b - 2| < \delta, |c - 3| < \delta$ and $|d - 4| < \delta$

3. Prove that to every $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R})} = \|\mathbf{y}\|_{\mathbb{R}^n}$.

4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable on I. For every $a \in I$, define $T_a : \mathbb{R} \to \mathbb{R}$ by
$$T_a(\lambda) := \lim_{h\to 0} \frac{f(a + \lambda h) - f(a)}{h} \quad \text{for every } \lambda \in \mathbb{R}.$$
   (a) Prove that for every $a \in I$, $T_a \in \mathcal{B}(\mathbb{R}, \mathbb{R})$ and find $\|T_a\|_{\mathcal{B}(\mathbb{R},\mathbb{R})}$.
   (b) Prove that for given $\epsilon > 0$ there exists $\delta > 0$ such that if $|a - b| < \delta$, then $\|T_a - T_b\|_{\mathcal{B}(\mathbb{R},\mathbb{R})} < \epsilon$.
   (c) Define $\Phi : I \to \mathcal{B}(\mathbb{R}, \mathbb{R})$ by $\Phi(a) = T_a$. Prove that Φ is continuous on I.

5. (a) Find the matrix representative of T if $T(x_1, x_2, \cdots, x_n) = (x_1 - x_n, x_n - x_1)$.
   (b) Find the matrix representative of T if $T(1, 1) = (3, \pi, 0)$ and $T(0, 1) = (4, 0, 1)$.

6. Let $f(x) = (x^2, \sin x)$. Find the matrix representative of a linear map $T \in \mathcal{B}(\mathbb{R}; \mathbb{R}^2)$ such that
$$\lim_{h\to 0} \frac{\|f(1 + h) - f(1) - Th\|_{\mathbb{R}^2}}{|h|} = 0.$$

7. Let (S, ρ) be a metric space and $a, b, c, d, e, f : S \to \mathbb{R}$ be continuous functions. Define $A : S \to M_{2\times 3}(\mathbb{R})$ by
$$A(p) = \begin{bmatrix} a(p) & b(p) & c(p) \\ d(p) & e(p) & f(p) \end{bmatrix}.$$
for every $p \in S$. Prove that $A : S \to \mathcal{B}(\mathbb{R}^3; \mathbb{R}^2)$ is continuous on S.

Part II:

1. Give an example of normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ such that

$$\mathcal{L}(X, Y) \supsetneqq \mathcal{B}(X, Y).$$

2. Let $L \in GL(n)$. Prove that $\|L^{-1}\|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)} = \dfrac{1}{\inf_{\|x\|_{\mathbb{R}^n}=1} \|Lx\|_{\mathbb{R}^n}}$.

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two finite dimensional normed vector spaces, say $\dim X = m$ and $\dim Y = n$. In Homework 6, we have knows that $\mathcal{B}(X; Y)$ is a vector space. Prove that the dimension of $\mathcal{B}(X; Y)$ is finite and find its dimensions.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_{2\times2}(\mathbb{R})$. For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} \in M_{2\times1}(\mathbb{R})$$

For a given $\mathbf{x} \in \mathbb{R}^2$, we want to regard $A\mathbf{x}$ as a linear map from $\mathbb{R}^2$ to $\mathbb{R}$ by defining

$$\underbrace{\big(A\mathbf{x}\big)}_{\in\, \mathcal{B}(\mathbb{R}^2;\mathbb{R})} (\mathbf{y}) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} [A\mathbf{x}] = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = y_1(x_1 + 2x_2) + y_2(3x_1 + 4x_2)$$

for every $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Find $\|A\mathbf{x}\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R})}$.
(Note: 此題是指給定一個 $\mathbf{x}$，則 “$A\mathbf{x}$” 為一個 $\mathbb{R}^2$ 中的向量，此向量可視為 $\mathcal{B}(\mathbb{R}^2;\mathbb{R})$ 中的一個 linear map 定義如上。因此一個 $\mathbf{x}$, 將對應 $\mathcal{B}(\mathbb{R}^2;\mathbb{R})$ 中的一個 linaer map。此題在問如果 $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ 給定, 則對應的 linear map “$A\mathbf{x}$” 的 operator norm 應該為何? 理當會以 x_1, x_2 表達出來。)

Homework 7

Part I:

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x^4 y, xe^z)$.

   (a) Find the Jacobian matrix of f at (a, b, c).

   (b) Use the definition of differentiation to show that f is differentiable at (a, b, c) and find the matrix representation of $Df(a, b, c)$.

2. Let

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^2 + y^2}} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

   Determine whether f is differentiable at $(0, 0)$.

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and $f, g : X \to Y$ be differentiable at $a \in X$. Prove that

$$D(f + g)(a) = Df(a) + Dg(a).$$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x^4 y, xe^z)$ and $a = (1, 1, 1)$. A point moves along a curve C with equation $r(t) = (t, t^2, t^3)$ and hence $r(1) = a$.

   (a) Find the tangent vector when the point passes a.

   (b) Consider another point moves along the curve $s(t) = f\big(r(t)\big)$. Find the tangent vector when the point passes $f(a)$.

   (c) Find the matrix representation of $Df(a)$ and check that

$$\big[s'(1)\big] = \frac{d}{dt}\big[f(r(t))\big]\Big|_{t=1} = \big[Df(a)\big]\,[r'(1)].$$

5. Let S be a surface in $\mathbb{R}^3$ with equation $z = x^2 + y^2$ and $\mathbf{a} = (1, 1, 2) \in S$.

   (a) Find a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that S is the range of f and $f(1, 1) = \mathbf{a}$.

   (b) Find a linear map $L \in \mathcal{B}(\mathbb{R}^2; \mathbb{R}^3)$ such that the corresponding affine plane

$$V_{\mathbf{a}} := \underbrace{f(1, 1)}_{vector} + \underbrace{Range(L)}_{vector\ space} = \big\{\mathbf{a} + \mathbf{v} \,\big|\, \mathbf{v} \in \ Range(L)\big\}$$

   is the tangent plane of S at $\mathbf{a}$.

   (c) Show that the value of $f(x, y)$ can be approximated by the value of $f(1, 1) + \underbrace{L(x - 1, y - 1)}_{L \text{ maps the vector } (x-1,y-1)}$ as (x, y) near $(1, 1)$. That is,

$$f(x, y) = f(1, 1) + L(x - 1, y - 1) + R(x, y)$$

   where $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\|R(x, y)\|_{\mathbb{R}^3}}{\|(x - 1, y - 1)\|_{\mathbb{R}^2}} = 0.$

Lecture Note :(Page 262)

6. Problem 6.4
7. Problem 6.10

Part II:

1. Let X and Y be two vector spaces and $f : X \to Y$ be a mapping. Suppose that $\|\cdot\|_X$ is a norm on X, and $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on Y. Prove that $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_1)$ is differentiable at a if and only if $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_2)$ is differentiable at a.

2. Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in \mathcal{U}$ and $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. Define the directional derivative of f at a in the direction $\mathbf{u}$ by

$$D_{\mathbf{u}}f(a) = \lim_{h\to 0} \frac{f(a + h\mathbf{u}) - f(a)}{h}.$$

   Use the definition of derivative of f to prove that $D_{\mathbf{u}}f(a) = \nabla f(a) \cdot \mathbf{u}$.

Lecture Note:(Page 262)

3. Problem 6.5
4. Problem 6.9

Homework 8

Part I:

1. Suppose that $\mathbf{f}, \mathbf{g} : \mathbb{R} \to \mathbb{R}^m$ are differentiable at a and there exists a $\delta > 0$ such that $\mathbf{g}(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $\mathbf{f}(a) = \mathbf{g}(a) = \mathbf{0}$ and $D\mathbf{g}(a) \neq \mathbf{0}$, prove that
$$\lim_{x \to a} \frac{\|\mathbf{f}(x)\|_{\mathbb{R}^m}}{\|\mathbf{g}(x)\|_{\mathbb{R}^m}} = \frac{\|D\mathbf{f}(a)\|_{\mathscr{B}(\mathbb{R};\mathbb{R}^m)}}{\|D\mathbf{g}(a)\|_{\mathscr{B}(\mathbb{R};\mathbb{R}^m)}}$$

2. Prove that
$$f(x,y) = \begin{cases} \dfrac{x^2 + y^2}{\sin \sqrt{x^2 + y^2}} & 0 < \|(x,y)\|_{\mathbb{R}^2} < \pi \\ 0 & (x,y) = (0,0) \end{cases}$$
is not differentiable at $(0,0)$.

3. Prove that
$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
is continuous on $\mathbb{R}^2$ and has first-order partial derivatives everywhere on $\mathbb{R}^2$, but f is not differentiable at $(0,0)$.

4. Let $\mathcal{U} \subseteq \mathbb{R}^n$. Prove that the following two norms on $C^1(\mathcal{U}, \mathbb{R}^m)$ are equivalent.
$$\|f\|_1 := \sup_{x \in \mathcal{U}} \|f(x)\|_{\mathbb{R}^m} + \sup_{x \in \mathcal{U}} \|Df(x)\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)}$$
and
$$\|f\|_2 := \sup_{x \in \mathcal{U}} \Big[\|f(x)\|_{\mathbb{R}^m} + \sum_{i=1}^{m} \sum_{j=1}^{n} \Big| \frac{\partial f_i}{\partial x_j}(x) \Big| \Big].$$

Lecture Note :(Page 263)

5. Problem 6.11
6. Problem 6.12
7. Problem 6.13

Part II:

1. Let $r > 0$, $f : B(\mathbf{0}, r) \subseteq \mathbb{R}^n \to \mathbb{R}$, and suppose that there exists an $\alpha > 1$ such that $|f(\mathbf{x})| \leq \|\mathbf{x}\|_{\mathbb{R}^n}^{\alpha}$ for all $\mathbf{x} \in B(\mathbf{0}, r)$. Prove that f is differentiable at $\mathbf{0}$. What happens to this result when $\alpha = 1$?

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

Prove that all directional derivatives of f at $(0,0)$ exists but f is not differentiable at $(0,0)$.

3. Let L be a linear map of $\mathbb{R}^n \to \mathbb{R}^m$, let $g : \mathbb{R}^n \to \mathbb{R}^m$ be such that $\|g(x)\|_{\mathbb{R}^m} \leq M\|x\|^{\mathbf{2}}_{\mathbb{R}^n}$, and $f(x) = L(x) + g(x)$. Prove that $Df(0) = L$.

4. Let $f(x,y) = (xy, y/x)$ and $\mathbf{h} \in \mathbb{R}^2$ be a vector.
   (a) Compute $\left[Df\right]_B$ (with respect to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$)
   (b) Compute the matrix of $\left[Df(x,y)\right]_{B_1}$ with respect to the basis $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
   (c) Write the two expressions of $\mathbf{h}$ with respect to the two basis B and B_1 respectively.
   (d) In Problem(c), we have two expressions of $\mathbf{h}$, say $\left[h_1, h_2\right]_B$ and $\left[u_1, u_2\right]_{B_1}$. Show that $\left[Df(1,1)\right]_B \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}_B$ and $\left[Df(1,1)\right]_{B_1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{B_1}$ represent the same vector in $\mathbb{R}^2$.

Homework 9

Part I:

1. Let $f(x,y) = x^2y$ and $g(s,t) = (t-s^2, ts^2)$. Define $h(s,t) = (f\circ g)(s,t)$.

   (a) Find $[Df(x,y)]$, $[Dg(s,t)]$ and $[Dh(s,t)]$.

   (b) Check that
   $$[Dh(s,t)] = [Df(g(s,t))][Dg(s,t)]$$
   .

2. Let $f(x,y) = xe^y$.

   (a) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at $(1,0,1)$.

   (b) Define $F(x,y) = (x,y,f(x,y))$. It is easy to see that the range of F is equal to the graph of f. Let $\mathbf{e}_1 =< 1,0 >$ and $\mathbf{e}_2 =< 0,1 >$. Find $(DF)(1,0)\mathbf{e}_1$ and $(DF)(1,0)\mathbf{e}_2$.

   (c) Suppose that $\mathbf{n}\in\mathbb{R}^3$ is the normal vector to the tangent plane in problem(a). Prove that for any vector $\mathbf{v}\in\mathbb{R}^2$, $(DF)(1,0)\mathbf{v}\perp\mathbf{n}$.

3. For every $\mathbf{z}\in\mathbf{R}^{n+m}$, we express $\mathbf{z} = (\mathbf{x},\mathbf{y}) = (x_1,\cdots,x_n,y_1,\cdots,y_m)$ where $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^m$. Suppose that $f:\mathbb{R}^{n+m}\to\mathbb{R}^m$ is differentiable everywhere and denote
$$[D_{\mathbf{x}}f(\mathbf{x},\mathbf{y})] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix}(\mathbf{x},\mathbf{y}) \text{ and } [D_{\mathbf{y}}f(\mathbf{x},\mathbf{y})] = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m}\end{bmatrix}(\mathbf{x},\mathbf{y}).$$

   (a) Define $F(\mathbf{x},\mathbf{y}) = \big(\mathbf{x}, f(\mathbf{x},\mathbf{y})\big):\mathbb{R}^{n+m}\to\mathbb{R}^{n+m}$. Prove that F is differentiable everywhere and
   $$[DF(\mathbf{x},\mathbf{y})] = \begin{bmatrix} \mathbb{I}_n & 0_{n\times m} \\ D_{\mathbf{x}}f(\mathbf{x},\mathbf{y}) & D_{\mathbf{y}}f(\mathbf{x},\mathbf{y})\end{bmatrix}$$
   where $\mathbb{I}_n$ is the $n\times n$ identity matrix and $0_{n\times m}$ is the $n\times m$ zero matrix.

   (b) If $[D_{\mathbf{y}}f(\mathbf{x},\mathbf{y})]$ is invertible, prove that $[DF(\mathbf{x},\mathbf{y})]$ is invertible.

4. If $f,g:\mathbb{R}^n\to\mathbb{R}$ are differentiable real functions, prove that $\nabla(fg) = f\nabla g + g\nabla f$ and $\nabla(1/f) = -\dfrac{\nabla f}{f^2}$.

5. Let $f:\mathcal{U}\subseteq\mathbb{R}^2\to\mathbb{R}$ be differentiable at $(x_0,y_0)\in\mathcal{U}$ and $z_0 = f(x_0,y_0)$. Define $F:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ by $F(x,y,z) = z - f(x,y)$. Use the gradient of F at (x_0,y_0,z_0) to prove the equation of the tangent plane to the graph of f at (x_0,y_0,z_0) is
$$z = z_0 + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0).$$

6. Let $f(x,y,z,w) = e^x\sin(\pi y) + \dfrac{z}{w}$. Use the linear approximation of f at $(0,1,2,3)$ to estimate $f(0.1,0.9,1.8,2.7)$.

Lecture Note :(Page 264)

7. Problem 6.14

Part II:

1. Suppose that f is a real-valued function defined in an open set $\mathcal{U} \subseteq \mathbb{R}^n$, and that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded in $\mathcal{U}$. Prove that f is continuous on $\mathcal{U}$.

2. Suppose that I is a nonempty, open interval and that $f : I \to \mathbb{R}^m$ is differentiable on I. If $f(I) \subseteq \partial B(0, r)$ for some fixed $r > 0$, prove that $f(t)$ is orthogonal to $f'(t)$ for all $t \in I$.

3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined by $L(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x + 2y, 3x + 4y)$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function satisfying

$$f(x, y) = L(x, y) + o\big(\|(x, y) - (0, 0)\|_{\mathbb{R}^2}\big) \quad \text{as} \ \ (x, y) \to (0, 0).$$

Prove that there exists $r > 0$ such that f is one-to-one in $B\big((0, 0), r\big)$.

Lecture Note:(Page 264)

4. Problem 6.15

Homework 10

Part I:

1. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable with A convex, and suppose $\|\nabla f(x)\|_{\mathbb{R}^n} \leq M$ for $x \in A$.

   (a) Prove that $|f(x) - f(y)| \leq M\|x - y\|_{\mathbb{R}^n}$.

   (b) Is the result still true if A is not convex?

2. Let $U \subseteq \mathbb{R}^n$ be a connected and open set. Suppose that $f : U \to \mathbb{R}^m$ is differentiable on U and $Df(x) = 0$ for every $x \in U$. Prove that f is a constant function.

3. Suppose that V is convex and open in $\mathbb{R}^n$ and that $\mathbf{f} : V \to \mathbb{R}^n$ is differentiable on V. If there exists an $\mathbf{a} \in V$ such that $D\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{a})$ for all $\mathbf{x} \in V$, prove that there exist a linear function $S \in \mathcal{B}(\mathbb{R}^n; \mathbb{R}^n)$ and a vector $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) = S(\mathbf{x}) + \mathbf{c}$ for all $\mathbf{x} \in V$.

4. If $f(x, y)$ is differentiable on a connected open set $S \subseteq \mathbb{R}^2$ and $\dfrac{\partial f(x, y)}{\partial x} = 0$ for all $(x, y) \in S$.

   (a) Show that if S is convex, then f is independent of x on S.

   (b) Show that the result of Part (a) is false if S is not convex.

5. For a point (r, θ, ϕ) in $\mathbb{R}^3$, define

$$F(r, \theta, \phi) = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r \cos\theta).$$

   At what point (r_0, θ_0, ϕ_0) in $\mathbb{R}^3$ does the Inverse Function Theorem apply to the mapping F?

6. Let $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. Show that the map $(x, y) \to (u, v)$ is locally invertible at all point $(x, y) \neq (0, 0)$.

7. Let $F(x, y, z) = (x + y + z,\ x^2y,\ xyz)$. Determine whether F has an inverse near the point $(1, 1, 0)$. If the inverse function F^{-1} exists and is defined on an open neighborhood of $F(1, 1, 0)$, find its derivative at $F(1, 1, 0)$.

Part II:

1. If $f : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function and if the directional derivative $D_{\mathbf{u}}f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^n$ and every direction $\mathbf{u}$. Prove that f is a constant function.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that for each unit vector $\mathbf{u} \in \mathbb{R}^n$, the directional derivative $D_{\mathbf{u}}f(\mathbf{a} + t\mathbf{u})$ exists for $t \in [0, 1]$. Prove that

$$f(\mathbf{a} + \mathbf{u}) - f(\mathbf{a}) = D_{\mathbf{u}}f(\mathbf{a} + t\mathbf{u})$$

   for some $t \in (0, 1)$.

3. Let $F(x,y) = (x^3y,\ x^5 - y^5,\ x^2 + y)$. Prove that there exists an open neighborhood U of $(1,1)$, an open neighborhood V of $(1,0)$ and a function $f : V \to \mathbb{R}$ such that the set $F(U)$ is equal to the graph of f on V. (**DO NOT** try to find the exact expression of f)
   (*Hint: (i) define $u(x,y) = x^3y$ and $v(x,y) = x^5 - y^5$, then the map $(x,y) \to (u,v)$ satisfies the Inverse Function Theorem; (ii) use the inverse map to solve x, y in terms of u, v theoretically (iii) define $f(u,v) = x^2(u,v) + y(u,v)$.*)

Lecture Note:(Page 267)

4. Problem 6.28 (6)(7)

Homework 11

Part I:

1. Investigate whether the system

$$\begin{aligned} u(x,y,z) &= x + xyz \\ v(x,y,z) &= y + xy \\ w(x,y,z) &= z + 2x + 3z^2 \end{aligned}$$

can be solved for x, y, z interms of u, v, w near $(0,0,0)$.

2. Give an example of a continuously differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property that there is no open subset U of $\mathbb{R}^n$ for which $F(U)$ is open in $\mathbb{R}^n$.
(*Hint: Do Problem 6.16 first*).

3. Suppose that the function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\psi : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable. Define, for $(x,y,z) \in \mathbb{R}^3$

$$F(x,y,z) = \big(\phi(x,y,z), \psi(x,y,z), \phi^2(x,y,z) + \psi^2(x,y,z)\big).$$

   (a) Explain analytically why there is no point $(x_0, y_0, z_0) \in \mathbb{R}^3$ at which the assumptions of Inverse Function Theorem hold for the mapping F.

   (b) Explain geometrically why there is no point $(x_0, y_0, z_0) \in \mathbb{R}^3$ at which the conclusion of the Inverse Function Theorem holds for the mapping F.

4. Let $\mathbf{f}(x,y) = (e^x \cos y, e^x \sin y)$ be a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$. Let $\mathbf{a} = (0, \pi/3)$ and $b = \mathbf{f}(\mathbf{a})$. Let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$ such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$.

   (a) Use the Inverse Function Theorem to find $D\mathbf{g}(\mathbf{b})$.

   (b) Find an explicit formula for $\mathbf{g}$ and compute $D\mathbf{g}(\mathbf{b})$ directly and check whether it equals the answer of Problem(a).

5. Let $L = (L_1, L_2) : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by

$$L(x_1, x_2, x_3, x_4, x_5) = (x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5,\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + 5x_5).$$

   (a) Show that

$$\begin{bmatrix} \dfrac{\partial L_1}{\partial x_2} & \dfrac{\partial L_1}{\partial x_5} \\[2ex] \dfrac{\partial L_2}{\partial x_2} & \dfrac{\partial L_2}{\partial x_5} \end{bmatrix}$$

   is invertible.

   (b) Find two maps $f_2, f_5 : \mathbb{R}^3 \to \mathbb{R}$ such that for every $(x_1, x_3, x_4) \in \mathbb{R}^3$

$$L\big(x_1, f_2(x_1, x_3, x_4), x_3, x_4, f_5(x_1, x_3, x_4)\big) = (1, 2)$$

   (That is, if the differentiation matrix with respect to x_2 and x_5 is invertible, then the preimage $L^{-1}\big((1,2)\big)$ can be expressed as the graph of a function $\mathbf{f} = (f_2, f_5)$ of the variables x_1, x_3 and x_4.)

6. Let U be an open set in $\mathbb{R}^2$ and $f : U \to \mathbb{R}^3$ be a continuous differentiable function on U defined by $f(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ where

$$\begin{aligned} x(u, v) &= u^2 v \\ y(u, v) &= u + v \\ z(u, v) &= uv^3 \end{aligned}$$

Let $S \subseteq \mathbb{R}^3$ be the range of f, then $f(1, 1) = (1, 2, 1) \in S$. Use the Inverse Function Theorem to show that, near $(1, 2, 1)$, S can be expressed as the graph of a function.
That is, there exists an open set V of $(1, 2)$ and a function $\phi : V \to \mathbb{R}$ such that $\phi(1, 2) = 1$ and near $(1, 2, 1)$, $S = \big\{\big(x, y, \phi(x, y)\big) \mid (x, y) \in V\big\}$.

Lecture Note :(Page 267)

7. Problem 6.16

Part II:

1. Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies that f is differentiable everywhere, $Df(x, y)$ is not continuous at $(0, 0)$ and the Inverse Function Theorem fails near $(0, 0)$.

2. ~~Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is continuous, the inverse function f^{-1} exists, but f is not an open mapping.~~

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T : X \to Y$ be a linear map. Prove that the following statements are equivalent.

   (1) The linear map T is an open mapping.

   (2) There exists a constant $K > 0$ such that, for all $y \in Y$, there exists $x \in X$ with $\|x\|_X \leq K\|y\|_Y$ such that $T(x) = y$.

4. Let $(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) \in \mathbb{R}^n$ and let $\mathbf{f} = (f_1, \cdots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\begin{aligned} f_1(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) &= r\cos(\phi_1) \\ f_2(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) &= r\sin(\phi_1)\cos(\phi_2) \\ f_3(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) &= r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ &\vdots \\ f_{n-1}(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1}) \\ f_n(r, \phi_1, \phi_2, \cdots, \phi_{n-1}) &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1}) \end{aligned}$$

Find the Jacobian of $\mathbf{f}$, $\dfrac{\partial(f_1, \cdots, f_n)}{\partial(r, \phi_1, \cdots, \phi_{n-1})}$.

Homework 12

Part I:

1. Construct a C^1 mapping $u(x,y) : \mathbb{R}^2 \to \mathbb{R}^2$, say $u = (u_1, u_2)$, such that $\left[Du(0,0)\right]$ is NOT invertible. But there exist an open neighborhood $\mathcal{U}$ of $(0,0)$ and an open neighborhood $\mathcal{V}$ of $u(0,0)$ such that $u : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto.
(Thus, we can still solve (x,y) in terms of (u_1,u_2) near $(0,0)$ even if $\left[Du(0,0)\right]$ is not invertible.)

2. Consider the transformation for spherical coordinates:

$$\begin{aligned} x(r,\phi,\theta) &= r\sin\phi\cos\theta \\ y(r,\phi,\theta) &= r\sin\phi\sin\theta \\ z(r,\phi,\theta) &= r\cos\phi \end{aligned}$$

   (a) Show that $\dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi$.

   (b) When can we solve for (r,ϕ,θ) in terms of (x,y,z)?

   (c) What happened for those point (r,ϕ,θ) which cannot be solved in terms of (x,y,z)? (Explain more details than just saying $r^2\sin\phi = 0$.)

3. For the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0, \end{aligned}$$

use the "Implicit Function Theorem" to determine whether any three of the four variables x, y, z, u can be solved in terms of the remaining one.

4. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = x^2y + e^x + z.$$

   (a) Show that there exists a differentiable function $g(y,z)$ in some neighborhood of $(1,-1)$ in $\mathbb{R}^2$ such that $g(1,-1) = 0$ and

$$f\big(g(y,z), y, z\big) = 0$$

   (b) Find $\dfrac{\partial g}{\partial y}(1,-1)$ and $\dfrac{\partial g}{\partial z}(1,-1)$.

5. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $f : X \to Y$. Suppose that f is twice differentiable at $a \in X$. Let $u, v_1, v_2 \in X$ and $c \in \mathbb{R}$. Prove that

   (a)

$$D^2f(a)(cv_1 + v_2)(u) = cD^2f(a)(v_1)(u) + D^2f(a)(v_2)(u)$$

(b)
$$D^2 f(a)(u)(cv_1 + v_2) = cD^2 f(a)(u)(v_1) + D^2 f(a)(u)(v_2)$$

6.

> **<u>Definition:</u>** Let X be a vector space. A map $B : X \times X \to \mathbb{R}$ is said to be a "**bilinear form**" if B satisfies
>
> (i) $B(cu_1 + u_2, v) = cB(u_1, v) + B(u_2, v)$ for every $u, v_1, v_2 \in X$ and $c \in \mathbb{R}$; and
>
> (ii) $B(u, cv_1 + v_2) = cB(u, v_1) + B(u, v_2)$ for every $u_1, u_2, v \in X$ and $c \in \mathbb{R}$.

Suppose that $\dim X = n$, $\{e_1, \cdots, e_n\}$ is a basis of X and $B : X \times X \to \mathbb{R}$ is a bilinear form. Prove that there exists a $n \times n$ matrix A such that for every $u = \sum_{i=1}^{n} u_i e_i$ and $v = \sum_{j=1}^{n} v_j e_j$

$$B(u, v) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

<u>Lecture Note :(Page 264)</u>

7. Problem 6.18

Part II:

1. As Problem 1 of Part I, for the system of equations
$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0, \end{aligned}$$
in order to satisfy the equation, use the "Inverse Function Theorem" to determine whether x, y, z can be solved in term of u.

2. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be smooth and satisfy the Cauchy-Riemann equations
$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$
   (a) Show that, at (x_0, y_0), $\frac{\partial(f_1, f_2)}{\partial(x, y)} = 0$ if and only if $Df(x_0, y_0) = \mathbf{0}$ and hence that f is locally invertible if and only if $Df(x, y) \neq \mathbf{0}$.
   (b) Prove that the inverse function also satisfies the Cauchy-Riemann equations.

<u>Lecture Note:(Page 264)</u>

3. Problem 6.19

4. Problem 6.20

Homework 13

Part I:

1. Let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{a} \in U$ and $\mathbf{u} =< u_1, u_2, u_3 >\in \mathbb{R}^3$.

   (a) If f is of class C^3, prove that

   $$D^3 f(\mathbf{a})(\mathbf{u},\mathbf{u},\mathbf{u}) = \sum_{\substack{0\le k,m,n\le 3\\ k+m+n=3}} \frac{3!}{k!m!n!} \cdot \frac{\partial^3 f(\mathbf{a})}{\partial x_1^k \partial x_2^m \partial x_3^n} u_1^k u_2^m u_3^n$$

   (b) If f is of class C^r, prove that

   $$D^r f(\mathbf{a})(\underbrace{\mathbf{u},\cdots,\mathbf{u}}_{\text{r copies}}) = \sum_{\substack{0\le k,m,n\le r\\ k+m+n=r}} \frac{r!}{k!m!n!} \cdot \frac{\partial^r f(\mathbf{a})}{\partial x_1^k \partial x_2^m \partial x_3^n} u_1^k u_2^m u_3^n$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at $\mathbf{a}$. Prove that all second partial derivatives $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$ exist for $i, j = 1, \cdots, n$.

3. Let $f(x, y) = e^x \sin y$.

   (a) Use the Taylor formula for the multi-variable functions to compute the second-order Taylor polynomial for f centered at $(0, 0)$.

   (b) Use the Taylor formula for the single variable functions to compute the Taylor polynomials for e^x and $\sin y$ centered at $x = 0$ and $y = 0$ respectively. Use them to compute the second Taylor formula for f centered at $(0, 0)$.

4.

Definition: Let A be a $n \times n$ matrix. We say that A is "*positive definite*" if

$$\mathbf{u}^T A\mathbf{u} > 0$$

for every $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$ and A is "*negative definite*" if

$$\mathbf{u}^T A\mathbf{u} < 0$$

for every $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$.

Let A be a $n \times n$ matrix be positive definite. Prove that there exists $c > 0$ such that for every $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$,

$$\mathbf{u}^T A\mathbf{u} \geq c\|\mathbf{u}\|^2_{\mathbb{R}^n}.$$

Moreover, prove that the smallest eigenvalue of A satisfies this number c provided A is symmetric.

Lecture Note :(Page 265)

5. Problem 6.21(1)(2)
6. Problem 6.22(1)
7. Problem 6.25

Part II:

1. Let $f(x, y) = x^2 + 2xy^2$. Determine the point (x, y) such that the Hessian matrix $H_f(x, y)$ is positive definite, negative definite or neither.
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \begin{cases} \exp(-\frac{1}{\|\mathbf{x}\|_{\mathbb{R}^2}^2}) & \text{if } \mathbf{x} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{x} = \mathbf{0} \end{cases}$$

where $\mathbf{x} = (x_1, x_2)$. Find the kth degree Taylor polynomial for f centered at $\mathbf{0}$.

Lecture Note:(Page 265)

3. Problem 6.21(3)
4. Problem 6.28(5)

Homework 14

Part I:

1. Let $f(x,y,z) = x^2 + y^2 + z^2 - xy + yz - xz$. Find all extreme value(s) of f.

2. Let $D = [0,2] \times [0,2]$ and $f(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1) \\ 0 & \text{if } (x,y) \in D \backslash (1,1). \end{cases}$ Determine whether f is integrable over D.

3. Let $D = [0,1] \times [0,1]$ and $A = \{(x,x) \mid 0 \le x \le 1\}$ be the diagonal in D. Suppose that $f(x,y)$ be an arbitrary bounded and integrable function on D. Define
$$g(x,y) = \begin{cases} 0 & (x,y) \in A \\ f(x,y) & (x,y) \in D \backslash A. \end{cases}$$
Prove that $g(x,y)$ is also integrable over D and $\int_D f(x,y)\, dA = \int_D g(x,y)\, dA$.

4. Let $D \subseteq \mathbb{R}^n$ be a compact box, that is, $D = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Suppose that $f : D \to \mathbb{R}$ is a continuous function on D. Prove that f is integrable over D.

5. Prove Theorem 7.9 (Page 272)

6. Let $D \subset \mathbb{R}^n$ be a bounded set and $f : D \to \mathbb{R}$ be a bounded function. If f is Riemann integrable over D, prove that $|f|$ is Riemann integrable over D.

Lecture Note :(Page 266)

7. Problem 6.26

Part II:

1. Let $f(x,y,z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2$.

   (a) Find the second degree Taylor polynomial for f centered at $(0,0,0)$.

   (b) Use the Taylor theorem to explain that f has a local minimum point at $(0,0,0)$.

2. Let $D = [0,1] \times [0,1] \subset \mathbb{R}^2$. Give an example of a function f defined on D such that
$$\underline{\int}_D f(x)\, d\mathbb{A} < \overline{\int}_D f(x)\, d\mathbb{A}.$$

3. Let $D = [0,1] \times [0,1]$ and $A = \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \times [0,1] \subseteq D$.
$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \in D \backslash A. \end{cases}$$
Determine whether f is integrable over D.

Lecture Note:(Page 266)

4. Problem 6.27

Homework 15

Part I:

1. Let $A_1, \cdots, A_k \subseteq \mathbb{R}^n$ have volumes. Prove that
$$V\Big(\bigcup_{i=1}^{k} A_i\Big) \le \sum_{i=1}^{k} V(A_i).$$

2. Prove that the set $\Big\{\frac{1}{n} \mid n \in \mathbb{N}\Big\} \subset [0,1]$ has volume zero.

3. Let $f : D \to \mathbb{R}$ be integrable over D such that $\int_D |f(x)|dx = 0$. Suppose that the sets $E = \big\{x \in D \mid f(x) \neq 0\big\}$ and $E_n := \Big\{x \in D \;\Big|\; |f(x)| > \frac{1}{n}\Big\}$ have volume for every $n \in \mathbb{N}$. Prove that $V(E) = 0$.

4. Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$, $E \subset D$ have volume zero and $f : D \to \mathbb{R}$ be a bounded function. Suppose that f is continuous on $D \backslash E$. Prove that f is integrable over D.

5. Let $n < m$ and $D \subseteq \mathbb{R}^n$ be a rectangle in $\mathbb{R}^n$. Suppose that $f : D \to \mathbb{R}^m$ is of class C^1. Prove that the set $f(D) \subset \mathbb{R}^m$ has volume zero.

6. Let $P = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ be a partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that
$$\underline{\int_a^b} f(x)\, dx = \sum_{i=1}^{n} \underline{\int_{x_{i-1}}^{x_i}} f(x)\, dx.$$

7. Let $D = [a,b] \times [c,d]$ and $f_k : D \to \mathbb{R}$ be continuous for all $k \in \mathbb{N}$ such that $\{f_k\}_{k=1}^{\infty}$ converges pointwise to a continuous function $f : D \to \mathbb{R}$. Suppose that $f_k \ge f_{k+1}$ for all $k \in \mathbb{N}$. Prove that
$$\lim_{k\to\infty} \int_D f_k(x,y)\, d\mathbb{A} = \int_D f(x,y)\, d\mathbb{A}.$$
(Ref: Theorem 5.19)

Part II:

1. Let $D \subseteq \mathbb{R}^n$ be an open set with volume $V(D) > 0$ and suppose that $f : D \to \mathbb{R}$ is continuous on D. Suppose that for every continuous function $g : D \to \mathbb{R}$, we have $\int_D (fg)(x)dx = 0$. Prove that $f \equiv 0$ on D.

2. Prove that a Cantor set has volume zero.

3. Let $A, A_1, A_2, \cdots \subseteq \mathbb{R}^n$ be bounded sets with volume.

   (a) Prove that ∂A has volume zero.

(b) Suppose that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$, $A = \bigcup_{k=1}^{\infty} A_k$. Prove that $\lim_{k\to\infty} V(A_k) = V(A)$.

Lecture Note:(Page 340)

4. Problem 7.4(2)

Homework 16

Part I:

1. Evaluate

   (a) $\iint_S (x+3y^3)\,d\mathbb{A}$, where S is the upper half ($y \geq 0$) of the unit disc $x^2+y^2 \leq 1$. (*Ans:* $\frac{4}{5}$)

   (b) $\iint_S (x^2 - \sqrt{y})\,d\mathbb{A}$, where S is the region between the parabola $x = y^2$ and the line $x = 2y$. (*Ans:* $\frac{32}{35}(5 - \sqrt{2})$)

   (c) Find the volume of the region above the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$ and below the surface $z = 6xy(1-x-y)$. (*Ans:* $\frac{1}{20}$)

   (d) Let $S \subset \mathbb{R}^3$ be the region between the paraboloid $z = x^2+y^2$ and the plane $z = 1$. Express the triple integral $\iiint_S f\,dV$ as an iterated integral with the order of integration (i) z, y, x; (ii) y, z, x; (iii) x, y, z.

2. Find the volume of the region inside both the sphere $x^2+y^2+z^2 = 4$ and the cylinder $x^2+y^2 = 1$. (*Ans:* $4\pi(\frac{8}{3} - \sqrt{3})$)

3. Calculate $\int_S (x+y)^4(x-y)^{-5}\,d\mathbb{A}$ where S is the square $-1 \leq x+y \leq 1$, $1 \leq x-y \leq 3$. (*Ans:* $\frac{4}{81}$)

4. Let S be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2-y^2 = 1$, and $x^2-y^2 = 4$. Compute $\int_S (x^2+y^2)\,d\mathbb{A}$. (*Ans:* 3)

5. Use cylindrical coordinates to evaluate the triple integral

$$\iiint_E x\,dV$$

   where E is the solid bounded by the planes $z = 0$ and $z = x+y+5$ and the cylindrical shells $x^2+y^2 = 4$ and $x^2+y^2 = 9$. (*Ans:* $\frac{65\pi}{4}$)

6. Use spherical coordinates to evaluate the triple integral

$$\iiint_H (x^2+y^2)\,dV$$

   where H is the solid that is bounded below by the xy-plane, and bounded above by the sphere $x^2+y^2+z^2 = 1$. (*Ans:* $\frac{4\pi}{15}$)

Lecture Note :(Page 342)

7. Problem 7.14 (1) (2)

Part II:

1. Find the centroid of the portion of the ball $x^2 + y^2 + z^2 \leq 1$ lying in the first octant $(x, y, z \geq 0)$. Note that the centroid $(\bar{x}, \bar{y}, \bar{z})$ on D is defined by $\bar{x} = \int_D x\, dV$ and similar for $\bar{y}$ and $\bar{z}$. (*Ans:* $(\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$)

2. Let $0 < a < b < \infty$ and $p \in \mathbb{R}$. Define $D := \left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; a \leq \|\mathbf{x}\|_{\mathbb{R}^n} \leq b\right\}$. Compute

$$\int_D \frac{1}{\|\mathbf{x}\|_{\mathbb{R}^n}^p}\, d\mathbf{x}.$$

Lecture Note:(Page 341)

3. Problem 7.9

4. Problem 7.14 (3)

Homework 17

Part I:

1. Compute $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\, dx = \sqrt{2\pi}|\sigma|$ for every $\sigma \neq 0$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function and $\{B_k\}_{k=1}^{\infty}$ be any bounded sequence of open sets with volume which satisfies

   (i) $B_k \subseteq B_{k+1}$ for every $k \in \mathbb{N}$

   (ii) For any $R > 0$, the ball $B(0, R) \subseteq B_k$ when k is sufficiently large.

   Prove that $\lim_{k\to\infty} \int_{[-k,k]^n} f(x)\, dx$ converges if and only if $\lim_{k\to\infty} \int_{B_k} f(x)\, dx$ converges. Moreover, the above limits are equal if they exist.

3. Let $p \in \mathbb{R}$ and $D = \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{R}^n} \geq 1\}$. Find the range of p such that the integral

   $$\int_D \frac{1}{\|x\|_{\mathbb{R}^n}^p}\, dx$$

   converges.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $f(x) = O(e^{-\|x\|_{\mathbb{R}^n}})$ as $\|x\| \to \infty$. Prove that f is integrable over $\mathbb{R}^n$.

5. Determine whether the following improper integrals converge, and evaluate them if they do.

   (a) $\iint_{x,y>0} \frac{1}{(1 + x^2 + y^2)^2}\, d\mathbb{A}$. (*Ans:* $\frac{1}{4}\pi$)

   (b) $\iint_{x>0} x e^{-(x^2+y^2)}\, d\mathbb{A}$. (*Ans:* $\sqrt{\pi}/2$)

   (c) $\iint_{x^2+y^2<1} \frac{x^2}{(x^2 + y^2)^2}\, d\mathbb{A}$. (*Ans: Diverges*)

Lecture Note :(Page 342)

6. Problem 7.12

7. Problem 7.15

Exams

高等微積分 (二) 第一次期中考

1102 Advanced Calculus (2) Midterm 1

April 7, 2022

Affiliation: __________ Name: __________ Student ID: __________

1. 本期中考共 10 道題 (含加分題)，總計 130 分，若總分超過 100 分以 100 計。
2. 考試過程中不可使用計算機、手機、3C 產品、參考書、筆記及個人計算紙。如需計算紙，請利用考卷背面。
3. 每道題都需寫出完整的過程，並請保持試卷乾淨及答案清楚。

1. (25 points) Let $A = [0,1] \times [0,1]$ be a closed square in $\mathbb{R}^2$, and $K : A \to \mathbb{R}$ be continuous on A. Define
$$T\big(f\big)(x) = \int_0^1 K(x,y)f(y)\;dy,$$
where f is a real-valued function defined on $[0,1]$ such that the integral makes sense.
   (a) (10 points) For a family of functions $\mathcal{F}$ consisting of f such that $T(f)$ is well-defined and $|f(y)| \le M$ for all $y \in [0,1]$, let $\mathcal{G} = T(\mathcal{F})$. Show that each sequence of $\mathcal{G}$ contains a uniformly convergent subsequence.
   (b) (8 points) Let $X = \Big(\mathcal{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\Big)$. Prove that $T \in \mathcal{B}(X;X)$.
   (c) (7 points) Assume that $K(x,y) \ge 0$. Find $\|T\|_{\mathcal{B}(X;X)}$.

2. (10 points) Let $C > 0$ be a number and
$$F = \Big\{f \in \mathcal{C}\big([-1,1];\ [0,\infty)\big)\ \Big|\ f(-1) = 1 = f(1) \text{ and } \big|f(x)-f(y)\big| \le C\big|x-y\big|\ \forall x,y \in [-1,1]\Big\}.$$
Define the area function A on $\mathcal{C}\big([-1,1];\mathbb{R}\big)$ by
$$A(f) = \int_{-1}^1 f(x)\;dx.$$
Determine whether A attains its minimum on F. That is, determine whether there exists $f_0 \in F$ such that $A(f_0) = \inf\limits_{f\in F} A(f)$.

3. (15 points)
   (a) (10 points) Prove that $\Big(\mathcal{C}_b\big([a,b];\mathbb{R}\big), \|\cdot\|_\infty\Big)$ is separable. (That is, $\mathcal{C}_b\big([a,b];\mathbb{R}\big)$ contains a countable dense subset.)
   (b) (5 points) Determine whether $\mathcal{C}_b(\mathbb{R};\mathbb{R})$ contains a countable dense subset.

4. (10 points) Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\;dx = 0.$$

5. (10 points) Determine whether every continuous function in $\mathcal{C}\big([0,1];\mathbb{R}\big)$ can be uniformly approximated by a sequence of even polynomials. (Even polynomial means all its terms are of even degree.)

6. (15 points) Let the equation $x^3 - x = 0$ be given.
   (a) (6 points) Use Newton's method with $x_1 = \frac{1}{3}$ to find x_3, the third approximation to the root of the equation.
   (b) (9 points) Find an interval I containing 0 such that if we choose an arbitrary point $x_1 \in I$ as the initial point, then the Newton iterations $\{x_n\}_{n=1}^\infty$ will converge to the root 0. Explain your reason.

7. (10 points) Consider the mapping $T : \mathcal{C}\big([1,r];\mathbb{R}\big) \to \mathcal{C}\big([1,r];\mathbb{R}\big)$ defined by

$$T\big(f\big)(x) = 1 + 3\int_1^x t^2 f(t)\ dt.$$

   (a) (5 points) Find a number $r > 1$ such that T is a contraction mapping on $\mathcal{C}\big([1,r];\mathbb{R}\big)$.

   (b) (5 points) What is its fixed point?

8. (15 points)

   (a) (6 points) Let $A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{R})$ be a 3×2 matrix with real-valued entries and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Define $A\mathbf{x} := \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2 \\ 0 \end{pmatrix}$. Prove that $\|A\|_{\mathcal{B}(\mathbb{R}^2;\mathbb{R}^3)} = \max(|a|, |b|)$.

   (b) (9 points) Let (S, d) be a metric space and $a_1, a_2, \cdots, a_6 : S \to \mathbb{R}$ be continuous functions. Define $A : S \to M_{2\times 3}(\mathbb{R})$ by

$$A(p) = \begin{bmatrix} a_1(p) & a_2(p) & a_3(p) \\ a_4(p) & a_5(p) & a_6(p) \end{bmatrix}.$$

   for every $p \in S$. Prove that $A : S \to \mathcal{B}\big(\mathbb{R}^3;\mathbb{R}^2\big)$ is continuous on S.

9. (10 points) Prove that to every $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R})$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\|_{\mathcal{B}(\mathbb{R}^n;\mathbb{R})} = \|\mathbf{y}\|_{\mathbb{R}^n}$.

Bonus Problem A: (10 points) Let

$$f(x,y) = (e^{y-2} - 3, \frac{1}{4}\sin x - 1).$$

(a) (4 points) Prove that f is a contraction mapping on $E := (-\infty, 0] \times (-\infty, 0]$.

(b) (6 points) Let $\mathbf{x}_0 = (0,0)$, $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ and $\mathbf{a} \in E$ be the fixed point for f. Find $N \in \mathbb{N}$ such that for $n \geq N$,

$$\|\mathbf{x}_n - \mathbf{a}\|_{\mathbb{R}^2} < \frac{1}{100}.$$

(*Hint: To show* $\|\mathbf{x}_n - \mathbf{a}\|_{\mathbb{R}^2} \leq \frac{c^n}{1-c}\|\mathbf{x}_0 - f(\mathbf{x}_0)\|_{\mathbb{R}^2}$ *where* c *is the contraction constant.*)

Bonus Problem B: (10 points) Let $K \subset \mathbb{R}$ be a compact subset and let $\mathcal{C}^1\big(K;\mathbb{R}\big)$ be the collection of all continuously differentiable functions on K with the norm $\|\cdot\|_{\mathcal{C}^1}$ defined by

$$\|f\|_{\mathcal{C}^1} := \|f\|_\infty + \|f'\|_\infty$$

Determine whether $\mathcal{P}(K)$, the collection of all polynomials on K, is dense in $\Big(\mathcal{C}^1\big(K;\mathbb{R}\big), \|\cdot\|_{\mathcal{C}^1}\Big)$.

I will do Problem __________.

高等微積分 (二) 第二次期中考

1102 Advanced Calculus (2) Midterm 2

May 12, 2022

Affiliation: __________ Name: __________ Student ID: __________

1. 本期中考共 10 道題 (含加分題)，總計 115 分，若總分超過 100 分以 100 計。
2. 考試過程中不可使用計算機、手機、3C 產品、參考書、筆記及個人計算紙。如需計算紙，請利用考卷背面。
3. 每道題都需寫出完整的過程，並請保持試卷乾淨及答案清楚。

1. (15 points) Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be a function where $f = (f_1, \cdots, f_m)$.
   (a) (5 points) State the Mean Value Theorem for f.
   (b) (10 points) State the Inverse Function Theorem and the Implicit Function Theorem

2. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = \begin{cases} \left(\dfrac{xy}{x^2+y^2}, x+y \right) & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) (5 points) Use the definition of differentiation to prove that f is differentiable at $(1,0)$.
   (b) (5 points) Determine whether f is differentiable at $(0,0)$ and explain it.

3. (10 points) Let $f(x,y) = (xy^3, x^2+y^2, 3x+2y)$ and the range of f, $S = Range(f)$, be a surface in $\mathbb{R}^3$. Then $f(1,1) = (1,2,5) \in S$. Find the equation of the tangent plane of S at $(1,2,5)$.

4. (10 points) Let $f(x,y,z) = xy^2z^3$.
   (a) (5 points) Use the linear approximation for f at $(3,2,1)$ to estimate $f(3.1, 1.8, 0.9)$.
   (b) (5 points) Let S be the level surface of f for the value 12. Prove that the gradient $\nabla f(3,2,1)$ is perpendicular to the surface S at $(3,2,1)$.

5. (15 points) Let $U \subseteq \mathbb{R}^2$ be a connected and open set.
   (a) (10 points) Suppose that $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable on U and $D\mathbf{f}(\mathbf{x}) = \mathbf{0} \in \mathcal{B}(\mathbb{R}^2; \mathbb{R}^m)$ for every $\mathbf{x} \in U$. Prove that $\mathbf{f}$ is a constant function.
   (b) (5 points) If $g(x,y)$ is differentiable on U and $\dfrac{\partial g(x,y)}{\partial x} = 0$ for all $(x,y) \in U$. Determine whether g is independent of x on U.

6. (15 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^x \sin y, e^x \cos y)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be of class $\mathcal{C}^1$ such that $[Dg(0,1)] = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$. Define $h(x,y) := \big(g \circ f\big)(x,y)$.
   (a) (5 points) Prove that there exist an open neighborhood U of $(0,0)$ and an open neighborhood V of $h(0,0)$ such that h is a bijection from U onto V.
   (b) (5 points) Let U and V be the open neighborhoods in Problem(a). Prove that $h : U \to V$ is an open mapping on U.
   (c) (5 points) Find the matrix representation of $(Dh^{-1})(y_0)$ at $y_0 = h(0,0)$.

7. (10 points) Define $f(x,y,z) = \big(x + yz + e^z, x^2 - y^2 + xz\big)$ on $\mathbb{R}^3$.
   (a) (5 points) Determine whether the zero set $Z = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0)\}$ near $(-1,1,0)$ can be written as the graph of some function g in the variables x. Explain the reason.

(b) (5 points) If the function g in Problem(a) exists, find the matrix representation of $Dg(-1)$.

8. (10 points) Let (r, ϕ, θ) be the spherical coordinate of $\mathbb{R}^3$ so that

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$

(a) (5 points) Find the Jacobian of the map $(r, \theta, \phi) \to (x, y, z)$. That is, find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

(b) (5 points) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function which only depends on r. Prove that $\nabla f(x, y, z)$ is parallel to $\langle x, y, z\rangle$ for every $(x, y, z) \neq (0, 0, 0)$.

9. (10 points) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T : X \to Y$ be a linear map. Suppose that there exists a constant $K > 0$ such that, for all $y \in Y$, there exists $x \in X$ with $\|x\|_X \leq K\|y\|_Y$ such that $T(x) = y$. Prove that T is an open mapping.

Bonus Problem: (10 points) Let $X := M_n(\mathbb{R})$ be the set of all $n \times n$ matrics and $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be given by $f(A) = A^2$ for $A \in M_n(\mathbb{R})$. Prove that f is differentiable on $M_n(\mathbb{R})$.
(Hint: For $A \in M_n(\mathbb{R})$, we want to prove that f is differentiable at A and find a linear map $Df(A) \in \mathcal{B}(X, X)$. Observe $f(A+H) - f(A)$ and think how to define $Df(A)(H)$.)

高等微積分 (二) 期末考

1102 Advanced Calculus (2) Final Exam

June 16, 2022

Name:____________________ Student ID:____________________

1. 本期中考共 10 道題 (含 1 道加分題)，總計 125 分。(若得分超過 100 分，以 100 分計算)
2. 加分題為二選一，若兩題都作答則只計算第一題分數。
3. 考試過程中不可使用計算機、手機、3C 產品、參考書、筆記及個人計算紙。如需計算紙，請利用考卷背面。
4. 每道題都需寫出完整的過程，並請保持試卷乾淨及答案清楚。

- In this exam, you may assume that every set has volume.
- n-dimensional spherical coordinate

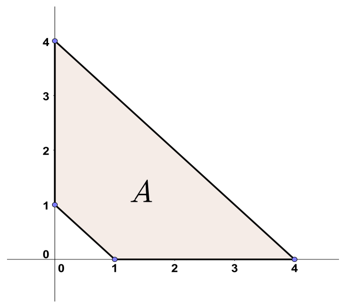
$$\begin{cases} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ x_{n-1} &= r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n &= r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{cases}$$

where $0 \le r < \infty$, $0 \le \theta_1, \cdots, \theta_{n-2} < \pi$ and $0 \le \theta_{n-1} < 2\pi$.

$$\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(r, \theta_1, \theta_2, \cdots, \theta_{n-1})} = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}$$

1. (15 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = e^{x\cos y}$, $a = (0,0)$, $\mathbf{u} =< 1,2 >$ and $\mathbf{v} =< 3,4 >$.
   (a) (8 points) Find the third degree Taylor polynomial for f centered at a.
   (b) (7 points) For the linear map $D^3 f(a)(\mathbf{u},\mathbf{v}) \in \mathcal{B}(\mathbb{R}^2,\mathbb{R})$, find its matrix representation.

2. (10 points) Let $f(x,y) = e^{(y-x)/(y+x)}$ and A be the region as the below graph. Find the average of f over A.



3. (10 points) Let
$$f(x,y,z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2.$$
Find all extreme point(s) and value(s) of f.

4. (20 points) Let $A \subset \mathbb{R}^n$ be a set with volume zero.
   (a) (5 points) Suppose $f : A \to \mathbb{R}$ be a bounded function. Prove that $\displaystyle\int_A f(x)\, dx = 0$.
   (b) (7 points) Determine whether the result of Problem(a) is still true if f is an unbounded and integrable function over A.
   (c) (8 points) Let $L \in \mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)$, $E := L(A)$ be a set in $\mathbb{R}^n$ and $g : E \to \mathbb{R}$ be an integrable function over E. Determine whether $\displaystyle\int_E g(x)\, dx = 0$.

5. (15 points) Let $A \subset \mathbb{R}^n$ be a bounded set with volume and $f_k : A \to \mathbb{R}$ be a sequence of integrable functions which uniformly converges to f on A.
   (a) (10 points) Prove that f is integrable over A and $\displaystyle\int_A f(x)\, dx = \lim_{k\to\infty} \int_A f_k(x)\, dx$.
   (b) (5 points) Determine whether the result of Problem(a) is still true if $A = \mathbb{R}^2$.

6. (15 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function.
   (a) (8 points) Suppose that f is integrable over $\mathbb{R}^2$. Prove that f^2 is also integrable over $\mathbb{R}^2$.
   (b) (7 points) Suppose that $\int_{\mathbb{R}^2} f^2(x)\, dx = 0$. Prove that the set $\{x \in \mathbb{R}^2 \mid f(x) \neq 0\}$ has volume zero. (*Note: You may assume that every set has volume.*)

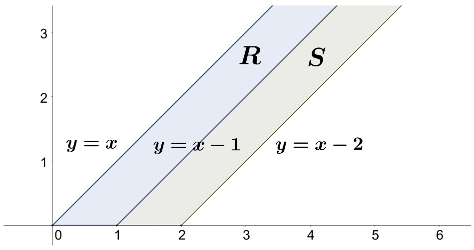
7. (10 points) Let $p \in \mathbb{R}$ and $D = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \|\mathbf{x}\|_{\mathbb{R}^n} > 1 \right\}$. Define $f : D \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|_{\mathbb{R}^n}^p}.$$

Determine the range of p such that f is Riemann integrable over D. (*Note: If necessary, you may assume that* $\int_0^\pi \sin^k t \, dt = c_k$ *is a positive constant for* $k \in \mathbb{N}$.)

8. (10 points) Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$. (*Hint: use cylindrical coordinates*)

9. (10 points) Let $Q = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x \geq 0,\ y \geq 0 \right\}$ be the first quadrant in $\mathbb{R}^2$.



Let

$$\begin{aligned} R &= \left\{ (x, y) \in Q \;\middle|\; x - 1 < y < x \right\} \text{ and} \\ S &= \left\{ (x, y) \in Q \;\middle|\; x - 2 < y < x - 1 \right\} \end{aligned}$$

Define $f(x, y) : Q \to \mathbb{R}$ by

$$f(x, y) := \mathbb{1}_S(x, y) - \mathbb{1}_R(x, y).$$

(a) (5 points) Check that $\displaystyle\int_0^\infty \int_0^\infty f(x, y) \, dydx \neq \int_0^\infty \int_0^\infty f(x, y) \, dxdy$.

(b) (5 points) Explain why the Fubini's Theorem does not apply on $f(x, y)$ over Q.

Bonus Problem A:(10 points) Prove that the double series

$$\sum_{m,n=1}^{\infty} \frac{1}{(m+n)^3}$$

converges. (That is, there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there are $M, N \in \mathbb{N}$ such that if $k \geq M$ and $l \geq N$, then $\left| \displaystyle\sum_{m=1}^{k} \sum_{n=1}^{l} \frac{1}{(m+n)^3} - L \right| < \epsilon$.)

Bonus Problem B:(10 points) Prove that

$$\sum_{m,n=1}^{\infty} \frac{1}{(m+n)^3} = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(m+n)^3} \right).$$

I will do Problem __________.

Reference

1. *An Introduction to Elementary Analysis*(基礎分析導論)，鄭經斅
2. *A Friendly Introduction to Analysis 2nd Edition*, Witold A. J. Kosmala.
3. *The Elements of Real Analysis 2nd Edition*, Robert G. Bartle.
4. *Elementary Classical Analysis*, J. E. Marsden & M. J. Hoffman.
5. *A First Course in Real Analysis*, M.H. Proter.
6. *Principles of Mathematical Analysis*, Walter Rudin.
7. *Fourier Analysis: An Introduction*, Elias M. Stein & Rami Shakarchi

Appendix

1. 泛音列: https://www.youtube.com/watch?v=0iJmDhNocaQ
2. Makewave: https://phet.colorado.edu/sims/html/wave-on-a-string/latest/wave-on-a-string_zh_TW.html
3. 傅立葉變換: https://www.youtube.com/watch?v=spUNpyF58BY
4. 傅立葉變換: https://www.youtube.com/watch?v=r18Gi8lSkfM
5. Convolution: https://www.youtube.com/watch?v=acAw5WGtzuk