1. Supremum and Infimum

Remark: In this sections, all the subsets of $\mathbb{R}$ are assumed to be nonempty.

Let E be a subset of $\mathbb{R}$. We say that E is bounded above if there exists a real number U such that $x \leq U$ for all $x \in E$. In this case, we say that U is an upper bound for E. We say that E is bounded below if there exists a real number L so that $x \geq L$ for all $x \in E$. In this case, we say that L is a lower bound for E. A subset E of $\mathbb{R}$ is said to be bounded if E is both bounded above and bounded below.

Let E be a subset of $\mathbb{R}$.

(1) Suppose E is bounded above. An upper bound U of set E is the least upper bounded of E if for any upper bound U' of E, $U' \geq U$. If U is the least upper bound of E, we denote U by $\sup E$. The least upper bound for E is also called supremum of E.

(2) Suppose E is bounded below. A lower bounded L of E is said to be the greatest lower bound of E if for any lower bound L' of E, $L \geq L'$. If L is the greatest lower bound for E, we denote L by $\inf E$. The greatest lower bound for E is also called the infimum of E.

Example 1.1. Let a, b be real numbers. The set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is bounded with $\sup[a, b] = b$ and $\inf[a, b] = a$.

Example 1.2. Let a be a real numbers. We denote $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$. Then $(-\infty, a)$ is bounded above but not bounded below.

Example 1.3. Given a sequence (a_n) of real numbers, let $\{a_n \in \mathbb{R} : n \geq 1\}$ be the image of (a_n), i.e. the set of all values of (a_n). Then (a_n) is bounded (bounded above, bounded below) if and only if the set $\{a_n \in \mathbb{R} : n \geq 1\}$ is bounded (bounded above, bounded below).

Theorem 1.1. (Property of $\mathbb{R}$) In $\mathbb{R}$, the following hold:

(1) Least upper bound property: Let S be a nonempty set in $\mathbb{R}$ that has an upper bound. Then S has a least upper bound.

(2) Greatest lower bound property: Let P be a nonempty subset in $\mathbb{R}$ that has a lower bound. Then P has a greatest lower bound in $\mathbb{R}$.

Example 1.4. Consider the set $S = \{x \in \mathbb{R} : x^2 + x < 3\}$. Find $\sup S$ and $\inf S$.

Example 1.5. Let $S = \{x \in \mathbb{Q} : x^2 < 2\}$. Find $\sup S$ and $\inf S$.

Example 1.6. Let $S = \{x \in \mathbb{R} : x^3 < 1\}$. Find $\sup S$. Is S bounded below?

Proposition 1.1. Let E be a bounded subset of $\mathbb{R}$ and $U \in \mathbb{R}$ is an upper bound of E. Then U is the least upper bound of E if and only if for any $\epsilon > 0$, there exists $x \in E$ so that $x \geq U - \epsilon$.

Proof. Suppose $U = \sup E$. Then for any $\epsilon > 0$, $U - \epsilon < U$. Hence $U - \epsilon$ is not an upper bound of E. Claim: there exists $x \in E$ so that $x > U - \epsilon$. If there is no x so that $x > U - \epsilon$, then $x \leq U - \epsilon$ for all $x \in E$. This implies that $U - \epsilon$ is again an upper bound for E. By the definition of the least upper bound, $U \leq U - \epsilon$ which is absurd since $\epsilon > 0$.

Conversely, let $U' = \sup E$. Since U is an upper bound of E, $U \geq U'$. Claim $U' = U$. For any $\epsilon > 0$, choose $x \in E$ so that $x > U - \epsilon$. Thus $U' \geq x > U - \epsilon$. We know that for any $U' > U - \epsilon$ for all $\epsilon > 0$. Since ϵ is arbitrary $U' \geq U$. We conclude that $U' = U$.

□

Lemma 1.1. Let E be a nonempty subset of $\mathbb{R}$.

(1) If E is bounded above and $\alpha < \sup E$, then there exists $x \in E$ so that $x > \alpha$.
(2) If E is bounded above and $\beta > \inf E$, then there exists $x \in E$ so that $x < \beta$.

Proof. The result follows from the definition. □

Corollary 1.1. Let E be a bounded subset of $\mathbb{R}$ and $L \in \mathbb{R}$ is a lower bound of E. Then L is the greatest upper bound of E if and only if for any $\epsilon > 0$, there exists $x \in E$ so that $x \leq L - \epsilon$.

Proof. We leave it to the reader as an exercise. □

Let E be a nonempty subset of $\mathbb{R}$. We say that M is a maximum of E if M is an element of E and $x \leq M$ for all $x \in E$. Using the definition, we immediately know that there is only one maximum of E if the maximum elements of E exists. In this case, we denote M by $\max E$. It also follows from the definition that if the maximum of E exists, it must be bounded above.

We say that m is the minimum of a nonempty subset E of $\mathbb{R}$ if m is an element of E and $x \geq m$ for all $x \in m$. We denote m by $\min E$. It follows from the definition that if a set has minimum, it must be bounded below.

Example 1.7. The following subsets of $\mathbb{R}$ are all bounded. Hence their greatest lower bound and their least upper bound exist. Determine whether their maximum or minimum exist.

(1) $E_1 = (0, 1)$.
(2) $E_2 = (0, 1]$.
(3) $E_3 = [0, 1)$.
(4) $E_4 = [0, 1]$.

Proposition 1.2. Suppose E is a nonempty subset of $\mathbb{R}$. If E is bounded above, then the maximum of E exists if and only if $\sup E \in E$. Similarly, if E is bounded below, then the minimum of E exists if and only if $\inf E \in E$.

Proof. The proof follows from the definition. □

Proposition 1.3. Let E be a nonempty subset of $\mathbb{R}$. Suppose E is a bounded set. Then $\inf E \leq \sup E$.

Proof. We leave it to the reader as an exercise. □

Proposition 1.4. Let E and F be nonempty subsets of $\mathbb{R}$. Suppose that $E \subset F$.

(1) If E and F are both bounded below, then $\inf F \leq \inf E$.
(2) If E and F are both bounded above, then $\sup E \leq \sup F$.

Proof. Let us prove (a). (b) is left to the reader.

For any $x \in F$, $x \geq \inf F$. Since E is a subset of F, $x \geq \inf F$ holds for all $x \in E$. Therefore $\inf F$ is a lower bound for E. Since $\inf E$ is the greatest lower bound for E, $\inf E \geq \inf F$. □

Theorem 1.2. Every bounded monotone sequence is convergent.

Proof. Without loss of generality, we may assume that (a_n) is a bounded nondecreasing sequence of real numbers. Let $a = \sup\{a_n : n \geq 1\}$. Given $\epsilon > 0$, there exists a_N so that $a \geq a_N > a - \epsilon$. Since (a_n) is nondecreasing, $a_n \geq a_N$ for every $n \geq N$. Hence $a_n > a - \epsilon$ for every $n \geq N$. In this case, $|a_n - a| = a - a_n < \epsilon$ for $n \geq N$. We prove that $\lim_{n \to \infty} a_n = a$. □

2. LIMSUP AND LIMINF

Let (a_n) be a bounded sequence of real numbers. Define a new sequence (x_n) by

$$x_n = \sup\{a_m : m \geq n\}, \quad n \geq 1,$$

Since (a_n) is bounded, x_n is a real number for each $n \geq 1$. We assume that $|a_n| \leq M$ for all $n \geq 1$. Then $|x_n| \leq M$ for all $n \geq 1$. This shows that (x_n) is also a bounded sequence. By Proposition 1.4, (x_n) is nonincreasing. By monotone sequence property, $x = \lim_{n\to\infty} x_n$ exists. We denote x by $\limsup_{n\to\infty} a_n$. Similarly, define a sequence (y_n) by

$$y_n = \inf\{a_m : m \geq n\}, \quad n \geq 1.$$

Then (y_n) is a nondecreasing sequence by Proposition 1.4. By monotone sequence property, $y = \lim_{n\to\infty} y_n$ exists. We denote y by $\liminf_{n\to\infty} a_n$.

From now on, we will simply denote by

$$a^* = \limsup_{n\to\infty} a_n, \quad a_* = \liminf_{n\to\infty} a_n.$$

Since (x_n) is nonincreasing and bounded below, its limit equals to $\inf\{x_n : n \geq 1\}$. Similarly, (y_n) is nondecreasing and bounded above, its limit equals to $\sup\{y_n : n \geq 1\}$.

Example 2.1. Find $\limsup_{n\to\infty}(-1)^n$ and $\liminf_{n\to\infty}(-1)^n$.

Solution: Let us compute these two numbers via definition. Denote $(-1)^n$ by a_n. For each $k \geq 1$, we know $\{a_n : n \geq k\} = \{-1, 1\}$. For $k \geq 1$, $x_k = \sup\{a_n : n \geq k\} = \sup\{-1, 1\} = 1$. Similarly, for $k \geq 1$, $y_k = \sup\{a_m : m \geq k\} = \inf\{-1, 1\} = -1$. Therefore $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} x_k = 1$ while $\liminf_{n\to\infty} a_n = \lim_{k\to\infty} y_k = -1$. Notice that the subsequence (a_{2n}) of (a_n) is convergent to 1 and the subsequence (a_{2n-1}) of (a_n) is convergent to -1. Later, we will prove that in general, the limit supremum and the limit infimum of a bounded sequence are always the limits of some subsequences of the given sequence.

Example 2.2. Let (a_n) be the sequence defined by

$$a_n = 1 - \frac{1}{n}, \quad n \geq 1.$$

Evaluate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Solution: The sequence (a_n) is increasing and bounded above by 1. Let us prove that $\sup\{a_n : n \geq 1\} = 1$. We have seen that 1 is an upper bound. Now, for each $\epsilon > 0$, choose $N_\epsilon = [1/\epsilon] + 1$. For $n \geq N_\epsilon$, we see

$$1 - \epsilon < 1 - \frac{1}{n}.$$

By Proposition 1.1, we find 1 is indeed the least upper bound for $\{a_n : n \geq 1\}$. Therefore $1 = \sup\{a_n : n \geq 1\}$. We also know that $\lim_{n\to\infty} a_n = 1$. In other words, we prove

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \sup\left\{1 - \frac{1}{n} : n \geq 1\right\}.$$

This gives us an example of Theorem 1.2. Similarly, for each $k \geq 1$, we can show that 1 is the least upper bound of the set $\{1 - 1/n : n \geq k\}$ and -1 is the greatest lower bound for

$\{1-1/n : n \geq k\}$. In other words, we find that

$$x_k = \sup\left\{1-\frac{1}{n} : n \geq k\right\} = 1, \quad y_k = \inf\left\{1-\frac{1}{n} : n \geq k\right\} = -1.$$

This shows that $\lim_{k\to\infty} x_k = 1$ and $\lim_{k\to\infty} y_k = -1$. In other words,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 1.$$

In this case, (a_n) is convergent to 1 and at the same time, both limit supremum and limit infimum of (a_n) are also equal to 1. This is not an accident. Later, we will prove that a bounded sequence is convergent if and only if its limit supremum equals to its limit infimum.

Lemma 2.1. Let (a_n) be a bounded sequence and $a \in \mathbb{R}$.

(1) If $a > a^*$, there exists $k \in \mathbb{N}$ such that $a_n < a$ for all $n \geq k$.
(2) If $a < a^*$, then for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $n \geq k$ such that $a_n > a$
(3) If $a < a_*$, there exists $k \in \mathbb{N}$ such that $a_n > a$ for all $n \geq k$.
(4) If $a > a_*$, then for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $a_n < a$.

Proof. Let (x_k) be the sequence defined as above, i.e. for each $k \geq 1$,

$$x_k = \sup\{a_n : n \geq k\}.$$

By definition, $a^* = \inf\{x_k : k \geq 1\}$.

(1) If $a > a^*$, a is not a lower bound for $\{x_k : k \geq 1\}$. Then there exists an element $x_{k_0} \in \{x_k : k \geq 1\}$ such that $x_{k_0} < a$. Since $a > x_{k_0}$ and x_{k_0} is the least upper bound for $\{a_n : n \geq k_0\}$, then a is an upper bound for $\{a_n : n \geq k_0\}$. Hence $a_n < a$ for all $n \geq k_0$.

(2) If $a < a^*$, then a is a lower bound for $\{x_k : k \geq 1\}$. Then for all $k \geq 1$, $x_k > a$. For each $k \geq 1$, x_k is the least upper bound for $\{a_n : n \geq k\}$. Since $a < x_k$, a is not an upper bound for $\{a_n : n \geq k\}$. Hence we can choose an element $a_n \in \{a_n : n \geq k\}$ so that $a_n > a$. In other words, we can find $n \in \mathbb{N}$ with $n \geq k$ such that $a_n > a$.

(3) and (4) are left to the reader.

□

Definition 2.1. We say that a real number x is a cluster point of a bounded sequence (a_n) if there exists a subsequence (a_{n_k}) of (a_n) whose limit is x.

Corollary 2.1. Let (a_n) be a bounded sequence. Then both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are cluster points.

Proof. Since $a^* + 1 > a^*$, there exists $n_1 \in \mathbb{N}$ so that $a_n < a^* + 1$ for all $n \geq n_1$ by Lemma 2.1. Since $a^* - 1 < a^*$, for the given n_1, we can find $m_1 \in \mathbb{N}$ with $m_1 \geq n_1$ such that $a_{m_1} > a^* - 1$. Since $m_1 \geq n_1$, $a_{m_1} < a^* + 1$. We find $a^* - 1 < a_{m_1} < a^* + 1$. Inductively, we obtain a subsequence (a_{m_k}) of (a_n) so that

$$a^* - \frac{1}{k} < a_{m_k} < a^* + \frac{1}{k}, \quad k \geq 1.$$

Since $\lim_{k\to\infty}\left(a^*-\frac{1}{k}\right)=\lim_{k\to\infty}\left(a^*+\frac{1}{k}\right)=a^*$, by the Sandwich principle, $\lim_{k\to\infty} a_{m_k}=a^*$. This shows that a^* is a cluster point.

Similarly, we can show that a_* is a cluster point. □

Lemma 2.2. If (a_n) is a convergent to a, the all the subsequences (a_{n_k}) is also convergent to a.

Proof. Suppose (a_n) is convergent to a. Then for any $\epsilon>0$, there exists $N_\epsilon\in\mathbb{N}$ so that

$$|a_n-a|<\epsilon,\quad \text{whenever } n\geq N_\epsilon.$$

Let (a_{n_k}) be any subsequence of (a_n). For any $k\geq N_\epsilon$, $n_k\geq k\geq N_\epsilon$. Then $|a_{n_k}-a|<\epsilon$ whenever $k\geq N_\epsilon$. This shows that $\lim_{k\to\infty} a_{n_k}=a$. □

Corollary 2.2. If (a_n) is convergent to a, then $a^*=a_*=a$.

Proof. Since a^* and a_* are both cluster points, Lemma 2.2 implies that $a^*=a=a_*$. □

In fact, the converse is also true:

Theorem 2.1. Let (a_n) be a bounded sequence of real numbers. Then (a_n) is convergent to a real number a if and only if $a^*=a_*=a$.

Proof. We have proved one direction. Conversely, let us assume $a_*=a^*=a$.

For any $\epsilon>0$, $a+\epsilon>a^*$. By Lemma 2.1, there exists $k_\epsilon\in\mathbb{N}$ so that for any $n\geq k_\epsilon$, $a_n<a^*+\epsilon$.

Since $a-\epsilon<a_*=a$, Lemma 2.1 implies that there exists $j_\epsilon\in\mathbb{N}$ so that $a_n>a-\epsilon$ whenever $n\geq j_\epsilon$. Denote $N_\epsilon=\max\{k_\epsilon,j_\epsilon\}$. Then for all $n\geq N_\epsilon$, we have

$$a-\epsilon<a_n<a+\epsilon.$$

Thus $|a_n-a|<\epsilon$ whenever $n\geq N_\epsilon$. This shows that $\lim_{n\to\infty} a_n=a$.

□

Let (a_n) be a bounded sequence. If E is the set of all cluster points of (a_n), we know that it is nonempty by Bolzano-Weierstrass theorem (since any bounded sequence has a convergent subsequence). For $x\in E$, choose a subsequence (a_{n_k}) of (a_n) so that $\lim_{k\to\infty} a_{n_k}=x$. By Theorem 2.1, we know

$$\limsup_{k\to\infty} a_{n_k}=\liminf_{k\to\infty} a_{n_k}=x.$$

By definition, for any $j\geq 1$, the set $\{a_{n_k}:k\geq j\}$ is a subset of $\{a_n:n\geq j\}$. This is because for $k\geq j$, $n_k\geq k\geq j$. For each $j\geq 1$,

$$\sup\{a_n:n\geq j\}\geq \sup\{a_{n_k}:k\geq j\}.$$

Denote the left hand side by x_j and the right hand side by α_j. Then $x_j \geq \alpha_j$ for all $j \geq 1$. Since both (x_j) and (α_j) are convergent, by Lemma **??**, we find

$$a^* = \lim_{j\to\infty} x_j \geq \lim_{j\to\infty} \alpha_j = x.$$

This shows $x \leq a^*$, i.e. E is bounded above by $\limsup_{n\to\infty} a_n$. Using a similar argument, we can show that $x \geq a_*$, i.e. E is bounded below by a_*. We obtain that E is a nonempty bounded subset of $\mathbb{R}$. Since both a^* and a_* are cluster points, i.e. $a^*, a_* \in E$, we obtain that:

Theorem 2.2. Let (a_n) be a bounded sequence and E be the set of all cluster points of (a_n). Then

$$\limsup_{n\to\infty} a_n = \max E, \quad \liminf_{n\to\infty} a_n = \min E.$$

Example 2.3. Construct a sequence with exact three different cluster points.

Solution: We leave it to the reader as an exercise.

Example 2.4. Find the limsup and liminf of the following sequence of real numbers.

(1) $a_n = (-1)^n + \dfrac{1}{n}$, $n \geq 1$.

(2) $a_n = \cos(n\pi + \dfrac{\pi}{6})$, $n \geq 1$.

(3) $a_n = \begin{cases} \frac{1}{n} & \text{if } n = 3k \\ 1 + \frac{1}{n} & \text{if } n = 3k+1 \\ -2 + \frac{1}{n} & \text{if } n = 3k+2. \end{cases}$

Theorem 2.3. Let (s_n) and (t_n) be bounded sequences of real numbers. Suppose that there exists $N > 0$ so that $s_n \leq t_n$ for $n \geq N$. Then

(1) $\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$.

(2) $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$.

Proof. We leave it to the reader as an exercise. □

It follows immediately from this theorem that

Corollary 2.3. Let (a_n) be a sequence of real numbers. Then

(1) If $a_n \leq M$ for all n, then $\limsup_{n\to\infty} a_n \leq M$.

(2) If $a_n \geq M$ for all n, then $\liminf_{n\to\infty} a_n \geq M$.