LOCAL EXISTENCE FOR SEMILINEAR WAVE EQUATIONS AND APPLICATIONS TO YANG–MILLS EQUATIONS

by

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ABSTRACT

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This thesis concerns several questions about the local existence of certain semilinear wave equations with minimal assumptions on the regularity of the initial data.

The first part deals with the Cauchy problem for equation of the type $\Box u = uDu$. This is partially motivated by the Yang-Mills equations which can schematically be written as $\Box A + ADA + A^3 = 0$. Assuming the initial data are in $H^{1+\epsilon}$ and H^{ϵ} for any $\epsilon > 0$, I prove a local result for the problem using a fixed point argument where the main ingredients are the standard energy estimate and an a priori estimate for the quadratic nonlinear term uDu. Then I apply

those to the Yang-Mills equations in the Lorentz gauge where the cubic terms are actually lower order and hence easier to handle.

The second part gives a local result for the Yang-Mills equations in the Coulomb gauge with initial data in H^1 and L^2, but it requires that the size of the initial data be sufficiently small. The reason for the improved regularity is that, in the Coulomb gauge, the quadratic terms of the equations can be written as null forms for which there are better estimates applicable. Null forms are special types of quadratic expressions in the first derivatives.

The final part concerns some estimates of null forms in the norm that is a sort of a local version of Hardy space's norm with initial data in H^1. The main idea is that a quadratic expression in the first derivatives is only L^1 in space for each fixed time but certain null forms have an improved integrability expressed by the fact that an appropriate Hardy norm is bounded by the energy.

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Chapter 0

The Main Result

0.1. Introduction.

In this thesis, we are interested in establishing certain local existence results for Yang-Mills equations with minimal assumptions on the regularity of the initial data. For this purpose, we want to study first the following type of system of semilinear wave equations:

$$\tag{0.1} \begin{cases} \Box u^i = a^i_{jkl} u^l \partial_j u^k, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3, \end{cases}$$

where $\Box$ denotes the D'Alembertian $-\partial_t^2 + \Delta$ in $\mathbb{R}^{3+1}$ with t as the time variable and x as the space variables, and summation over repeating indices is implied. In equations that arise from physical problems a certain quantity called energy is conserved, hence it is desirable to have an existence theorem so that the regularity of the initial data will imply that the energy is finite. For equations (0.1), the energy is $E = \int |\partial_t u^i|^2 + |\nabla u^i|^2 dx$, so that the minimal regularity for initial data is $f^i \in H^1(\mathbb{R}^3)$, $g^i \in L^2(\mathbb{R}^3)$.

A Yang-Mills field is a Lie algebra which takes values of 2-form F satisfying Bianchi-type equations. The field F is derived from a potential 1-form A which is also taking values in the same Lie algebra. The Yang-Mills equations are the basic equations of gauge theory, which is believed by physicists to best describe the weak and strong interactions of elementary particles. The basic concept of gauge theory goes back to H. Weyl in the 1920's. The equations arise as the

Euler-Lagrange equations corresponding to variations of the Lagrangian of the field. There has been many research done on the equations.

In [16], Segal established a local existence result of classical solution for the Cauchy problem of Yang-Mills equations in temporal gauge, but in order to have classical solutions, one needs to assume the initial data are in C^3. After Segal's result, the main result is the proof of the global existence of a classical solution for the Cauchy problem of Yang-Mills equations in temporal gauge, with initial data in $H^{2+k}(\mathbb{R}^3)$ and $H^{1+k}(\mathbb{R}^3)$ for $k \geq 3$, by Eardley and Moncrief [3]. In [7], Glassey and Strauss found a class of global weak solutions of Yang-Mills equations in the temporal gauge with initial data which are radial symmetric and have finite energy and whose potential vanishes at spacial infinity. Schirmer [15] proved global existence of spherically symmetric solutions which are small in a specified norm. Ponce and Sideris [14] recently proved a local existence result for an equation similar to (0.1) where the nonlinear term is quadratic in the first derivatives and they have to assume that the initial data are in $H^{2+\epsilon}(\mathbb{R}^3)$ and $H^{1+\epsilon}(\mathbb{R}^3)$ respectively for $\epsilon > 0$. Klainerman and Machedon [12] proved global existence for the Yang-Mills equations in the temporal gauge with finite energy initial data in $\mathbb{R}^3$ which is also locally H^1.

0.2. Main Result.

We will prove a local existence result for the system (0.1) assuming that the regularity of initial data is just a little better than energy norm, namely $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$, $g^i \in H^{\epsilon}(\mathbb{R}^3)$, where $\epsilon > 0$. The main ingredients in the proof are the standard energy estimate and an a priori estimate given by the following theorem.

Theorem A. *(A Priori Estimate for Equations (0.1)) Let u^i be a solution* *of the equations*

$$
(0.2) \qquad \begin{cases} \Box u^i = b^i, & & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u^i(0,x) = f^i(x), & u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3, \end{cases}
$$

where $i = 1, 2, \ldots, N$. *Then, for every* $\frac{1}{2} > \epsilon > 0$, *the following estimate holds*

(0.3)

$$
\left(\int_0^T \|D^\epsilon(u^l Du^k)\|^q_{L^2(\mathbb{R}^3)} dt\right)^{\frac{1}{q}} \leqslant C(\epsilon) \left(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^l\|_{L^2} dt\right) \cdot \left(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^k\|_{L^2} dt\right),
$$

where $l,\ k = 1,\ 2,\ \ldots,\ N$, $q = \frac{2}{1-2\epsilon}$ *and* $C(\epsilon)$ *blows up as* ϵ *decreases to zero, actually* $C(\epsilon) = O(1/\sqrt{\epsilon})$, *and* D^ϵ *is a fractional derivative defined by Fourier transform.*

Once the above estimate is established, we can use a standard iteration argument to prove the local existence result. This result is probably optimal, namely the local existence theorem may not be true if $\epsilon = 0$ (see [13] H. Lindblad, for a related result).

Next, we want to apply the above ideas to the Yang-Mills equations which can be written schematically as a system of the following form:

$$
(0.4) \qquad \Box A + ADA + A^3 = 0,
$$

where A^a_μ are the gauge potentials, $a = 1, 2, 3$, and $\mu = 0, 1, 2, 3$, and DA represents any kind of first order derivatives of A. The Yang-Mills equations are under-determined and in order to solve them, we need an extra condition known as the gauge condition. For more details about the exact form of the equations and the gauge conditions see chapter 2.

For the Yang-Mills equations in the Lorentz gauge, we can prove a local existence assuming that initial data is in $H^{1+\epsilon}(\mathbb{R}^3)$, for each $\frac{1}{2} > \epsilon > 0$. The main ingredients in the proof are the standard energy estimate and theorem A.

On the other hand, if we use Coulomb gauge, then the term ADA can be rewritten as null quadratic form and there is a better estimate (see [11] Klainerman and Machedon). Consider the systems

$$\begin{cases} \Box\phi_k = \psi_k, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ \phi_k(0,\cdot) = g_k, \quad \partial_t\phi_k(0,\cdot) = h_k, & x \in \mathbb{R}^3, \end{cases} \tag{0.5}$$

where $k = 1, 2$, we have the following estimate.

Theorem B. *(A Priori Estimate for Equations (0.5)) Let*

$$Q_{ij}(\phi,\psi) = \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi,$$

for i, j = 1, 2, 3. If ∇g_1, h_1, $\nabla^2 g_2$, and $\nabla h_2 \in L^2(\mathbb{R}^3)$, then we have $Q_{ij}(\phi_1,\phi_2) \in L^2(\mathbb{R}^3 \times [0,T])$, and

(0.6)

$$\|Q(\phi_1,\phi_2)\|_{L^2(\mathbb{R}^3\times[0,T])} \le C\left(\|\nabla g_1\|_{L^2(\mathbb{R}^3)} + \|h_1\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)}dt\right)$$
$$\cdot\left(\|\nabla^2 g_2\|_{L^2(\mathbb{R}^3)} + \|\nabla h_2\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\nabla\psi_2\|_{L^2(\mathbb{R}^3)}dt\right).$$

For the same system, if ∇g_1, h_1, ∇g_2, and $h_2 \in L^2(\mathbb{R}^3)$, where ∇ is the gradient, then $(-\Delta)^{-\frac{1}{2}}Q(\phi_1,\phi_2) \in L^2(\mathbb{R}^3 \times [0,T])$ and

(0.7)

$$\|(-\Delta)^{\frac{-1}{2}}Q(\phi_1,\phi_2)\|_{L^2(\mathbb{R}^3\times[0,T])} \le C\left(\|\nabla g_1\|_{L^2} + \|h_1\|_{L^2} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)}dt\right)$$
$$\cdot\left(\|\nabla g_2\|_{L^2} + \|h_2\|_{L^2} + \int_0^T \|\psi_2\|_{L^2(\mathbb{R}^3)}dt\right),$$

where Δ is the Laplacian.

For the Yang-Mills equations in the Coulomb gauge, we can prove a local existence result assuming only that the energy of the initial data, namely $\|A_j^a\|_{H^1(\mathbb{R}^3)} + \|\partial_t A_j^a\|_{L^2(\mathbb{R}^3)}$, is small. The reason for the restriction on the size of the energy comes from the fact that the equation for $A_0^a(t,x)$ is elliptic, $\Delta A_0 = F(A, ADA)$, and in order to control the appropriate norms of A_0, we need the energy of initial data to be smaller than a certain constant.

0.3. Outline.

The outline of this paper is as follows. In chapter 1, we introduce the notation used throughout the work and give a local existence result for the Cauchy problem of equations (0.1). In chapter 2, we give a brief introduction of gauge theory and the Yang-Mills equations, together with the various gauge conditions. Then we apply the method in chapter 1 to prove a local existence result for the Yang-Mills equations in the Lorentz gauge. In chapter 3, we use the estimates (see [11]), in theorem.B, to derive the local existence for Yang-Mills equations in the Coulomb gauge, with the energy of the initial data sufficiently small. In chapter 4, we give the proof for the estimates in Theorem A and Theorem B. At last in chapter 5, some estimates of null forms in the Hardy space's norm with initial data in H^1 will be proved.

Chapter I
Semilinear Wave Equation

1.1. Introduction.

In this chapter, we will prove a local existence result for a certain kind of nonlinear wave equations which can be applied to the Yang-Mills equations. Consider the system equations of the following form.

$$\begin{cases} \Box u^i = a^i_{jkl} u^l \partial_j u^k, & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\Box = \partial_{tt} - \sum \partial_{jj}$ and the initial data $f^i \in H^{1+\epsilon}(\mathbb{R}^3), g^i \in H^{\epsilon}(\mathbb{R}^3), a^i_{jkl}$ constant and $\frac{1}{2} > \epsilon > 0$. Note there is a summation over repeating indices and all the indices i, j, k and l run from 0 up to 3.

1.2. Preliminaries.

Let us introduce some notation. The space $H^s(\mathbb{R}^n)$ is defined to be

$$H^s(\mathbb{R}^n) = \{f | (1-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R}^n)\}, \tag{1.2}$$

where $1 - \Delta$ is the mutiplier operator associated with $1 + |\xi|^2$. The space is equipped with the norm $\| \cdot \|_{H^s(\mathbb{R}^n)}$ defined by

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

The space $\dot{H}^s(\mathbb{R}^n)$ is defined to be

$$\dot{H}^s(\mathbb{R}^n) = \{f | (-\Delta)^{\frac{s}{2}} \hat{f} \in L^2(\mathbb{R}^n)\}, \tag{1.3}$$

with norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \||\xi|^s \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

The fractional derivative D^ϵ is defined by Fourier transform to be

$$\widehat{D^\epsilon f}(\xi) = |\xi|^\epsilon \hat{f}(\xi). \tag{1.4}$$

We use the notation D to represent the partial derivatives ∂_t or ∂_j throughout the paper. The case we are interested is $n = 3$.

1.3. The Local Existence.

Theorem 1.1. *There exists a $T > 0$ and C depending only on $\|f^i\|_{H^{1+\epsilon}}$, $\|g^i\|_{H^\epsilon}$, where ϵ is any positive number, such that the system of equations (1.1) has solutions $u^i : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ with*

$$\sup_{[0,T]} \|u^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leqslant C, \tag{1.5}$$

where i = 1, 2, 3, and

$$\int_0^T \|D^\epsilon(u^l D u^k)\|^q_{L^2(\mathbb{R}^3)} dt \leqslant C, \tag{1.6}$$

where l,k = 1, 2, 3 and $q = \frac{2}{1-2\epsilon}$.

To prove theorem 1.1, we need the following main estimate.

Theorem 1.2. *Let u^i be the solutions of the system*

$$\begin{cases} \Box u^i = b^i, & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3 \end{cases} \tag{1.7}$$

If $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$, $g^i \in H^\epsilon(\mathbb{R}^3)$, and $0 < \epsilon < \frac{1}{2}$, then for the system we have the following estimate

(1.8)

$$\Big(\int_0^T \|D^\epsilon(u^l D u^k)\|^q_{L^2(\mathbb{R}^3)} dt\Big)^{\frac{1}{q}} \leqslant C(\epsilon)\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^l\|_{L^2} dt\Big) \cdot \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^k\|_{L^2} dt\Big),$$

where i, l, k $= 0, 1, 2, 3$, $q = \frac{2}{1-2\epsilon}$ *and* $C(\epsilon) = O\big(1/\sqrt{\epsilon}\big)$.

The proof of the above theorem will be given in chapter 4.

Remark. ϵ can not be zero. There is a counterexample in [13] for the estimate (1.8) if $\epsilon = 0$. Therefore, this is the optimal result for local existence in the viewpoint of using the estimate (1.8).

The proof of theorem 1.1 is based on theorem 1.2 and the standard energy estimate which states below.

Lemma 1.3. *Let* u^i *be the solutions of the system (1.7). If* $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$ *and* $g^i \in H^{\epsilon}(\mathbb{R}^3)$ *where* ϵ *is any positive number, then the following estimate holds*

$$\sup_{[0,T]} \|u^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leqslant C\Big(\|f^i\|_{H^{1+\epsilon}} + \|g^i\|_{H^{\epsilon}} + \int_0^T \|D^{\epsilon} b^i\|_{L^2(\mathbb{R}^3)} dt\Big). \tag{1.9}$$

Proof. We omit indices of the u^i for simplicity. It suffices to prove the followings:

$$\|Du(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|Du(0,\cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|b(\tau,\cdot)\|_{L^2(\mathbb{R}^3)} d\tau\Big), \tag{1.10}$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|u(0,\cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|Du(\tau,\cdot)\|_{L^2(\mathbb{R}^3)} d\tau\Big). \tag{1.11}$$

where $D = \partial_t$ or ∂_j. For the estimate (1.10), multiplying (1.7) by u_t, and then integrating by parts on $[0,T] \times \mathbb{R}^3$, also employing the divergence theorem, thus we have

$$\int_{t=T} e(u)dx - \int_{t=0} e(u)dx = \int_0^T \int_{\mathbb{R}^3} b(\tau, x) u_t dx dt, \tag{1.12}$$

where $e(u) = \frac{1}{2}(u_t^2 + |\nabla u|^2)$.

Let $E(t) = \int_{t=t} e(u)dx$, then we have

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}^3} bu_t dx \leq C\|b(t,\cdot)\|_{L^2} E^{\frac{1}{2}}(t). \tag{1.13}$$

Dividing both sides by $E^{\frac{1}{2}}$ and integrating them on $[0,t]$ give (1.10). In order to prove the estimate (1.11), one use the formula

$$u(t) - u(0) = \int_0^t \partial_t u d\tau \tag{1.14}$$

which comes from the fundamental theorem of calculus and takes L^2-norm on both sides and applies Minkowski inequality to get the desired result. $\square$

Now we can prove the Theorem 1.1.

Proof of theorem 1.1. The existence proof is based on a simple iteration procedure. Let $u_{-1}^i \equiv 0$. Define

$$\Box u_n^i = a_{jkl}^i u_{n-1}^l \partial_j u_{n-1}^k, \tag{1.18}$$

with initial data given by

$$u_n^i(0,x) = f^i(x), \quad \partial_t u_n^i(0,x) = g^i(x), \tag{1.19}$$

for all $n \geq 0$. Our goal is to prove that $u = \lim u_n$ exists and satisfies the equation and the desired estimates. It suffices to prove that there exists $T \leq 1, \triangle \geq 1$, such that, for all n,

$$\sup_{[0,T]} \|u_n^i(t) - u_{n-1}^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq \frac{\triangle}{2^{n+1}}, \tag{1.20}$$

$$\sup_{[0,T]} \|u_n^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C, \tag{1.21}$$

for some constant C that depends on $\triangle$ but not on n. Also

$$\int_0^T \|D^\epsilon[u_n^l \partial_j(u_n^k - u_{n-1}^k)]\|^q_{L^2(\mathbb{R}^3)} dt \le \frac{\triangle^{2q}}{2^{nq}}, \tag{1.22}$$

$$\int_0^T \|D^\epsilon[(u_n^l - u_{n-1}^l)\partial_j u_{n-1}^k]\|^q_{L^2(\mathbb{R}^3)} dt \le \frac{\triangle^{2q}}{2^{nq}}, \tag{1.23}$$

$$\int_0^T \|D^\epsilon(u_n^l \partial_j u_n^k)\|^q_{L^2(\mathbb{R}^3)} dt \le \triangle^{2q}. \tag{1.24}$$

For $n = 0$, it is easy to check that (1.20)-(1.24) are satisfied.

Next, we assume they are true for some $n \ge 0$ and prove (1.20)-(1.24) are true for $n+1$ with a $T \le 1$, T is independent of n and the same $\triangle$. $C = C(\|f^i\|_{H^{1+\epsilon}}, \|g^i\|_{H^\epsilon})$.

Using (1.24) and (1.8) give

$$\begin{aligned}
&\Big(\int_0^T \|D^\epsilon(u_{n+1}^l \partial_j u_{n+1}^k)\|^q_{L^2(\mathbb{R}^3)} dt\Big)^{1/q} \\
\le& C\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^l\|_{L^2} dt\Big) \\
&\cdot \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^k\|_{L^2} dt\Big) \\
\le& \left(\frac{\triangle}{2} + CT^{1/p}\left(\int_0^T \|D^\epsilon(u_n^r D u_n^s)\|^q_{L^2} dt\right)^{1/q}\right)^2 \\
\le& \Big(\frac{\triangle}{2} + CT^{1/p}\triangle\cdot\triangle\Big)^2 \le \triangle^2,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, hence it is enough to have $CT^{1/p}\triangle \le 1/2$.

Since

$$b_n^k - b_{n-1}^k = \big(u_n^r D u_n^s - u_{n-1}^r D u_n^s\big) + \big(u_{n-1}^r D u_n^s - u_{n-1}^r D u_{n-1}^s\big),$$

thus we can use (1.22) and (1.8) to get

$$\left(\int_0^T \|D^\epsilon[u_{n+1}^l \partial_j(u_{n+1}^k - u_n^k)]\|_{L^2(\mathbb{R}^3)}^q dt\right)^{\frac{1}{q}}$$

$$\leq C\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^l\|_{L^2} dt\Big)\Big(\int_0^T \big\|D^\epsilon(b_n^k - b_{n-1}^k)\big\|_{L^2} dt\Big)$$

$$\leq C\Big(\frac{\triangle}{2} + T^{1/p}\triangle^2\Big)\cdot$$

$$\Big(\int_0^T \big\|D^\epsilon\big[u_{n-1}^r D(u_n^s - u_{n-1}^s)\big]\big\|_{L^2} dt + \int_0^T \big\|D^\epsilon\big[(u_n^r - u_{n-1}^r) D u_n^s\big]\big\|_{L^2} dt\Big)$$

$$\leq \triangle\Big[CT^{1/p}\big(\frac{\triangle^2}{2^n} + \frac{\triangle^2}{2^n}\big)\Big] \leq \frac{\triangle^2}{2^{n+1}}.$$

The computation of (1.23) is similar.

Using (1.20) and (1.9) give

$$\begin{aligned}\sup_{[0,T]} \|u_{n+1}^i(t) - u_n^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} &\leq C\Big(\int_0^T \|D^\epsilon(b_n^i - b_{n-1}^i)\|_{L^2} dt\Big)\\ &\leq 2CT^{1/p}\frac{\triangle^2}{2^n} \leq \frac{\triangle}{2^{n+2}}.\end{aligned}$$

Using (1.21) and (1.9) give

$$\sup_{[0,T]} \|u_{n+1}^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C\Big(\|f^i\|_{H^{1+\epsilon}} + \|g^i\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^i\|_{L^2(\mathbb{R}^3)} dt\Big)$$

$$\leq \frac{\triangle}{2} + CT^{1/p}\triangle^2 \leq \triangle.$$

Therefore the sequence of functions $\{u_n^i\}$ is a Cauchy sequence under the norm $\sup_{[0,T]} \|\cdot\|_{H^{1+\epsilon}(\mathbb{R}^3)}$. $\square$

We will also prove the following technical proposition.

Proposition 1.4. *If f has compact support in B_R, a ball with center at the origin and with radius R, then for any $\epsilon > 0$ there is a constant C depending only on R and ϵ such that*

$$\|f\|_{L^2(\mathbb{R}^3)} \leqslant C(R,\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}. \tag{1.25}$$

Proof. Using Plancherel theorem, we get

$$\left\|D^\epsilon f\right\|_{L^2} = \left\||\xi|^\epsilon \hat{f}\right\|_{L^2}.$$

and since the Fourier transform of $\frac{1}{|x|^{3-\epsilon}}$ is $\frac{1}{|\xi|^\epsilon}$ so that the function f can be written as follows.

$$f = \mathfrak{F}^{-1}\Big(\frac{1}{|\xi|^\epsilon}\widehat{D^\epsilon f}\Big) = \frac{1}{|x|^{3-\epsilon}} * D^\epsilon f. \tag{1.26}$$

Hence, from [17], we have

$$\|f\|_{L^{\frac{6}{3-2\epsilon}}(\mathbb{R}^3)} \leqslant C(\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}. \tag{1.27}$$

Since $\frac{6}{3-2\epsilon} > 2$ and f has compact support in B_R, we have

$$\|f\|_{L^2(\mathbb{R}^3)} \leqslant C(R,\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}.$$

□

Remark. Thus $\|f\|_{H^\epsilon(\mathbb{R}^3)}$ is equivalent to $\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}$ if f has compact support. Therefore, the proof in this paper also works for the norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^3)}$.

Chapter II
An Application to Yang-Mills Equations

2.1. Introduction.

In this chapter, we are going to apply the estimates (1.8) and (1.9) to prove a local existence result for the Yang-Mills equations in the Lorentz gauge. After that, we will also prove a local existence result for the equations in the Coulomb gauge using different estimates but relaxing the regularity of the initial data.

The gauge potentials are A_μ which $A_\mu \in \mathcal{G}$, Lie algebra of a group G, where $\mu = 0, 1, 2, 3$. The Lagrangian for the gauge field, $F_{\mu\nu}$, is

$$\mathfrak{L}(A) = \int F_{\mu\nu} F^{\mu\nu} dx dt$$

. The operation between $F_{\mu\nu}$ and $F^{\mu\nu}$ is the inner product of two elements in a Lie algebra $\mathcal{G}$. The Minkowski metric is $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$, the gauge fields are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where $[A_\mu, A_\nu]$ denotes the commutator in the Lie algebra. In this work, $G = SU(2)$ is the gauge group we are interested in and the Lie algebra is $su(2)$. Since the Lie algebra, $su(2)$, is isomorphic with $\mathbb{R}^3$, thus the inner product in Lie algebra, $su(2)$, can be regarded as the usual inner product and the commutator is just the usual cross product of two three-vectors. In this case, A_μ can be regarded as vector valued functions from $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$ i.e. $A_\mu = (A_\mu^1, A_\mu^2, A_\mu^3)$, where A_μ^a are functions from $\mathbb{R}^4$ to $\mathbb{R}$, and $[A_\mu, A_\nu] = A_\mu \wedge A_\nu$, $\wedge$ means the usual cross product. We will use the metric $\eta_{\mu\nu}$ to raise and lower indices.

For example, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. For computation need, let us introduce some notations, $\vec{A} = (A_1, A_2, A_3)$, $A = (A_0, \vec{A})$.

2.2. The Yang-Mills Equations.

The Yang-Mills equations are a variation of the Maxwell's equations,

$$\begin{cases} \partial_t E = \nabla \wedge B, & \nabla \cdot E = 0; \\ \partial_t B = -\nabla \wedge E, & \nabla \cdot B = 0, \end{cases}$$

where E is the electric field and B the magnetic field. They are the Euler-Lagrange equations corresponding to variations of the formal Lagrangian

$$\mathcal{L} = \frac{1}{2}\int |E|^2 - |B|^2 dxdt$$

subject to the constraints

$$B = -\nabla \wedge A, \qquad E = \partial_t A + \nabla A_0$$

for some 3 dimensional vector $A = (A_1, A_2, A_3)$ and some scalar A_0. The 4 dimensional vector (A_0, A_1, A_2, A_3) is the potential. It is not unique; one can add to it a 4 dimensional gradient without affecting the fields E and B. The process of adding such a gradient is called a gauge transformation. Notice that the two Lagrangians mentioned are equavilent. Denote the covariant derivative operators $D_k = \partial_k + A_k\wedge$, $D_0 = -\partial_t + A_0\wedge$ and $D = (D_1, D_2, D_3)$. The equations can be written as

$$\begin{cases} D_0 E = -D \wedge B, & D \wedge E = 0, \\ D_0 B = -D \wedge E, & D \wedge B = 0, \\ E_k = \partial_k A_0 + \partial_t A_k + A_k \wedge A_0, & \text{and} \\ B_1 = \partial_3 A_2 - \partial_2 A_3 + A_2 \wedge A_3, & \cdots \quad \text{etc..} \end{cases}$$

On the other hand, one can derive the equations in the standard way. Since the equations are the critical point of the Lagrangian, $\mathfrak{L}$, thus we have

$$\frac{d}{d\delta}\mathfrak{L}(A+\delta B)\Big|_{\delta=0}=0.$$

Since

$$\frac{d}{d\delta}\mathfrak{L}(A+\delta B)\Big|_{\delta=0}=$$

$$\sum_{\mu<\nu}\eta^{\mu\mu}\eta^{\nu\nu}\int 2F_{\mu\nu}(A)[\partial_\mu B_\nu-\partial_\nu B_\mu+B_\mu\wedge(A_\nu+\delta B_\nu)+(A_\mu+\delta B_\mu)\wedge B_\nu]dxdt\Big|_{\delta=0}$$

$$=\sum_{\mu<\nu}\eta^{\mu\mu}\eta^{\nu\nu}\int 2F_{\mu\nu}(A)\big[\partial_\mu B_\nu-\partial_\nu B_\mu+B_\mu\wedge A_\nu+A_\mu\wedge B_\nu\big]dxdt=0.$$

From the above computation, choosing suitable functions B_μ, the general Yang-Mills equations are

$$\Box A_0+\partial_0(\nabla\cdot\vec{A}-\partial_0 A_0)+\Big[A_0\wedge\nabla\cdot\vec{A}+\sum_{k=1}^{3}(\partial_k A_0\wedge A_k+A_k\wedge\partial_0 A_k)\Big]$$
$$+\sum_{k=1}^{3}A_k\wedge(A_0\wedge A_k)=0,$$

$$\Box A_j+\partial_j(\nabla\cdot\vec{A}-\partial_0 A_0)+\Big[\partial_0 A_0\wedge A_j+2A_0\wedge\partial_0 A_j-A_0\wedge\partial_j A_0+ \tag{2.1}$$
$$A_j\wedge\nabla\cdot\vec{A}+\sum_{k=1}^{3}(2\partial_k A_j\wedge A_k+A_k\wedge\partial_j A_k)\Big]+\Big[A_0\wedge(A_0\wedge A_j)$$
$$+\sum_{k=1}^{3}A_k\wedge(A_j\wedge A_k)\Big]=0.$$

These are the original physical equations proposed by Yang and Mills in 1954. The underlying metric of space-time is the Minkowski metric.

Since The Yang-Mills equations are under-determined, we need an extra condition on them, which known as gauge conditions. There are many gauge conditions that have been posed, such as the temporal gauge, the Lorentz gauge, the Coulomb gauge, ... etc.

In the Lorentz gauge : $-\partial_0 A_0 + \nabla \cdot \vec{A} = 0$, the Yang-Mills equations are

(2.2a)

$$\Box A_0 + A_0 \wedge \partial_0 A_0 + \sum \big(2\partial_k A_0 \wedge A_k + A_k \wedge \partial_0 A_k\big) + \sum A_k \wedge (A_0 \wedge A_k) = 0,$$

$$\Box A_j + \Big[2A_0 \wedge \partial_0 A_j + \partial_j A_0 \wedge A_0 + \sum \big(2\partial_k A_j \wedge A_k + A_k \wedge \partial_j A_k\big)\Big] +$$

$$\Big[A_0 \wedge (A_0 \wedge A_j) + \sum A_k \wedge (A_j \wedge A_k)\Big] = 0,$$

with initial conditions $A_\mu(0,x) = f_\mu(x)$, $\partial_t A_\mu(0,x) = g_\mu(x)$. Let us call

$$\begin{aligned} F_0(A,\partial A) =& A_0 \wedge \partial_0 A_0 + \sum \big(2\partial_k A_0 \wedge A_k + A_k \wedge \partial_0 A_k\big) + \sum A_k \wedge (A_0 \wedge A_k), \\ F_j(A,\partial A) =& \Big[2A_0 \wedge \partial_0 A_j + \partial_j A_0 \wedge A_0 + \sum \big(2\partial_k A_j \wedge A_k + A_k \wedge \partial_j A_k\big)\Big] \\ &+ \Big[A_0 \wedge (A_0 \wedge A_j) + \sum A_k \wedge (A_j \wedge A_k)\Big], \end{aligned}$$

thus we can write the equations as $\Box A_\mu = F_\mu$. Using the notation $F_{\mu\nu}$, we can also rewrite the Yang-Mills equations (2.2a) as

(2.2b)

$$\begin{cases} \Box A_0 + A_0 \wedge \partial_0 A_0 + \sum \big(\partial_k A_0 \wedge A_k + A_k \wedge F_{0k}\big) = 0, \\ \Box A_j + \Big[A_0 \wedge \partial_0 A_j + A_0 \wedge F_{0j} + \sum \big(\partial_k A_j \wedge A_k + A_k \wedge F_{jk}\big)\Big] = 0. \end{cases}$$

2.3. Local Existence for the Equations in Lorentz Gauge.

Theorem 2.1. *Let us assume that the initial data $f_\mu \in H^{1+\epsilon}(\mathbb{R}^3)$ and $g_\mu \in H^{\epsilon}(\mathbb{R}^3)$. Then, for any $0 < \epsilon < \frac{1}{2}$ and the Yang-Mills equations in the Lorentz* *gauge, there is a $T > 0$, $C > 0$ depends on $\|f_\mu\|_{H^{1+\epsilon}(\mathbb{R}^3)}$ and $\|g_\mu\|_{H^{\epsilon}(\mathbb{R}^3)}$ such that the system (2.2a) has solutions $A_\mu : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ with*

$$\sup_{[0,T]} \|A_\mu(t,\cdot)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C, \tag{2.4}$$

$$\int_0^T \left\|D^\epsilon(A_\mu \wedge DA_\alpha)\right\|^q_{L^2(\mathbb{R}^3)} dt \leqslant C, \tag{2.5}$$

and

$$\sup_{[0,T]} \left\|D^\epsilon[A_\mu \wedge (A_\mu \wedge A_\alpha)](t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \leq C. \tag{2.6}$$

In order to prove the above theorem, we need the lemma stated below.

Lemma 2.2. *(Ponce & Sideris) If f, g $\in \mathcal{S}(\mathbb{R}^n)$ and $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}$, $i = 1, 2$, with $2 < p_i \leq +\infty$, then ,for $s > 0$,*

$$\|fg\|_{H^s} \leq C\Big(\|f\|_{L^{p_1}}\left\|(1-\Delta)^{\frac{s}{2}}g\right\|_{L^{q_1}} + \|g\|_{L^{p_2}}\left\|(1-\Delta)^{\frac{s}{2}}f\right\|_{L^{q_2}}\Big). \tag{2.7}$$

The proof of Lemma 2.2 is basically in Lemma X4 of [8] which is based on the ideas of Coifman and Meyer in [1]. The proof of the theorem 2.1 is similar to that of theorem 1.1. Besides those estimates (1.8), and (1.9), we need an estimate for the cubic terms in the equations (2.2a). A related result for operator $(-\Delta)^{\frac{1}{2}}$ can be found in [2].

Theorem 2.3. *If f, g, and h are all in $H^{1+\epsilon}(\mathbb{R}^3)$, then for any $\epsilon > 0$ there is a constant C depends on ϵ such that*

$$\|D^\epsilon(fgh)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{H^{1+\epsilon}(\mathbb{R}^3)}\|g\|_{H^{1+\epsilon}(\mathbb{R}^3)}\|h\|_{H^{1+\epsilon}(\mathbb{R}^3)}. \tag{2.8}$$

Proof. Here we employ (2.7), Hölder inequality, Sobolev inequality and (1.25) to estimate $D^\epsilon(fgh)$ in L^2-norm.

$$\begin{aligned}\|D^\epsilon(fgh)\|_{L^2} &\le \|fgh\|_{H^\epsilon}\\ &\le \|f\|_{L^6}\big\|(1-\Delta)^{\epsilon/2}(gh)\big\|_{L^3} + \|gh\|_{L^3}\big\|(1-\Delta)^{\epsilon/2}f\big\|_{L^6}\\ &\le \|f\|_{H^{1+\epsilon}}\big\|(1-\Delta)^{\epsilon/2}(gh)\big\|_{H^{\frac12}} + \|g\|_{L^6}\|h\|_{L^6}\big\|(1-\Delta)^{\epsilon/2}f\big\|_{H^1}\\ &\le \|f\|_{H^{1+\epsilon}}\|gh\|_{H^{\frac12+\epsilon}} + \|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}\|f\|_{H^{1+\epsilon}}.\end{aligned}$$

In the same vein, we estimate the product gh in $H^{\frac12+\epsilon}$-norm,

$$\begin{aligned}\|gh\|_{H^{\frac12+\epsilon}} &\le \|g\|_{L^6}\big\|(1-\Delta)^{\frac14+\frac{\epsilon}{2}}h\big\|_{L^3} + \|h\|_{L^6}\big\|(1-\Delta)^{\frac14+\frac{\epsilon}{2}}g\big\|_{L^3}\\ &\le \|g\|_{H^{1+\epsilon}}\big\|(1-\Delta)^{\frac14+\frac{\epsilon}{2}}h\big\|_{H^{\frac12}} + \|h\|_{H^{1+\epsilon}}\|g\|_{H^{1+\epsilon}}\\ &\le 2\|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}.\end{aligned}$$

Hence, we have

$$\|D^\epsilon(fgh)\|_{L^2} \le C\|f\|_{H^{1+\epsilon}}\|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}.$$

$\square$

Proof of theorem 2.1. The proof is based on a simple iteration like that of theorem 1.1. Define A^n_μ by setting $A^{-1}_\mu \equiv 0$ and for $n \ge 0$,

$$\Box A^n_\mu = F_\mu(A^{n-1}, \partial A^{n-1}), \tag{2.9}$$

with initial conditions

$$A^n_\mu(o,x) = f_\mu(x), \quad \partial_t A^n_\mu(0,x) = g_\mu(x). \tag{2.10}$$

It suffices to prove that there exists $T \le 1$, $\triangle \ge 1$ such that, for all n,

$$\sup_{[0,T]} \big\|(A^n_\mu - A^{n-1}_\mu)(t,\cdot)\big\|_{H^{1+\epsilon}} \le \frac{\triangle}{2^{n+1}}, \tag{2.11}$$

$$\sup_{[0,T]} \|A^n_\mu(t,\cdot)\|_{H^{1+\epsilon}} \le \triangle, \tag{2.12}$$

$$\int_0^T \left\| D^\epsilon \big[A_\mu^n \wedge D(A_\alpha^n - A_\alpha^{n-1}) \big] \right\|_{L^2(\mathbb{R}^3)}^q dt \le \frac{\triangle^{2q}}{2^{nq}}, \tag{2.13}$$

$$\int_0^T \left\| D^\epsilon \big[(A_\mu^n - A_\mu^{n-1}) \wedge D A_\alpha^{n-1} \big] \right\|_{L^2(\mathbb{R}^3)}^q dt \le \frac{\triangle^{2q}}{2^{nq}}, \tag{2.14}$$

$$\sup_{[0,T]} \left\| D^\epsilon \big[A_\mu^n \wedge \big(A_\mu^n \wedge (A_\alpha^n - A_\alpha^{n-1}) \big) \big] (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \le \frac{\triangle^3}{2^n}, \tag{2.15}$$

$$\sup_{[0,T]} \left\| D^\epsilon \big[A_\mu^n \wedge \big((A_\mu^n - A_\mu^{n-1}) \wedge A_\alpha^{n-1} \big) \big] (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \le \frac{\triangle^3}{2^n}, \tag{2.16}$$

$$\sup_{[0,T]} \left\| D^\epsilon \big[(A_\mu^n - A_\mu^{n-1}) \wedge (A_\mu^{n-1} \wedge A_\alpha^{n-1}) \big] (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \le \frac{\triangle^3}{2^n}, \tag{2.17}$$

$$\left(\int_0^T \left\| D^\epsilon (A_\mu^n \wedge D A_\alpha^n) \right\|_{L^2(\mathbb{R}^3)}^q dt \right)^{1/q} \le \triangle^2, \tag{2.18}$$

$$\sup_{[0,T]} \left\| D^\epsilon \big[A_\mu^n \wedge (A_\mu^n \wedge A_\alpha^n) \big] (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \le \triangle^3. \tag{2.19}$$

For $n = 0$, the above inequalities follow from the energy estimate (1.9) and theorem 1.2. Next, we assume that inequalities (2.11)-(2.19) are true for some $n \ge 0$ and prove they are true for $n+1$. The calculation here is similar to those in the proof of theorem 1.1. Inequalities (2.13), (2.14) and (2.18) follow from the proof of theorem 1.2. Using theorem 2.3 gives inequalities (2.15)-(2.17), and (2.19) for all $n \ge 0$. Using the energy estimate (1.9), the estimates (1.8) and (2.8) give inequalities (2.11), (2.12) for all $n \ge 0$. The proof is complete. $\square$

Chapter III

Another Application to Yang-Mills Equation

3.1. Introduction.

In this chapter, we are going to discuss another local existence result for Yang-Mills equations in the Coulomb gauge. Since some terms in the equations can be written as null forms and we have estimates for the null forms. This enables us to relax the regularity of initial data to energy norm, merely assuming energy is finite, but we need to require initial data, the energy, to be small for compensation.

3.2. Yang-Mills equations in the Coulomb gauge.

In the Coulomb gauge, $\nabla \cdot \vec{A} = 0$, the Yang-Mills equations can be written as follows.

(3.0a)

$$-\Delta A_0 + \sum \big(2\partial_k A_0 \wedge A_k + A_k \wedge \partial_0 A_k\big) + \sum A_k \wedge (A_0 \wedge A_k) = 0,$$

$$\Box A_j - \partial_{0j} A_0 + \Big[\partial_0 A_0 \wedge A_j + 2A_0 \wedge \partial_0 A_j - A_0 \wedge \partial_j A_0 +$$

$$\sum \big(2\partial_k A_j \wedge A_k + A_k \wedge \partial_j A_k\big)\Big] + \Big[A_0 \wedge (A_0 \wedge A_j) + \sum A_k \wedge (A_j \wedge A_k)\Big] = 0,$$

with initial conditions $A_j(0,x) = g_j(x)$, $\partial_t A_j(0,x) = h_j(x)$. For the same reason, we would like to rewrite the Yang-Mills equations (3.0a) as follows

$$(3.0\text{b}) \qquad \begin{cases} -\Delta A_0 + & \sum \big(\partial_k A_0 \wedge A_k + A_k \wedge F_{0k}\big) = 0, \\ \Box A_j - & \partial_{0j} A_0 + \Big[\partial_0 A_0 \wedge A_j + A_0 \wedge \partial_0 A_j + A_0 \wedge F_{0j} + \\ & \sum \big(\partial_k A_j \wedge A_k + A_k \wedge F_{jk}\big)\Big] = 0, \end{cases}$$

Similarly, we use the same abbreviation notation for equations (3.0a), $-\Delta A_0 = F_0$ and $\Box A_j = F_j$.

3.3. Local Existence for the Equations in the Coulomb Gauge.

The main result of this chapter is the following theorem.

Theorem 3.1. *Let us assume that the initial data $g_j \in H^1(\mathbb{R}^3)$ and $h_j \in L^2(\mathbb{R}^3)$. Then, if $\|g_j\|_{H^1} + \|h_j\|_{L^2}$ is sufficiently small, for the Yang-Mills equations in Coulomb gauge, there is a $T > 0$, $C > 0$ depending only on $\|g_j\|_{H^1}$ and $\|h_j\|_{L^2}$ such that the system (3.0a) has solutions $A_\mu : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ with*

$$\sum_{\mu=0}^{3} \|\partial_\mu A_j(t)\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|g_j\|_{H^1(\mathbb{R}^3)}, \|h_j\|_{L^2(\mathbb{R}^3)}\Big), \tag{3.1}$$

$$\sum_{\mu=0}^{3} \|\partial_\mu A_0(t)\|_{L^3(\mathbb{R}^3)} \leq C, \tag{3.2}$$

for all $0 \leq t \leq T$, and

$$\int_0^T \int_{\mathbb{R}^3} \Big|Q_{kl}(A_j^b, W_m^c)\Big|^2 dxdt \leqslant C, \tag{3.3}$$

$$\int_0^T \int_{\mathbb{R}^3} \Big|(-\Delta)^{-\frac{1}{2}} Q_{qr}(A_k^b, A_k^c)\Big|^2 dxdt \leqslant C, \tag{3.4}$$

where $Q_{kl}(B,V) = \partial_k B \partial_l V - \partial_k V \partial_l B$ and $\vec{W}$ satisfies $\vec{A} = \nabla \wedge \vec{W}$.

In order to prove theorem 3.1, we need the following ingredients. First, we need an estimate for the following system.

$$\begin{cases} \Box\phi = f, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ \phi(0,x) = f_0, \quad \partial_t\phi(0,x) = f_1, & x \in \mathbb{R}^3. \end{cases} \tag{3.5}$$

Theorem 3.2. *If $f_0 \in H^1(\mathbb{R}^3)$, $f_1 \in L^2(\mathbb{R}^3)$, and ϕ is a solution of the above system, then we have*

$$\text{(3.6)} \qquad \|\phi\|_{L^4_{\frac{1}{2}}(\mathbb{R}^3\times[0,T])} \leqslant C\left(\|f_0\|_{H^1(\mathbb{R}^3)} + \|f_1\|_{L^2(\mathbb{R}^3)} + \int_0^T \|f\|_{L^2(\mathbb{R}^3)}dt\right).$$

Proof. To prove (3.6), it is sufficient to prove the following estimate

$$\text{(3.7)} \qquad \left\|(D^{\frac{1}{2}}\phi)^2\right\|_{L^2(\mathbb{R}^3\times[0,T])} \leq C\left(\|f_0\|_{H^1} + \|f_1\|_{L^2} + \int_0^T \|f\|_{L^2}dt\right)^2.$$

The proof of the estimate is similar to the proof of theorem 1.2 and much simpler. It is a special case of the proof of (1.8) if we take ϵ=0. Therefore we get the estimate (3.6). For the proof see (4.23) in chapter 4. $\square$

The estimates of the null forms for the two systems

$$\text{(3.8)} \qquad \begin{array}{ll} \Box\phi_1 = \psi_1, & \Box\phi_2 = \psi_2 \\ \phi_1(0,\cdot) = g_1, \quad \partial_t\phi_1(0,\cdot) = h_1, & \phi_2(o,\cdot) = g_2, \quad \partial_t\phi_2(0,\cdot) = h_2, \end{array}$$

are the following

Theorem 3.3. *If ∇g_1, h_1, $\nabla^2 g_2$, and $\nabla h_2 \in L^2(\mathbb{R}^3)$, then, for systems (3.8), we have $Q(\phi_1,\phi_2) \in L^2(\mathbb{R}^3 \times [0,T])$, and*

(3.9)

$$\|Q(\phi_1,\phi_2)\|_{L^2(\mathbb{R}^3\times[0,T])} \leq C\left(\|\nabla g_1\|_{L^2(\mathbb{R}^3)} + \|h_1\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)}dt\right)$$

$$\cdot\left(\|\nabla^2 g_2\|_{L^2(\mathbb{R}^3)} + \|\nabla h_2\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\nabla\psi_2\|_{L^2(\mathbb{R}^3)}dt\right).$$

If ∇g_1, h_1, ∇g_2, and $h_2 \in L^2(\mathbb{R}^3)$, then, for systems (3.8), $(-\Delta)^{-\frac{1}{2}} Q(\phi_1, \phi_2) \in L^2(\mathbb{R}^3 \times [0,T])$, and

(3.10)

$$\left\| (-\Delta)^{-\frac{1}{2}} Q(\phi_1, \phi_2) \right\|_{L^2(\mathbb{R}^3 \times [0,T])} \leq C \left(\|\nabla g_1\|_{L^2} + \|h_1\|_{L^2} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)} dt \right) \cdot \left(\|\nabla g_2\|_{L^2} + \|h_2\|_{L^2} + \int_0^T \|\psi_2\|_{L^2(\mathbb{R}^3)} dt \right).$$

The proof of theorem 3.3 will be given in chapter 4. For the gauge potential A_0, since it satisfies an elliptic equation, therefore we need some estimates for the elliptic equations.

Lemma 3.4. *If $\Delta\phi = f$, f has compact support and is integrable, regular at $|x| \to \infty$, then, for every $\epsilon > 0$,*

$$\|\nabla\phi\|_{L^3(\mathbb{R}^3)} \leq C\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \tag{3.11}$$

$$\|\phi\|_{L^3(\mathbb{R}^3)} \leq C\left\|(-\Delta)^{-\frac{1}{2}} f\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \tag{3.12}$$

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \leq C_\epsilon\Big(\|f\|_{L^{\frac{3}{2}+\epsilon}(\mathbb{R}^3)} + \|\phi\|_{L^6(\mathbb{R}^3)}\Big). \tag{3.13}$$

Proof. These are standard elliptic estimates so we will only outline the proofs of (3.11) and (3.12). First, we prove part (3.11), since $\Delta\phi = f$, thus

$$\phi = \int \frac{1}{|x-y|} f(y)dy.$$

Therefore, we have

$$|\nabla\phi| \leq C\int \frac{1}{|x-y|^2}|f(y)|dy = I * f.$$

From [17], we have

$$\|\nabla\phi\|_{L^3} \leq C\|I * f\|_{L^3} \leq C\|f\|_{L^{\frac{3}{2}}}.$$

The proof of part (3.12) is similar. □

Besides the above theorems and lemma, we need the standard energy estimate and some Sobolev's inequalities

$$\|\phi\|_{L^\infty(\mathbb{R}^4)} \le C\|D\phi\|_{L^4_{\frac{1}{2}}(\mathbb{R}^4)}, \tag{3.14}$$

$$\|\phi\|_{L^4(\mathbb{R}^4)} \le C\|\phi\|_{H^1(\mathbb{R}^4)}, \tag{3.15}$$

$$\|\phi\|_{L^8(\mathbb{R}^4)} \le C\|\phi\|_{L^4_{\frac{1}{2}}(\mathbb{R}^4)}. \tag{3.16}$$

To eliminate the term, $\partial_{0j}A_0$, we need to define an operator $\mathcal{P}$ which is a projection onto divergence free vector field to be

$$\mathcal{P}\vec{B} = (-\Delta)^{-1}\big(\nabla \wedge (\nabla \wedge \vec{B})\big).$$

If $\nabla \cdot \vec{B} = 0$, then $\mathcal{P}\vec{B} = \vec{B}$. For simplicity sake, we will abuse the notation by writing $\mathcal{P}_j\vec{B} = (\mathcal{P}\vec{B})_j$. After this projection, the equations become

$$\Box A_j = \mathcal{P}_j\vec{F} = \mathcal{P}_j\vec{B} = F_j, \tag{3.17}$$

where $B_j = F_j - \partial_{0j}A_0$ and $\vec{B} = (B_1, B_2, B_3)$, so is $\vec{F}$. For F_j, we have $\nabla \cdot \vec{F} = 0$. This will be proved later. Since $\vec{A}$ and $\vec{F}$ are both divergence free , we can find $\vec{W}$ and $\vec{G}$ such that

$$\vec{A} = \nabla \wedge \vec{W}, \quad \vec{F} = \nabla \wedge \vec{G} \quad \text{and} \quad \Box W_j = G_j.$$

In order to estimate the quadratic terms, $\partial_k A_j \wedge A_k$ and $A_k \wedge \partial_j A_k$, we rewrite them into a null form(see [9] and [11])

$$\begin{aligned} \partial_k A_j \wedge A_k &= \partial_k A_j \wedge (\nabla \wedge \vec{W})_k = \epsilon_{klm}\partial_k A_j \wedge \partial_l W_m \\ &= \epsilon_{mkl}\partial_k A_j \wedge \partial_l W_m \\ &= \epsilon_{mkl}\mathcal{E}_{abc}Q_{kl}(A_j^b, W_m^c), \end{aligned} \tag{3.18}$$

and

(3.19)

$$\nabla \wedge \Big(A_k \wedge \nabla A_k\Big) = \nabla \wedge \Big(\epsilon_{lmn} A_k^m \nabla A_k^n\Big) = \epsilon_{pqr}\partial_q\Big(\epsilon_{lmn} A_k^m \partial_r A_k^n\Big)$$
$$= \epsilon_{lmn}\Big[\partial_q(A_k^m\partial_r A_k^n) - \partial_r(A_k^m\partial_q A_k^n)\Big]$$
$$= \epsilon_{lmn}\Big[\partial_q A_k^m\partial_r A_k^n - \partial_r A_k^m\partial_q A_k^n\Big]$$
$$= \epsilon_{lmn} Q_{qr}(A_k^m, A_k^n),$$

where ϵ_{lmn}, ϵ_{klm} and ϵ_{abc} are totally antisymmetric tensors.

Lemma 3.5. *For the Yang-Mills equations in the Coulomb gauge, $\nabla \cdot \vec{A} = 0$,*

$$-\Delta A_0 = F_0, \qquad \Box A_j = F_j, \tag{3.20}$$

if $\vec{F} = (F_1, F_2, F_3)$, then

$$\nabla \cdot \vec{F} = 0. \tag{3.21}$$

Proof. Here we need the equations (3.0a) and the following identities

$$A \wedge (B \wedge C) = (C \cdot A)B - (B \cdot A)C \tag{3.22}$$

and

$$A \wedge (B \wedge C) + B \wedge (C \wedge A) + C \wedge (A \wedge B) = 0, \tag{3.23}$$

where A, B and C are arbitrary 3-dimensional vectors. Using (3.0), (3.2), (3.22) and the Coulomb gauge condition give

$$-\nabla\cdot\vec{F}=\sum_j -\partial_j F_j$$

$$=\sum_j\Big\{-\partial_j\partial_{0j}A_0+\partial_{0j}A_0\wedge A_j+2\partial_jA_0\wedge\partial_0A_j-A_0\wedge\partial_{jj}A_0+$$

$$\sum_k\Big(2\partial_kA_j\wedge\partial_jA_k+A_k\wedge\partial_{jj}A_k\Big)+\partial_jA_0\wedge(A_0\wedge A_j)+A_0\wedge(\partial_jA_0\wedge A_j)+$$

$$\sum_k\Big[\partial_jA_k\wedge(A_j\wedge A_k)+A_k\wedge(A_j\wedge\partial_jA_j)\Big]\Big\}$$

$$=-\sum_k\Big\{A_k\wedge\Box A_k+\partial_{0k}A_0\wedge A_k+\Big[\partial_0A_k\wedge(A_0\wedge A_k)+A_k\wedge(\partial_0A_0\wedge A_k)+$$

$$A_k\wedge(A_0\wedge\partial_0A_k)+A_0\wedge(\partial_kA_0\wedge A_k)+A_0\wedge(A_k\wedge\partial_0A_k)$$

$$-(A_k\cdot A_0)(A_0\wedge A_k)-\partial_kA_0\wedge(A_0\wedge A_k)\Big]\Big\}$$

$$+\sum_{jk}\Big[\partial_jA_k\wedge(A_j\wedge A_k)+A_k\wedge(A_j\wedge\partial_jA_k)\Big]$$

$$=\sum_k\Big[A_k\wedge(A_0\wedge\partial_0A_k)-A_k\wedge(A_0\wedge\partial_kA_0)\Big]+\sum_{jk}\Big[2A_k\wedge(\partial_jA_k\wedge A_j)+$$

$$A_k\wedge(A_j\wedge\partial_kA_j)\Big]-\sum_k(A_k\cdot A_0)(A_0\wedge A_k)+\sum_{jk}(A_k\cdot A_j)(A_k\wedge A_j)-$$

$$\sum_k\partial_0A_k\wedge(A_0\wedge A_k)+\sum_k\Big[-A_0\wedge(\partial_kA_0\wedge A_k)-A_0\wedge(A_k\wedge\partial_0A_k)+$$

$$(A_k\cdot A_0)(A_0\wedge A_k)+\partial_kA_0\wedge(A_0\wedge A_k)\Big]+\sum_{jk}\Big[\partial_jA_k\wedge(A_j\wedge A_k)$$

$$+A_k\wedge(A_0\wedge\partial_jA_k)\Big]=0.$$

$\Box$

To prove theorem 3.1, we need one more lemma.

Lemma 3.6. *If vectors, $\vec{F}$ and $\vec{G}$, satisfy the relations given above, then*

$$\left\|\nabla\vec{G}\right\|_{L^2(\mathbb{R}^3)} \sim \left\|\vec{F}\right\|_{L^2(\mathbb{R}^3)}. \tag{3.24}$$

Proof. Since $\vec{\nabla}\wedge\vec{G} = \vec{F}$, we have

$$\xi\wedge\hat{F} = \xi\wedge(\xi\wedge\hat{G}), \tag{3.25}$$

here we use the notations F and G for $\vec{F}$ and $\vec{G}$.

Let $D_{ij} = \xi_i\xi_j - \delta_{ij}|\xi|^2$, the eigenvalues of matrix D are $|\xi|^2$, $|\xi|^2$, and 0. There is a matrix, R, such that

$$RDR^{-1}R\hat{G} = R\xi\wedge\hat{F}, \tag{3.26}$$

where RDR^{-1} is a diagonal matrix.

$$RDR^{-1}R\hat{G} = \begin{pmatrix} |\xi|^2 & 0 & 0 \\ 0 & |\xi|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} R\hat{G} = R\xi\wedge\hat{F}. \tag{3.27}$$

We can choose $(R\hat{G})_3 = 0$ so that

$$\begin{cases} |\xi|^2(R\hat{G})_1 & = (R\xi\wedge\hat{F})_1, \\ |\xi|^2(R\hat{G})_2 & = (R\xi\wedge\hat{F})_2, \\ 0 & = (R\xi\wedge\hat{F})_3. \end{cases} \tag{3.28}$$

Hence,

$$||\xi|\hat{G}| = |\xi||R\hat{G}| = |\xi|\sqrt{|(R\hat{G})_1|^2 + |(R\hat{G})_2|^2} \leq \left|\frac{\xi\wedge\hat{F}}{|\xi|}\right| \leq |\hat{F}|.$$

Thus

$$\|\nabla\vec{G}\|_{L^2} = \left\||\xi|\hat{G}\right\|_{L^2} \leq \|\hat{F}\|_{L^2} = \|\vec{F}\|_{L^2}.$$

$\square$

Proof of theorem 3.1. The proof is similar to that of theorem 1.1 and 2.1, based on a simple iteration procedure. We can construct a sequence of functions, say $\{A_\mu^n\}_{n=0}^\infty$, satisfying $A^{-1} \equiv 0$ and

$$-\Delta A_0^n = F_0(A_0^{n-1}, A_j^{n-1}), \tag{3.29}$$

$$\Box A_j^n = F_j(A_0^{n-1}, A_j^{n-1}),$$

with initial conditions

$$A_j(0,x) = g_j(x), \qquad \partial_t A_j(0,x) = h_j(x), \tag{3.30}$$

such that

$$\left\| A_j^n - A_j^{n-1} \right\|_{L^4_{\frac{1}{2}}(\mathbb{R}^3 \times (0,T))} \le \frac{\triangle}{2^{n+1}}, \tag{3.31}$$

$$\|A_j^n\|_{L^4_{\frac{1}{2}}(\mathbb{R}^3 \times (0,T))} \le C, \tag{3.32}$$

$$\int_0^T \int_{\mathbb{R}^3} \left| Q_{kl}(A_j^n - A_j^{n-1}, W_m^n) \right|^2 dxdt \le \frac{\triangle^4}{2^{2n}}, \tag{3.33}$$

$$\int_0^T \int_{\mathbb{R}^3} \left| Q_{kl}(A_j^{n-1}, W_m^n - W_m^{n-1}) \right|^2 dxdt \le \frac{\triangle^4}{2^{2n}}, \tag{3.34}$$

$$\int_0^T \int_{\mathbb{R}^3} \left| (-\Delta)^{-\frac{1}{2}} Q_{qr}(A_k^n - A_k^{n-1}, A_k^n) \right|^2 dxdt \le \frac{\triangle^4}{2^{2n}}, \tag{3.35}$$

$$\int_0^T \int_{\mathbb{R}^3} \left| (-\Delta)^{-\frac{1}{2}} Q_{qr}(A_k^{n-1}, A_k^n - A_k^{n-1}) \right|^2 dxdt \le \frac{\triangle^4}{2^{2n}}, \tag{3.36}$$

$$\sum_{\mu=0}^{3} \left\| \partial_\mu (A_j^n(t) - A_j^{n-1}(t)) \right\|_{L^2(\mathbb{R}^3)} \le \frac{\triangle}{2^n}, \tag{3.37}$$

$$\left\| \nabla (A_0^n - A_0^{n-1})(t) \right\|_{L^3(\mathbb{R}^3)} \le \frac{\triangle}{2^n}, \tag{3.38}$$

$$\left\| \partial_t (A_0^n - A_0^{n-1})(t) \right\|_{L^3(\mathbb{R}^3)} \le \frac{\triangle}{2^n}, \tag{3.39}$$

$$\int_0^T \int_{\mathbb{R}^3} \Big|Q_{kl}(A_j^n, W_m^n)\Big|^2 dxdt \le \triangle^4, \tag{3.40}$$

$$\int_0^T \int_{\mathbb{R}^3} \Big|(-\Delta)^{-\frac{1}{2}} Q_{qr}(A_k^n, A_k^n)\Big|^2 dxdt \le \triangle^4, \tag{3.41}$$

$$\sum_{\mu=0}^{3} \|\partial_\mu A_j^n(t)\|_{L^2(\mathbb{R}^3)} \le \triangle, \tag{3.42}$$

$$\|\nabla A_0^n(t)\|_{L^3(\mathbb{R}^3)} \le \triangle, \tag{3.43}$$

$$\|\partial_t A_0^n(t)\|_{L^3(\mathbb{R}^3)} \le \triangle, \tag{3.44}$$

For (3.44), We need an identity for $\partial_0 F_0$. Denote $B_j = F_j - \partial_{0j} A_0$. Using (3.21), we have

$$0 = \partial^j F_j = \partial_0 \Delta A_0 + \partial^j B_j = \partial_0 F_0 + \partial^j B_j. \tag{3.45}$$

Hence we have the identity $\partial_0 F_0 = -\partial^j B_j$. To prove (3.44), we use (3.12), Sobolev inequality and the identity to get

$$\begin{aligned}
&\|\partial_t A_0^{n+1}\|_{L^3(\mathbb{R}^3)} \le C\Big\|(-\Delta)^{-\frac{1}{2}}\partial_0 F_0^n\Big\|_{L^{\frac{3}{2}}} \le C\|B_j^n\|_{L^{\frac{3}{2}}} \\
\le& C\Big[\|\partial_0 A_0^n\|_{L^3}\|A_j^n\|_{L^3} + \|A_0^n\|_{L^6}\|\partial_0 A_j^n\|_{L^2} + \|A_0^n\|_{L^3}\|\partial_j A_0^n\|_{L^3} \\
&+ \|\partial_k A_j^n\|_{L^2}\|A_k^n\|_{L^6} + \|A_k^n\|_{L^6}\|\partial_j A_k^n\|_{L^2} + \|A_0^n\|_{L^6}^2\|A_j^n\|_{L^3} + \|A_k^n\|_{L^6}^2\|A_j^n\|_{L^3}\Big] \\
\le& C\Big[\triangle\|\nabla A_j^n\|_{L^2} + \|\nabla A_0^n\|_{L^3}\triangle + \|\nabla A_0^n\|_{L^3}\triangle + \triangle\|\nabla A_k^n\|_{L^2} \\
&+ \|\nabla A_k^n\|_{L^2}\triangle + \|\nabla A_0^n\|_{L^3}^2\|\nabla A_j^n\|_{L^2} + \|\nabla A_k^n\|_{L^2}^2\|\nabla A_j^n\|_{L^2}\Big] \le \triangle.
\end{aligned}$$

Note here we assume that $C(5\triangle + 2\triangle^2) \le 1$ and $\|A_0^n\|_{L^3} \le C\|\nabla A_0^n\|_{L^3}$,for some constant C.

For (3.43), using (3.11), Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
&\|\nabla A_0^{n+1}\|_{L^3(\mathbb{R}^3)} \le C\|F_0^n\|_{L^{\frac{3}{2}}} \\
&\le C\Big(\|\partial_k A_0^n \wedge A_k^n\|_{L^{\frac{3}{2}}} + \|A_k^n \wedge \partial_0 A_k^n\|_{L^{\frac{3}{2}}} + \|A_k^n \wedge (A_0^n \wedge A_k^n)\|_{L^{\frac{3}{2}}}\Big) \\
&\le C\Big(\|\nabla A_0^n\|_{L^3}\|A_k^n\|_{L^3} + \|A_k^n\|_{L^6}\|\partial_0 A_k^n\|_{L^2} + \|A_k^n\|_{L^4}^2\|A_0^n\|_{L^6}\Big) \\
&\le C\Big(\triangle\|\nabla A_k^n\|_{L^2} + \|\nabla A_k^n\|_{L^2}\triangle + \|\nabla A_k^n\|_{L^2}^2\|\nabla A_0^n\|_{L^3}\Big) \le \triangle.
\end{aligned}$$

Assume that $C(2\triangle + \triangle^2) \le 1$.

For (3.42), we need to estimate the gauge potential A_0 in L^∞-norm first. For this purpose, we use (3.13), Hölder inequality, Sobolev inequality, and letting $\epsilon = \frac{1}{10}$ to get

$$\begin{aligned}
&\|A_0\|_{L^\infty} \le C\Big(\|F_0\|_{L^{\frac{3}{2}+\epsilon}} + \|A_0\|_{L^6}\Big) \\
&\le C\Big(\|\nabla A_0\|_{L^3}\|A_k\|_{L^{\frac{24}{7}}} + \|A_k\|_{L^8}\|\partial_0 A_k\|_{L^2} + \|A_k\|_{L^{\frac{48}{11}}}^2\|A_0\|_{L^6} + \|\nabla A_0\|_{L^3}\Big) \\
&\le C\Big(\|\nabla A_0\|_{L^3}\|A_k\|_{L^8} + \|A_k\|_{L^8}\|\partial_0 A_k\|_{L^2} + \|A_k\|_{L^8}^2\|\nabla A_0\|_{L^3} + \|\nabla A_0\|_{L^3}\Big).
\end{aligned}$$

Now we can prove the estimate (3.42). Using (3.6), (3.9)-(3.13) gives

$$\begin{aligned}
&\|\partial_0 A_j^{n+1}\|_{L^2} + \|\nabla A_j^{n+1}\|_{L^2} \le C\Big(\|g_j\|_{H^1} + \|h_j\|_{L^2} + \int_0^T \|\mathcal{P}_j F^n\|_{L^2} dt\Big) \\
&\le \frac{\triangle}{2} + C\int_0^T \Big[\|\partial_0 A_0^n\|_{L^3}\|A_j^n\|_{L^6} + \|A_0^n\|_{L^\infty}\|\partial_0 A_j^n\|_{L^2} + \|A_0^n\|_{L^6}\|\nabla A_0^n\|_{L^3} \\
&\quad + \|Q_{kl}(A_j^n, W_m^n)\|_{L^2} + \Big\|(-\Delta)^{-\frac{1}{2}} Q_{qr}(A_k^n, A_k^n)\Big\|_{L^2} + \|A_0^n\|_{L^6}^2\|A_j^n\|_{L^6} + \\
&\quad \|A_k^n\|_{L^6}^2\|A_j^n\|_{L^6}\Big] dt \\
&\le \frac{\triangle}{2} + C\int_0^T \Big[\big(\|A_k^{n-1}\|_{L^8} + \|A_k^{n-1}\|_{L^8}^2 + 5\big)\triangle^2 + 2\triangle^3\Big] dt \le \triangle.
\end{aligned}$$

Assume $CT(5\triangle + 3\triangle^2 + \triangle^3) \le \frac{1}{2}$.

The computations for (3.37)-(3.39) are similar to those for (3.42)-(3.44). The computations for (3.31)-(3.36) are similar to those in theorem 1.1 and 2.1. The proof is complete. $\square$

For the gauge field $F_{\mu\nu} = \partial_\mu A_\nu$ - $\partial_\nu A_\mu + A_\mu \wedge A_\nu$, since they are in $L^2(\mathbb{R}^3)$, actually their L^2-norms are constant, for all time t. $\partial_\mu A_j$ are also in $L^2(\mathbb{R}^3)$, but only for all $0 \leq t \leq T$, therefore it is reasonable to expect that we can have an estimate of ∇A_0 in L^2-norm.

Proposition. *If A_μ are solutions of the system (3.0a) as in theorem 3.1. Then ∇A_0 is in $L^2(\mathbb{R}^3)$ for all $0 \leq t \leq T$.*

Proof. We multiply the equation (3.0b) by A_0 and integrate over $\mathbb{R}^3$. Using integration by parts give

$$\int |\nabla A_0|^2 dx \leq \sum \int \Big|(\partial_k A_0 \wedge A_k) \cdot A_0\Big| + \Big|(A_k \wedge F_{0k}) \cdot A_0\Big| dx.$$

Using Hölder inequality and Sobolev inequality give

$$\begin{aligned} \|\nabla A_0\|_{L^2}^2 &\leq \|\nabla A_0\|_{L^2}\|\partial A_k\|_{L^2}\|\nabla A_0\|_{L^3} + \|\nabla A_k\|_{L^2}\|F_{0k}\|_{L^2}\|\nabla A_0\|_{L^3} \\ &\leq \frac{1}{2}\|\nabla A_0\|_{L^2}^2 + 2\|\nabla A_k\|_{L^2}^2\|\nabla A_0\|_{L^3}^2 + C(F)\|\nabla A_k\|_{L^2}\|\|\nabla A_0\|_{L^3}, \end{aligned}$$

where $C(F)$ is a constant depends on the L^2-norm of F_{0k}, which is also constant. Hence we have

$$\|\nabla A_0\|_{L^2}^2 \leq C(F) + C\|\nabla A_k\|_{L^2}^2\|\nabla A_0\|_{L^3}^2. \tag{3.46}$$

$\square$

Chapter IV
Proof of the Estimates

In this part, we will prove theorems 1.2 and 3.3.

4.1. Estimate in Chapter II.

Theorem 1.2. *If $f^i \in H^{1+\epsilon}(\mathbb{R}^3), g^i \in H^{\epsilon}(\mathbb{R}^3)$, and $0 < \epsilon < \frac{1}{2}$, then for the system*

$$\begin{cases} \Box u^i = b^i, \\ u^i(0,x) = f^i(x), \qquad u^i_{,t}(0,x) = g^i(x), \end{cases} \tag{4.1}$$

we have the following estimate,

(4.2)

$$\left(\int_0^T \|D^\epsilon(u^l Du^k)\|^q_{L^2(\mathbb{R}^3)}dt\right)^{\frac{1}{q}} \leqslant C(\epsilon)\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^l\|_{L^2}dt\Big)$$
$$\cdot \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^k\|_{L^2}dt\Big).$$

where $C(\epsilon) = O(\frac{1}{\sqrt{\epsilon}})$ and $q = \frac{2}{1-2\epsilon}$.

Proof. By Duhamel's principle, it suffices to prove the following case: $\Box u^i = 0$, $u^i(0,x) = 0$, $u^i_{,t}(0,x) = f^i(x)$. For the case, $D = \partial_j$, since $|\frac{\eta_j}{|\eta|}| \leq 1$, it is enough to prove the estimate for $D = \partial_t$. Let $A(\xi) = 2\pi i(t|\xi| + x\cdot\xi)$, $B(\xi) = 2\pi i(-t|\xi| + x\cdot\xi)$. For simplicity, we skip the superscripts. Then the solution and its time derivative can be written as

$$\begin{cases} u(t,x) &= \int \frac{1}{|\xi|}\left(e^{A(\xi)} - e^{B(\xi)}\right)\hat{f}(\xi)d\xi = u_+(t,x) - u_-(t,x), \\ \partial_t u(t,x) &= \int \left(e^{A(\eta)} + e^{B(\eta)}\right)\hat{f}(\eta)d\eta = \partial_t u_+(t,x) + \partial_t u_-(t,x). \end{cases} \tag{4.3}$$

Then for $u_+\partial_t u_+$, we have

$$u_+\partial_t u_+ = \iint \frac{1}{|\xi|} e^{A(\xi)+A(\eta)} \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta. \tag{4.4}$$

Consider its inner product with an arbitrary function g,

$$\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle = \iint \frac{|\xi+\eta|^\epsilon}{|\xi|} \hat{g}_e \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta, \tag{4.5}$$

where $\hat{g}_e = \hat{g}(-|\xi|-|\eta|, -\xi-\eta)$. Taking absolute value on both sides of the above and using Hölder's inequality, we can get the inequality

$$\Big|\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle\Big| \le \Big(\iint \frac{|\xi+\eta|^{2\epsilon}}{|\xi|^{2+2\epsilon}|\eta|^{2\epsilon}} |\hat{g}_e|^2 d\xi d\eta\Big)^{\frac{1}{2}} \|f\|^2_{H^\epsilon}. \tag{4.6}$$

Let $\xi = \rho\omega$ where ω is a unit vector, $\xi+\eta = z$, and $|\xi|+|\eta| = \tau$. Then we have the formula for changes of variables

$$\begin{cases} \rho & = \frac{1}{2}\frac{\tau^2-|z|^2}{\tau - z\cdot\omega}, \\ |z-\rho\omega| & = \frac{1}{2}\frac{\tau^2-2\tau z\cdot\omega+|z|^2}{\tau-z\cdot\omega}, \\ \frac{d\rho}{d\tau} & = \frac{1}{2}\frac{\tau^2-2\tau z\cdot\omega+|z|^2}{(\tau-z\cdot\omega)^2}. \end{cases} \tag{4.7}$$

Since

$$d\xi d\eta = d\xi dz = \rho^2 d\rho d\omega dz = \rho^2 \frac{d\rho}{d\tau} d\omega d\tau dz,$$

we can change the coordinates for the integral in (4.6) to get

$$\begin{aligned} &\Big|\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle\Big| \\ \le &\Big(\iiint \int \frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}} \rho^2 \frac{d\rho}{d\tau} d\omega |\hat{g}(-\tau,-z)|^2 d\tau dz\Big)^{\frac{1}{2}} \|f\|^2_{H^\epsilon}. \end{aligned}$$

We denote

$$E(\tau, z) = \int \frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}} \rho^2 \frac{d\rho}{d\tau} d\omega \tag{4.8}$$

and use (4.7) and some substitution to estimate it.

$$
\begin{aligned}
E(\tau, z) &= \int \frac{|z|^{2\epsilon}}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega \leq \tau^{2\epsilon}\int \frac{1}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega \\
&= \frac{\tau^{2\epsilon}}{(\tau^2-|z|^2)^{2\epsilon}}\int \frac{(\tau^2+|z|^2-2\tau z\cdot\omega)^{1-2\epsilon}}{(\tau - z\cdot\omega)^{2-4\epsilon}}d\omega \\
&\leq \frac{C}{\tau^{2\epsilon}}\frac{1}{(1-\lambda)^{2\epsilon}}\int_0^{\pi}\frac{(1+\lambda^2-2\lambda\cos\theta)^{1-2\epsilon}}{(1-\lambda\cos\theta)^{2-4\epsilon}}\sin\theta d\theta \\
&\leq \frac{C}{\tau^{2\epsilon}}\int_1^{\frac{1+\lambda}{1-\lambda}}\frac{(s-\frac{1+\lambda}{2})^{1-2\epsilon}}{s^{2-4\epsilon}}ds \leq \frac{C(\epsilon)}{\tau^{2\epsilon}(1-\lambda)^{2\epsilon}} \leq \frac{C(\epsilon)}{(\tau-|z|)^{2\epsilon}}
\end{aligned}
$$

where the used substitution $\lambda = \frac{|z|}{\tau}$, $z\cdot\omega = |z|\cos\theta$ and $s = \frac{1-\lambda\cos\theta}{1-\lambda}$. Here $\epsilon > 0$, $\mathrm{C}(\epsilon) = \mathrm{O}(\frac{1}{\epsilon})$ and we only care about λ is closed to 1. It is easy to show that $E(\tau, z)$ is bounded by $\frac{C}{\tau^{2\epsilon}}$ when λ is closed to 0. Therefore, we have

$$
\left|\left\langle D^{\epsilon}(u_+\partial_t u_+), g\right\rangle\right| \leq C(\epsilon)\left(\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz\right)^{\frac{1}{2}}\|f\|^2_{H^{\epsilon}}, \tag{4.9}
$$

where $\mathrm{C}(\epsilon) = \mathrm{O}(\frac{1}{\sqrt{\epsilon}})$. Now we want to estimate the integral

$$
\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz.
$$

Since we know that if $0 < 2\epsilon < 1$, the Fourier transform of $\frac{1}{|\tau|^{2\epsilon}}$ is $\frac{C_\epsilon}{|t|^{1-2\epsilon}}$, where C_ϵ is some constant depends on ϵ (see [17] Stein). Using Hardy-Littlewood inequality, we have the following

$$\begin{aligned}
&\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz = \Big\langle \frac{1}{|\tau-|z||^{2\epsilon}}\hat{g},\hat{g}\Big\rangle \\
=&\Big\langle \mathcal{F}_t^{-1}\Big(\frac{1}{|\tau-|z||^{2\epsilon}}\Big) * \mathcal{F}_x(g), \mathcal{F}_x(g)\Big\rangle = C_\epsilon \Big\langle \frac{e^{it|z|}}{|t|^{1-2\epsilon}} * \mathcal{F}_x(g), \mathcal{F}_x(g)\Big\rangle \\
=&C_\epsilon \iiint \frac{e^{i(t-s)|z|}}{|t-s|^{1-2\epsilon}} \mathcal{F}_x(g)(s,-z)\overline{\mathcal{F}_x(g)}(t,-z) ds dt dz \\
\leq& C_\epsilon \iint \frac{1}{|t-s|^{1-2\epsilon}} \|\mathcal{F}_x(g)(s,\cdot)\|_{L^2}\|\mathcal{F}_x(g)(t,\cdot)\|_{L^2} ds dt \\
\leq& C_\epsilon \left(\int \|g(t,\cdot)\|_{L^2}^p dt\right)^{\frac{2}{p}},
\end{aligned}$$

where p $= \frac{2}{1+2\epsilon}$, $\mathcal{F}_t^{-1}$ is the inverse Fourier transform over time t and $\mathcal{F}_x$ the Fourier transform over space variable x. Therefore, we have

$$\Big|\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle\Big| \leq C\left(\int \|g(t,\cdot)\|_{L^2}^p dt\right)^{\frac{1}{p}} \|f\|_{H^\epsilon}^2. \tag{4.10}$$

Hence, we get

$$\left(\int \Big\|D^\epsilon(u_+^l D u_+^k)\Big\|_{L^2(\mathbb{R}^3)}^q dt\right)^{1/q} \leq C(\epsilon)\|f^l\|_{H^\epsilon}\|f^k\|_{H^\epsilon}, \tag{4.11}$$

where C(ϵ) = O$(\frac{1}{\sqrt{\epsilon}})$ and q $= \frac{2}{1-2\epsilon}$.

The calculation for the term $u_-\partial_t u_-$ is similar to the above.

For the term, $u_+\partial_t u_-$, we have

(4.12)

$$\begin{aligned}
\Big\langle D^\epsilon(u_+\partial_t u_-), g\Big\rangle &= \iiiint \frac{|\xi+\eta|^\epsilon}{|\xi|} e^{A(\xi)+B(\eta)} \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta g(t,x) dt dx \\
&= \iint \frac{|\xi+\eta|^{2\epsilon}}{|\xi|} \hat{g}_h \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta
\end{aligned}$$

We can split the above integral into two parts, one is $|\xi| > |\eta|$ and another is $|\xi| < |\eta|$. Let

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_+ = \iint_{\{|\xi|>|\eta|\}} \frac{|\xi+\eta|^\epsilon}{|\xi|}\hat{g}_h\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta, \tag{4.13}$$

and

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_- = \iint_{\{|\xi|<|\eta|\}} \frac{|\xi+\eta|^\epsilon}{|\xi|}\hat{g}_h\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta. \tag{4.14}$$

For (4.13), using coordinate change, we can rewrite it as

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_+ = \iint \int_\Sigma \frac{|z|^\epsilon}{\rho}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz, \tag{4.15}$$

where $\Sigma = \{\omega : |\omega| = 1, |z| \geq \omega\cdot z \geq \tau > 0\}$, $\hat{g}_h = \hat{g}(-|\xi|+|\eta|, -\xi-\eta)$, $\xi = \rho\omega$, $z = \xi+\eta$, and $\tau = |\xi|-|\eta|$. It is enough to consider $\tau > 0$. Thus, we have the formula for changes of variables that

$$\begin{cases} \rho & = \frac{1}{2}\frac{|z|^2-\tau^2}{z\cdot\omega-\tau}, \\ |z-\rho\omega| & = \frac{1}{2}\frac{|z|^2-2\tau z\cdot\omega+\tau^2}{z\cdot\omega-\tau}, \\ \frac{d\rho}{d\tau} & = \frac{1}{2}\frac{|z|^2-2\tau z\cdot\omega+\tau^2}{(z\cdot\omega-\tau)^2} \end{cases} \tag{4.16}$$

and the Fourier transform of $\partial^\epsilon(u_+\partial_t u_-)$ is

$$\mathcal{F}(\partial^\epsilon(u_+\partial_t u_-))(-\tau,-z) = \int_\Sigma \frac{|z|^\epsilon}{|\rho|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega. \tag{4.17}$$

Let $\Sigma = \Sigma_1 \cup \Sigma_2$, which

$$\Sigma_1 = \{z :\, |z| \geq \omega\cdot z \geq \tau + \delta(1-\lambda)|z|\},$$

and

$$\Sigma_2 = \{z :\, \tau + \delta(1-\lambda)|z| \geq \omega\cdot z \geq \tau > 0\},$$

where δ is any small positive number. We will estimate the integral on Σ_1 and Σ_2 separately. Thus we can split the integral into two parts so that

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_+ = \iint\int_{\Sigma_1}\cdots + \iint\int_{\Sigma_2}\cdots = I_1 + I_2. \tag{4.18}$$

For the term I_1, we have

$$I_1 = \iint\int_{\Sigma_1}\frac{|z|^\epsilon}{|\rho|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz.$$

Then

$$|I_1| \leq \left(\iint\int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}(-\tau,-z)|^2 d\tau dz\right)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}. \tag{4.19}$$

We denote the notation

$$H_{\Sigma_1}(\tau,z) = \int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega.$$

Thus, we can use (4.16) and some sbtitution to get

$$\begin{aligned}
H_{\Sigma_1}(\tau,z) &= \int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega = |z|^{2\epsilon}\int\frac{1}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega \\
&= \frac{|z|^{2\epsilon}}{(|z|^2-\tau^2)^{2\epsilon}}\int\frac{(\tau^2+|z|^2-2\tau z\cdot\omega)^{1-2\epsilon}}{(z\cdot\omega-\tau)^{2-4\epsilon}}d\omega \\
&\leq \frac{C}{|z|^{2\epsilon}(1-\lambda)^{2\epsilon}}\int_0^{\cos^{-1}(\lambda+\delta(1-\lambda))}\frac{(1+\lambda^2-2\lambda\cos\theta)^{1-2\epsilon}}{(\cos\theta-\lambda)^{2-4\epsilon}}\sin\theta d\theta \\
&\leq \frac{C}{|z|^{2\epsilon}}\int_\delta^1\frac{(1+\lambda-2\lambda s)^{1-2\epsilon}}{s^{2-4\epsilon}}ds < \frac{C(\delta)}{|z|^{2\epsilon}} < \frac{C(\delta)}{(|z|-\tau)^{2\epsilon}},
\end{aligned}$$

where we used the substitution $\lambda = \frac{\tau}{|z|}$, $z\cdot\omega = |z|\cos\theta$, and $s = \frac{\cos\theta-\lambda}{1-\lambda}$. Here $C(\delta) = \frac{C}{(1-4\epsilon)\delta^{1-4\epsilon}}$ and C is some constant. Therefore, we get

$$|I_1| \leq \left(\iint\frac{C(\delta)}{||z|-\tau|^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz\right)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}. \tag{4.20}$$

Now we want to estimate the second term, I_2. In order to do this we need two things. First, we note that on Σ_2 there exists constants C_1, C_2, such that

$$C_1 < \frac{|\xi|}{|\eta|} < C_2. \tag{4.21}$$

This is because

$$\frac{|\xi|}{|\eta|} = \frac{\rho}{|z-\rho\omega|} = \frac{|z|^2-\tau^2}{|z|^2+\tau^2-2\tau z\cdot\omega} = \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos\theta},$$

and

$$\lambda < \cos\theta < \lambda + \delta(1-\lambda)$$

on Σ_2. Hence, if we choose $\delta = \frac{1}{2}$, then we get $1 < \frac{|\xi|}{|\eta|} < 2$. We also have $|\xi+\eta| \le |\xi| + |\eta| \le C|\xi|$ (or $C|\eta|$). Second, let's consider the following system of equations

$$\begin{cases} \Box\phi &= 0, \\ \phi(0) &= 0, \qquad \partial_t\phi(0) = f', \end{cases} \tag{4.22}$$

where f' satisfies $\hat{f}' = |\hat{f}|$, then we have the result

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_{\pm})^2\right\|_{L^2(\mathbb{R}^4)} \le C\left\|D^{\epsilon}f'\right\|^2_{L^2(\mathbb{R}^3)}, \tag{4.23}$$

for any $\epsilon \ge 0$. The proof of (4.23) can be done directly in the same manner as above and will be given outline later.

For the term I_2, employing (4.21) and (4.23), we have

$$\begin{aligned} |I_2| &\le C\iint \frac{1}{|z|^{\epsilon}}\int_{\Sigma_2} \frac{|z|^{\epsilon}|z|^{\epsilon}}{|\rho|}\left|\hat{f}(\rho\omega)\right|\left|\hat{f}(z-\rho\omega)\right|\rho^2\frac{d\rho}{d\tau}d\omega\left|\hat{g}(-\tau,-z)\right|d\tau dz \\ &\le C\iint \frac{1}{|z|^{\epsilon}}\int_{\Sigma_2} \frac{1}{|\rho|^{\frac{1}{2}-\epsilon}|z-\rho\omega|^{\frac{1}{2}-\epsilon}}\left|\hat{f}(\rho\omega)\right|\left|\hat{f}(z-\rho\omega)\right|\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}|d\tau dz \end{aligned}$$

The integral over Σ_2 is basically the Fourier transform of $D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-$ which can be seen as follows.

(4.24)
$$
\begin{aligned}
&\left\langle D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-, g\right\rangle\\
&= \iint \frac{1}{|\xi|^{\frac{1}{2}-\epsilon}|\eta|^{\frac{1}{2}-\epsilon}} \hat{g}(-|\xi|+|\eta|, -\xi-\eta)\hat{f}'(\xi)\hat{f}'(\eta)d\xi d\eta\\
&= \iint \int_{\Sigma} \frac{1}{|\rho|^{\frac{1}{2}-\epsilon}|z-\rho\omega|^{\frac{1}{2}-\epsilon}} \hat{f}'(\rho\omega)\hat{f}'(z-\rho\omega)\rho^2 \frac{d\rho}{d\tau}\hat{g}(-\tau,-z)d\tau dz.
\end{aligned}
$$

Therefore we can estimate I_2 as follows.

$$
\begin{aligned}
|I_2| &\le C \iint \frac{1}{|z|^{\epsilon}} \mathcal{F}\Big(D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-\Big)|\hat{g}|d\tau dz\\
&\le C\left(\iint \frac{1}{|z|^{2\epsilon}}\chi_{\{0<\tau<|z|\}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}} \left\| D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-\right\|_{L^2(\mathbb{R}^4)}\\
&\le C\left(\iint \frac{1}{||z|-\tau|^{2\epsilon}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}} \left\| D^{\epsilon} f'\right\|^2_{L^2}.
\end{aligned}
$$

Combining the estimates for the terms I_1 and I_2, we have

$$
\left|\left\langle D^{\epsilon}(u_+\partial_t u_-), g\right\rangle\right| \le C(\epsilon,\delta)\left(\iint \frac{1}{||z|-\tau|^{2\epsilon}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}} \|f\|^2_{H^{\epsilon}}. \tag{4.25}
$$

Thus, with the same reason as in the estimate of the previous term, $D^{\epsilon}(u_+\partial_t u_+)$, we have

$$
\left|\left\langle D^{\epsilon}(u_+\partial_t u_-), g\right\rangle\right| \le C\left(\int \|g(t,\cdot)\|^p_{L^2}dt\right)^{\frac{1}{p}} \|f\|^2_{H^{\epsilon}}.
$$

Hence, we get

$$
\left(\int \left\| D^{\epsilon}(u^l_+ D u^k_-)\right\|^q_{L^2(\mathbb{R}^3)} dt\right)^{1/q} \le C(\epsilon)\|f^l\|_{H^{\epsilon}}\|f^k\|_{H^{\epsilon}}, \tag{4.26}
$$

where q $= \frac{2}{1-2\epsilon}$.

Finally, we will discuss the estimate of $\langle D^\epsilon(u_+\partial_t u_-), g\rangle_-$ briefly.

$$(4.27) \qquad \left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_- = \iint_{\{|\xi|<|\eta|\}} \frac{|\xi+\eta|^\epsilon}{|\xi|}\hat{g}_h\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta.$$

Let's switch the variables ξ with η, thus, we get

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_- = \iint\int_\Sigma \frac{|z|^\epsilon}{|z-\rho\omega|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz.$$

For this case, we compute H_{Σ_1} as follows.

$$\begin{aligned}
H_{\Sigma_1}(\tau,z) &= \int_{\Sigma_1}\frac{|z|^{2\epsilon}\rho^{2-2\epsilon}}{|z-\rho\omega|^{2+2\epsilon}}\frac{d\rho}{d\tau}d\omega \\
&\le \frac{C}{|z|^{2\epsilon}}\int_\delta^1 \frac{1}{(1+\lambda-2\lambda s)^{1+2\epsilon}s^{2-4\epsilon}}ds \\
&\le \frac{C}{|z|^{2\epsilon}}\left(\frac{1}{2\epsilon}\frac{1}{(1-\lambda)^{2\epsilon}}+\frac{1}{1-4\epsilon}\frac{1}{\delta^{1-4\epsilon}}\right) \\
&\le \frac{C(\epsilon,\delta)}{(|z|-\tau)^{2\epsilon}}.
\end{aligned}$$

Thus, it works for the term, I_1. For another term, I_2, the proof is similar. The proof for the term $D^\epsilon(u_-\partial_t u_+)$ is similar to those of the term $D^\epsilon(u_+\partial_t u_-)$. Hence, the proof is complete. $\square$

4.2. Estimate in Chapter III.

Next, We want to prove the estimates of the null forms for the two systems

$$(4.28) \qquad \begin{aligned}
&\Box\phi_1 = \psi_1, && \Box\phi_2 = \psi_2, \\
&\phi_1(0,\cdot) = g_1, \quad \partial_t\phi_1(0,\cdot) = h_1, && \phi_2(0,\cdot) = g_2, \quad \partial_t\phi_2(0,\cdot) = h_2,
\end{aligned}$$

from chapter III.

Theorem 3.3. *If ∇g_1, h_1, $\nabla^2 g_2$, and $\nabla h_2 \in L^2(\mathbb{R}^3)$, then, for the system (4.28), we have $Q(\phi_1, \phi_2) \in L^2(\mathbb{R}^3 \times [0,T])$, and*

(4.29)
$$\Big\|Q(\phi_1,\phi_2)\Big\|_{L^2(\mathbb{R}^3\times[0,T])} \le C\Big(\|\nabla g_1\|_{L^2(\mathbb{R}^3)} + \|h_1\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)}dt\Big)$$
$$\cdot\Big(\|\nabla^2 g_2\|_{L^2(\mathbb{R}^3)} + \|\nabla h_2\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\nabla\psi_2\|_{L^2(\mathbb{R}^3)}dt\Big).$$

If ∇g_1, h_1, ∇g_2, and $h_2 \in L^2(\mathbb{R}^3)$, then, for the system (4.28), the null form $(-\Delta)^{-\frac{1}{2}}Q(\phi_1,\phi_2)$ is in the space $L^2(\mathbb{R}^3 \times [0,T])$ and satisfies

(4.30)
$$\Big\|(-\Delta)^{-\frac{1}{2}}Q(\phi_1,\phi_2)\Big\|_{L^2(\mathbb{R}^3\times[0,T])} \le C\Big(\|\nabla g_1\|_{L^2} + \|h_1\|_{L^2} + \int_0^T \|\psi_1\|_{L^2(\mathbb{R}^3)}dt\Big)$$
$$\cdot\Big(\|\nabla g_2\|_{L^2} + \|h_2\|_{L^2} + \int_0^T \|\psi_2\|_{L^2(\mathbb{R}^3)}dt\Big).$$

Proof. For (4.29), see [10]. We only prove the estimate (4.30). Since the general case can be proved by using Duhamel's principle, it is enough to prove the following case

$$\begin{cases} \Box\phi &= 0, \qquad \phi(0) = 0, \qquad \partial_t\phi(0) = f, \\ \Box\psi &= 0, \qquad \psi(0) = 0, \qquad \partial_t\psi(0) = h. \end{cases} \tag{4.31}$$

As the proof in theorem 1.2, we let $A(\xi) = 2\pi i(t|\xi|+x\cdot\xi)$ and $B(\xi) = 2\pi i(-t|\xi|+x\cdot\xi)$ and the solutions for the above system are

(4.32)
$$\phi(t,x) = \int \frac{1}{|\xi|}\Big(e^{A(\xi)} - e^{B(\xi)}\Big)\hat{f}(\xi)d\xi = \phi_+(t,x) - \phi_-(t,x),$$
$$\psi(t,x) = \int \frac{1}{|\xi|}\Big(e^{A(\xi)} - e^{B(\xi)}\Big)\hat{h}(\xi)d\xi = \psi_+(t,x) - \psi_-(t,x).$$

For the term $Q_{ij}(\phi_+, \psi_+)$, we have

$$(4.33) \qquad \begin{aligned} Q_{ij}(\phi_+, \psi_+) &= \partial_i\phi_+\partial_j\psi_+ - \partial_i\psi_+\partial_j\phi_+ \\ &= \iint \frac{\xi_i\eta_j - \eta_i\xi_j}{|\xi||\eta|} e^{A(\xi)+A(\eta)} \hat{f}(\xi)\hat{h}(\eta) d\xi d\eta. \end{aligned}$$

Consider its inner product with an arbitrary function g and take absolute value on the inner product, we can get

(4.34)

$$\begin{aligned} \left|\left\langle (-\Delta)^{-\frac{1}{2}} Q_{ij}(\phi_+, \psi_+), g \right\rangle\right| &\leq \iint \frac{|\xi \wedge \eta|}{|\xi||\eta|} \frac{1}{|\xi + \eta|} |\hat{g}_e||\hat{f}(\xi)||\hat{h}(\eta)| d\xi d\eta \\ &\leq \left(\iint \frac{|\xi \wedge \eta|^2}{|\xi|^2|\eta|^2} \frac{1}{|\xi + \eta|^2} |\hat{g}_e|^2 d\xi d\eta \right)^{\frac{1}{2}} \|f\|_{L^2}\|h\|_{L^2}. \end{aligned}$$

Let $\xi = \rho\omega$ where ω is a unit vector, $\xi + \eta =$ z, and $|\xi| + |\eta| = \tau$. Then we have the same formula for change of variables as (4.7). Thus we can change the coordinates for the integral in (4.34) to get

$$\begin{aligned} &\left|\left\langle (-\Delta)^{-\frac{1}{2}} Q_{ij}(\phi_+, \psi_+), g \right\rangle\right| \\ \leq &\left(\iint \int \frac{|\omega \wedge z|^2}{|z - \rho\omega|^2|z|^2} \rho^2 \frac{d\rho}{d\tau} d\omega |\hat{g}_e|^2 d\xi d\eta \right)^{\frac{1}{2}} \|f\|_{L^2}\|h\|_{L^2}. \end{aligned}$$

We use the notation

$$E(\tau, z) = \int \frac{|\omega \wedge z|^2}{|z - \rho\omega|^2|z|^2} \rho^2 \frac{d\rho}{d\tau} d\omega$$

and have
(4.35)

$$
\begin{aligned}
E(\tau,z) &= \int \frac{|\omega\wedge z|^2}{|z-\rho\omega|^2|z|^2}\rho^2\frac{d\rho}{d\tau}d\omega \\
&= \int \frac{|\omega\wedge \hat{z}|^2(\tau^2-|z|^2)^2}{(\tau^2+|z|^2-2\tau z\cdot\omega)(\tau-z\cdot\omega)^2}d\omega \\
&= \frac{(\tau^2-|z|^2)^2}{\tau^4}\int_0^\pi \frac{\sin^3\theta}{(1+\lambda^2-2\lambda\cos\theta)(1-\lambda\cos\theta)^2}d\theta \\
&= C(1-\lambda)\int_1^{\frac{1+\lambda}{1-\lambda}} \frac{s-1}{(s-\frac{1+\lambda}{2})^2 s^2}ds < \infty,
\end{aligned}
$$

where $\lambda = \frac{|z|}{\tau}$, $z\cdot\omega = |z|\cos\theta$ and $s = \frac{1-\lambda\cos\theta}{1-\lambda}$, when $\epsilon > 0$. It is easy to show that $E(\tau,z)$ is bounded when λ is close to 0. Therefore, we have

$$
\left|\left\langle (-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_+), g\right\rangle\right| \le \|g\|_{L^2(\mathbb{R}^4)}\|f\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)}. \tag{4.36}
$$

The proof is similar for the term $(-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_-,\psi_-)$.

For the term $(-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)$, we use the formula of change of variables in (4.16) to get
(4.37)

$$
\begin{aligned}
&\left\langle (-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-), g\right\rangle \\
&= \iint (-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)(t,x)g(t,x)dtdx \\
&= \iint \frac{|\xi\wedge\eta|}{|\xi||\eta|}\frac{1}{|\xi+\eta|}\hat{g}_h\hat{f}(\xi)\hat{h}(\eta)d\xi d\eta \\
&= \iint\int_\Sigma \frac{\omega\wedge\hat{z}}{|z-\rho\omega|}\hat{f}(\rho\omega)\hat{h}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz.
\end{aligned}
$$

From the above, we can see that the Fourier transform of $(-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)$ is

$$
\mathcal{F}\Big((-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)\Big) = \int_\Sigma \frac{\omega\wedge\hat{z}}{|z-\rho\omega|}\hat{f}(\rho\omega)\hat{h}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega. \tag{4.38}
$$

Hence, we have
(4.39)
$$\Big\|(-\Delta)^{\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)\Big\|_{L^2}^2=\Big\|\mathcal{F}\Big((-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_+,\psi_-)\Big)\Big\|_{L^2}^2$$
$$=\iint\Big|\int_\Sigma\frac{\omega\wedge\hat{z}}{|z-\rho\omega|}\hat{f}(\rho\omega)\hat{h}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\Big|^2d\tau dz.$$

Let $\Sigma=\Sigma_1\cup\Sigma_2$, which
$$\Sigma_1=\{z:\,|z|\geq\omega\cdot z\geq\tau+\delta(1-\lambda)|z|\}$$
and
$$\Sigma_2=\{z:\,\tau+\delta(1-\lambda)|z|\geq\omega\cdot z\geq\tau\},$$
where δ is any small positive number. Thus the integral in (4.39) can be split into two parts, say I_1 and I_2, corresponding to Σ_1 and Σ_2 respectively. We will estimate the integral over Σ_1 and Σ_2 separately.

For the term I_1, we have
$$I_1\leq\iint\int_{\Sigma_1}\big|\hat{f}(\rho\omega)\big|^2\big|\hat{h}(z-\rho\omega)\big|^2\rho^2\frac{d\rho}{d\tau}d\omega\int_{\Sigma_1}\frac{|\omega\wedge\hat{z}|^2}{|z-\rho\omega|^2}\rho^2\frac{d\rho}{d\tau}d\omega d\tau dz. \tag{4.40}$$

We denote $H_{\Sigma_1}(\tau,z)=\int_{\Sigma_1}\frac{|\omega\wedge\hat{z}|^2}{|z-\rho\omega|^2}\rho^2\frac{d\rho}{d\tau}d\omega$ and use (4.17) and some substitution to get
(4.41)
$$H_{\Sigma_1}(\tau,z)=\int_{\Sigma_1}\frac{|\omega\wedge\hat{z}|^2}{|z-\rho\omega|^2}\rho^2\frac{d\rho}{d\tau}d\omega$$
$$=\frac{\big(|z|^2-\tau^2\big)^2}{|z|^4}\int_{\Sigma_1}\frac{|\omega\wedge\hat{z}|^2}{(\tau^2+|z|^2-2\tau\omega\cdot z)(\tau-z\cdot\omega)^2}d\omega$$
$$=C(1-\lambda)^2\int_0^{\cos^{-1}(\lambda+\delta(1-\lambda))}\frac{\sin^3\theta}{(1+\lambda^2-2\lambda\cos\theta)(\lambda-\cos\theta)^2}d\theta$$
$$=C(1-\lambda)\int_\delta^1\frac{\big(1+\lambda+(1-\lambda)s\big)(1-s)}{(\frac{1+\lambda}{2\lambda}-s)s^2}ds$$
$$\leq C(1-\lambda)\int_\delta^1 s^{-2}ds\leq C(\delta)<\infty,$$

where we used the $\lambda = \frac{\tau}{|z|}$, $z\cdot\omega = |z|\cos\theta$, and $s = \frac{\cos\theta - \lambda}{1-\lambda}$. Note that on Σ_2 there exists constants C_1, C_2, such that $C_1 < \sqrt{\frac{|\xi|}{|\eta|}} < C_2$ and $|\xi+\eta| \leq |\xi| + |\eta| \leq C|\xi|$ (or $C|\eta|$). Consider the the equations

(4.42)
$$\Box\phi' = 0, \quad \phi'(0) = 0, \quad \partial_t\phi'(0) = f', \quad \text{where} \quad \hat{f}' = |\hat{f}|;$$
$$\Box\psi' = 0, \quad \psi'(0) = 0, \quad \partial_t\psi'(0) = h', \quad \text{where} \quad \hat{h}' = |\hat{h}|.$$

Then, we have the same type of estimates as in (4.23). Thus, for the term I_2, we have

$$\begin{aligned} I_2 &= \iint \Big| \int_{\Sigma_2} \frac{\omega \wedge \hat{z}}{|z-\rho\omega|} \hat{f}(\rho\omega)\hat{h}(z-\rho\omega)\rho^2 \frac{d\rho}{d\tau} d\omega \Big|^2 d\tau dz \\ &\leq C \iint \Big(\int_{\Sigma_2} \frac{1}{\sqrt{\rho|z-\rho\omega|}} |\hat{f}(\rho\omega)||\hat{h}(z-\rho\omega)|\rho^2 \frac{d\rho}{d\tau} d\omega \Big)^2 d\tau dz. \end{aligned}$$

It is similar to (4.24) that

$$\int_{\Sigma} \frac{1}{\sqrt{\rho|z-\rho\omega|}} |\hat{f}(\rho\omega)||\hat{h}(z-\rho\omega)|\rho^2 \frac{d\rho}{d\tau} d\omega$$

is basically the Fourier transform of $D^{\frac{1}{2}}\phi'_+ D^{\frac{1}{2}}\psi'_-$. Therefore we can estimate I_2 as follows.

(4.43)
$$\begin{aligned} I_2 &\leq C \Big\| D^{\frac{1}{2}}\phi'_+ D^{\frac{1}{2}}\psi'_- \Big\|^2_{L^2} \\ &\leq C \Big\| (D^{\frac{1}{2}}\phi'_+)^2 \Big\|_{L^2} \Big\| (D^{\frac{1}{2}}\psi'_-)^2 \Big\|_{L^2} \\ &\leq C\|f'\|^2_{L^2}\|h'\|^2_{L^2} \leq C\|f\|^2_{L^2}\|h\|^2_{L^2}. \end{aligned}$$

Combining the estimates for the terms I_1 and I_2, we have

(4.44)
$$\Big\| (-\Delta)^{-\frac{1}{2}} Q_{ij}(\phi_+, \psi_-) \Big\|^2_{L^2} \leq C(\delta)\|f\|_{L^2}\|h\|_{L^2}.$$

The proof is similar for the term $(-\Delta)^{-\frac{1}{2}}Q_{ij}(\phi_-,\psi_+)$. $\square$

Proof of (4.23). We only outline the proof. Since

$$\left(D^{\frac{1}{2}+\epsilon}\phi_+\right)^2 = \iint \frac{1}{|\xi|^{\frac{1}{2}-\epsilon}+|\eta|^{\frac{1}{2}-\epsilon}} e^{A(\xi)+A(\eta)}\hat{f}'(\xi)\hat{f}'(\eta)d\xi d\eta,$$

thus we have

$$\begin{aligned}&\left|\left\langle \left(D^{\frac{1}{2}+\epsilon}\phi_+\right)^2, g\right\rangle\right|^2 \\ \le& \iiint \int \frac{1}{\rho|z-\rho\omega|}\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}(-\tau,-z)|^2 d\tau dz \left\|D^\epsilon f'\right\|^4_{L^2(\mathbb{R}^3)}.\end{aligned}$$

The inner-most integral called $E(\tau,z)$ can be computed as follows.

$$E(\tau,z) = \int \frac{1}{\rho|z-\rho\omega|}\rho^2\frac{d\rho}{d\tau}d\omega = \int_0^\pi \frac{1-\lambda^2}{(1-\lambda\cos\theta)^2}\sin\theta d\theta = \frac{1+\lambda}{\lambda}\int_1^{\frac{1+\lambda}{1-\lambda}}\frac{1}{s^2}ds$$

which is bounded by some constant. Hence we have

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_+)^2\right\|_{L^2(\mathbb{R}^4)} \le C\left\|D^\epsilon f'\right\|^2_{L^2(\mathbb{R}^3)}.$$

Similar calculation gives

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_-)^2\right\|_{L^2(\mathbb{R}^4)} \le C\left\|D^\epsilon f'\right\|^2_{L^2(\mathbb{R}^3)}.$$

$\square$

Chapter V
A Priori Estimates in Hardy Space

5.1. Introduction.

Hardy spaces $\mathcal{H}^p$ can be regarded as an extension of L^p spaces. It is a known result (see [18]) that $\mathcal{H}^p$ is equivalent to L^p if $p > 1$ and for $p \leq 1$, $\mathcal{H}^p$ is contained in L^p. Hardy spaces play better role in harmonic analysis than L^p spaces. This is the motivation that we want to investigate a priori estimates of wave equation in a norm related to Hardy space.

In this part, we want to show some estimates of null forms under a norm related to Hardy space. Let's consider the system

$$\begin{cases} \Box\phi = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.1}$$

where n = 2, 3.

5.2. Estimate of a Null Form.

The first null form we consider here is

$$Q_0(\phi,\phi)(t,x) = |\nabla\phi|^2(t,x) - |\partial_t\phi|^2(t,x). \tag{5.3}$$

Let's denote

$$H(Q_0)(t_0,x_0) = \sup_{0<r\leq 1} \frac{1}{r^{n+1}} \Big| \int \varphi\Big(\frac{t-t_0}{r}, \frac{x-x_0}{r}\Big) Q_0(t,x)\,dtdx \Big|, \tag{5.4}$$

where φ is a smooth function with support in the unit ball, and the norm of Hardy space

$$\|Q\|_{\mathcal{H}^1} = \sup_{t_0} \int_{\mathbb{R}^n} H(Q)(t_0,x)\,dx.$$

Theorem 5.1. *Let the initial data, ϕ_0 and ϕ_1, of the system (5.1) be in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ respectively. Then, for $n = 2, 3$, there is a constant C such that*

$$\|Q_0(\phi,\phi)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\Big(\|\phi_0\|^2_{H^1(\mathbb{R}^n)} + \|\phi_1\|^2_{L^2(\mathbb{R}^n)}\Big). \tag{5.5}$$

Proof. In order to compute $H(Q_0)$, the function ϕ needs to be localized. Let $q(x)$ be a smooth function such that $q(x) = 1$, for $|x| \leq 2$ and $= 0$, for $|x| \geq 3$. Thus, we choose initial data for the system

$$\begin{cases} \Box\psi(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ \psi(t_0,x) = q(\frac{x-x_0}{r})[\phi(t_0,x) - \bar{\phi}], & x \in \mathbb{R}^n, \\ \psi_t(t_0,x) = q(\frac{x-x_0}{r})[\phi_t(t_0,x)], & x \in \mathbb{R}^n, \end{cases} \tag{5.6}$$

where $\bar{\phi}$ is a constant. In the support of $\varphi(\frac{t-t_0}{r}, \frac{x-x_0}{r})$, one can notice that $\phi = \psi + C$ and $Q_0(\phi,\phi) = Q_0(\psi,\psi)$.

The solution ψ of the equation (5.6) can be written as

(5.7)

$$\psi(t,x) = \int_{\mathbb{R}^n} \frac{\sin(t-t_0)|\xi|}{|\xi|} e^{ix\cdot\xi}\hat{\psi}_t(t_0,\xi)d\xi + \int_{\mathbb{R}^n} \cos(t-t_0)|\xi| e^{ix\cdot\xi}\hat{\psi}(t_0,\xi)d\xi.$$

For simplicity, let's denote

$$A(\xi) = i\Big[(t-t_0)|\xi| + x\cdot\xi\Big], \quad B(\xi) = i\Big[-(t-t_0)|\xi| + x\cdot\xi\Big], \tag{5.8}$$

$$J_e = 1 - \frac{\xi\cdot\eta}{|\xi||\eta|}, \quad J_h = 1 + \frac{\xi\cdot\eta}{|\xi||\eta|}, \tag{5.9}$$

and

$$\hat{\psi}^+(\xi) = e^{A(\xi)}\hat{\psi}(t_0,\xi), \quad \hat{\psi}^-(\xi) = e^{B(\xi)}\hat{\psi}(t_0,\xi). \tag{5.10}$$

Compute $Q_0(\psi,\psi)$, we have

(5.11)

$$
\begin{aligned}
&4Q_0(\psi,\psi)\\
&=\int\Big\{J_e\Big[\hat{\psi}_t^+(\xi)\hat{\psi}_t^+(\eta)+\hat{\psi}_t^-(\xi)\hat{\psi}_t^-(\eta)\Big]+J_h\Big[\hat{\psi}_t^+(\xi)\hat{\psi}_t^-(\eta)\\
&+\hat{\psi}_t^-(\xi)\hat{\psi}_t^+(\eta)\Big]\Big\}d\xi d\eta-\int|\xi||\eta|\Big\{J_e\Big[\hat{\psi}^+(\xi)\hat{\psi}^+(\eta)+\hat{\psi}^-(\xi)\hat{\psi}^-(\eta)\Big]\\
&-J_h\Big[\hat{\psi}^+(\xi)\hat{\psi}^-(\eta)+\hat{\psi}^-(\xi)\hat{\psi}^+(\eta)\Big]\Big\}d\xi d\eta+\int i|\eta|\Big\{J_e\Big[\hat{\psi}_t^+(\xi)\hat{\psi}^+(\eta)\\
&-\hat{\psi}_t^-(\xi)\hat{\psi}^-(\eta)\Big]-J_h\Big[\hat{\psi}_t^+(\xi)\hat{\psi}^-(\eta)-\hat{\psi}_t^-(\xi)\hat{\psi}^+(\eta)\Big]\Big\}d\xi d\eta\\
&+\int i|\xi|\Big\{J_e\Big[\hat{\psi}^+(\xi)\hat{\psi}_t^+(\eta)-\hat{\psi}^-(\xi)\hat{\psi}_t^-(\eta)\Big]+J_h\Big[\hat{\psi}^+(\xi)\hat{\psi}_t^-(\eta)\\
&-\hat{\psi}^-(\xi)\hat{\psi}_t^+(\eta)\Big]\Big\}d\xi d\eta.
\end{aligned}
$$

Let's denote Q_0^k, where k = 1, 2, ..., 16, the terms in (5.11) as follows

$$
\left\{
\begin{array}{l}
Q_0^1(\psi,\psi)=\int J_e\hat{\psi}_t^+(\xi)\hat{\psi}_t^+(\eta)d\xi d\eta,\\
Q_0^2(\psi,\psi)=\int J_e\hat{\psi}_t^-(\xi)\hat{\psi}_t^-(\eta)d\xi d\eta,\\
Q_0^3(\psi,\psi)=\int J_h\hat{\psi}_t^+(\xi)\hat{\psi}_t^-(\eta)d\xi d\eta,\\
\text{...etc.}
\end{array}
\right.
\tag{5.12}
$$

Consider the term $Q_0^3(\psi,\psi)$, we have

(5.13)

$$
\begin{aligned}
&H_r[Q_0^3](t_0,x_0)=\frac{1}{r^{n+1}}\Big|\int\varphi\Big(\frac{t-t_0}{r},\frac{x-x_0}{r}\Big)Q_0^3(t,x)dtdx\Big|\\
&=\frac{1}{r^{n+1}}\Big|\int\varphi\Big(\frac{t-t_0}{r},\frac{x-x_0}{r}\Big)\int J_h\hat{\psi}_t^+(\xi)\hat{\psi}_t^-(\eta)d\xi d\eta dtdx\Big|\\
&=\Big|\int J_h\hat{\varphi}\Big(r(|\xi|-|\eta|),r(\xi+\eta)\Big)e^{ix_0\cdot(\xi+\eta)}\hat{\psi}_t(\xi)\hat{\psi}_t(\eta)d\xi d\eta\Big|\\
&\le\Big(\int|\hat{\psi}_t(\xi)\hat{\psi}_t(\eta)|^q d\xi d\eta\Big)^{\frac{1}{q}}\Big(\int J_h^p\Big|\hat{\varphi}\Big(r(|\xi|-|\eta|),r(\xi+\eta)\Big)\Big|^p d\xi d\eta\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Notice that $\frac{1}{p}+\frac{1}{q}=1$, $1\le p\le 2$, and $\hat{\varphi}(\tau,z)$ actually should be $\hat{\varphi}(-\tau,z)$. Let's denote the first integral term by I_1 and the second integral term by I_2.

For I_1, we use Hausdorff-Young inequality and (5.6) to get

$$I_1 \le \Big(\int |\psi_t(t_0,x)|^p dx\Big)^{\frac{2}{p}} \le \Big(\int_{B(x_0,3r)} |\phi_t(t_0,x)|^p dx\Big)^{\frac{2}{p}}.$$

Notice that $1<p\le 2$.

To estimate the term I_2, we need change of variables. Let $\xi=\rho\omega$ where ω is a unit vector, $\xi+\eta=$ z, and $|\xi|-|\eta|=\tau$. Then we have formula (4.16). Elementary calculation gives

$$\begin{aligned} I_2^p &= \int J_h^p |\hat{\varphi}(r\tau,rz)|^p \rho^{n-1} d\rho d\omega dz \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\{z\cdot\omega>\tau\}} \rho^{n-1} J_h^{p-1} d\omega |\hat{\varphi}(r\tau,rz)|^p d\tau dz, \end{aligned}$$

since $z\cdot\omega-\tau=\tau\cdot\omega+|\tau|>0$ for all τ and z. Let's call

$$I_3=\int_{\{z\cdot\omega>\tau\}} \rho^{n-1} J_h^{p-1} d\omega$$

and employ (4.16) so that we have

$$I_3 = C(|z|^2-\tau^2)^{n-1}\int_{\{z\cdot\omega>\tau\}} \frac{(z\cdot\omega-\tau)^{2p-n-1}}{(|z|^2-2\tau z\cdot\omega+\tau^2)^{p-1}} d\omega. \tag{5.15}$$

Set $z\cdot\omega=|z|\cos\theta$ and $\lambda=\frac{\tau}{|z|}$. Thus

$$I_3 = C|z|^{n-1}(1-\lambda^2)^{n-1}\int_0^{\arccos\lambda} \frac{(\cos\theta-\lambda)^{2p-n-1}}{(1-2\lambda\cos\theta+\lambda^2)^{p-1}} \sin^{n-2}\theta d\theta. \tag{5.16}$$

Notice that $|\tau|<|z|$, thus $|\lambda|<1$. Our main concern is that when $|\lambda|$ is close to 1.

For $\lambda > 0$ and is close to 1, we set change of variable $s = \frac{\cos\theta - \lambda}{1-\lambda}$ to get

$$I_3 = C|z|^{n-1}(1-\lambda)^{p+\frac{n-3}{2}} \int_0^1 \frac{s^{2p-n-1}(1-s)^{\frac{n-3}{2}}}{(1+\lambda-2\lambda s)^{p-1}} ds. \tag{5.16a}$$

Notice that $1 < 1+\lambda+(1-\lambda)s < 2$ for s $\in$ (0,1). Since $1-\lambda < 1+\lambda-2\lambda s < 2$, thus

$$I_3 = C|z|^{n-1}(1-\lambda)^{\frac{n-1}{2}} \int_0^1 s^{2p-n-1}(1-s)^{\frac{n-3}{2}} ds. \tag{5.16b}$$

For n = 2, we need that $p \in (1,2)$ such that $I_3 \leq C|z|$. For n = 3, we need that $p \in (\frac{3}{2}, 2)$ such that $I_3 \leq C|z|^2$. For n $\geq$ 4, it will require that $p > \frac{n}{2}$ to bound I_3 which contradicts to $p \leq 2$.

For $\lambda < 0$ and is close to -1, we set change of variable $s = \cos\theta - \lambda$ to get

$$I_3 = C|z|^{n-1}(1+\lambda)^{n-1} \int_0^{1-\lambda} \frac{s^{2p-n-1}(1+\lambda+s)^{\frac{n-3}{2}}(1-\lambda-s)^{\frac{n-3}{2}}}{(1-\lambda^2-2\lambda s)^{p-1}} ds. \tag{5.16c}$$

For n = 2, we split the integral, $\int_0^{1-\lambda}$, in (5.16c) into three parts, $\int_0^{1+\lambda} + \int_{1+\lambda}^1 + \int_1^{1-\lambda}$, and estimate them separately, also use the substitution $t = \frac{s}{1+\lambda}$ in the second part to get $I_3 \leq C(|z| + |\tau|)$, provided $1 < p < 2$. For n = 3, the computation is easier than the case n = 2 and we have $I_3 \leq c(|z|+|\tau|)^2$ provided that $\frac{3}{2} < p < 2$. For $n \geq 4$, we have to have $p > \frac{n}{2}$ in order to bound the term,I_3.

For $|\lambda|$ is close to 0, we have

$$I_3 \sim C|z|^{n-1} \int_0^{\cos^{-1}\lambda} (\cos\theta - \lambda)^{2p-n-1} \sin^{n-2}\theta d\theta. \tag{5.16d}$$

For n = 2, we set the substitution, $s = \cos\theta - \lambda$, then split the integral, $\int_0^{1-\lambda}$, into two parts, $\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-\lambda}$, to get that $I_3 \leq C(|z| + |\tau|)$, provided $1 < p < 2$. For n = 3, we can easily get $I_3 \leq C(|z| + |\tau|)^2$, provided $1 < p < 2$.

Therefore, for $|\lambda| < 1$, if n = 2, then we have $I_3 \leq C(|z| + |\tau|)$, provided $1 < p < 2$; if n = 3, then we have $I_3 \leq C(|z| + |\tau|)^2$, provided $\frac{3}{2} < p < 2$.

Thus, for the term I_2^p, we have

$$I_2^p \leq \frac{1}{r^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}} C(|z| + |\tau|)^{n-1} |\hat{\varphi}(\tau, z)|^p d\tau dz = \frac{1}{r^{2n}} C(\varphi)^p. \tag{5.17}$$

Hence, for $H_r[Q_0^3]$, we have

$$H_r[Q_0^3](t_0, x_0) \leq C\Big(\frac{1}{r^n} \int_{B(x_0, 3r)} |\phi_t(t_0, x)|^p dx\Big)^{\frac{2}{p}} \leq CM\Big(|\phi_t(t_0, \cdot)|^p\Big)^{\frac{2}{p}}(x_0).$$

Therefore, we get

$$\int_{R^n} H[Q_0^3](t_0, x) dx = \int \sup_{0<r\leq 1} H_r[Q_0^3] dx \leq C \int |\phi_t(t_0, x)|^2 dx. \tag{5.18}$$

The estimate for $H_r[Q_0^4]$ is the same as (5.18).

Consider the term $Q_0^7(\psi, \psi)$, we have

$$\begin{aligned} & H_r[Q_0^7](t_0, x_0) \\ & \leq \Big(\int \Big|\hat{\psi}(\xi)\hat{\psi}(\eta)|\xi||\eta|\Big|^q d\xi d\eta\Big)^{\frac{1}{q}} \Big(\int J_h^p \Big|\hat{\varphi}\Big(r(|\xi| - |\eta|), r(\xi + \eta)\Big)\Big|^p d\xi d\eta\Big)^{\frac{1}{p}} \\ & \leq \Big(\int \Big|\hat{\psi}(t_0, \xi)|\xi|\Big|^q d\xi\Big)^{\frac{2}{q}} \frac{1}{r^{\frac{2n}{p}}} C(\varphi). \end{aligned}$$

Let's denote I the above integral term. Since

$$\widehat{(R_j f)}(x) = i\frac{x_j}{|x|}\hat{f}(x),$$

where R_j is the Riesz transform, and using Hausdorff-Young inequality and

Poincaré inequality, we have

$$
\begin{aligned}
I &= \Big(\int \Big|\hat{\psi}(t_0,\xi)|\xi|\Big|^q d\xi\Big)^{\frac{2}{q}} \leq \Big(\int \Big|\sum_{j=1}^n \frac{i\xi_j}{|\xi} \hat{\psi}(t_0,\xi)\Big|^q d\xi\Big)^{\frac{2}{q}} \\
&\leq \Big(\int \Big|\sum_{j=1}^n R_j \partial_j \psi(t_0,\xi)\Big|^p d\xi\Big)^{\frac{2}{p}} \\
&\leq \sum_{j=1}^n \Big(\int \Big|\partial_j \psi(t_0,\xi)\Big|^p d\xi\Big)^{\frac{2}{p}} \leq \sum \Big(\int \Big|\partial_j q\Big(\frac{x-x_0}{r}\Big)[\phi(t_0,x)-\bar{\phi}]\Big|^p d\xi\Big)^{\frac{2}{p}} \\
&\leq C\Big[\Big(\int_{B(x_0,3r)} \frac{1}{r^p}|\phi(t_0,x)-\bar{\phi}|^p dx\Big)^{\frac{2}{p}} + \Big(\int_{B(x_0,3r)} |\partial_j \phi|^p dx\Big)^{\frac{2}{p}}\Big] \\
&\leq C\Big(\int_{B(x_0,3r)} |D\phi|^p dx\Big)^{\frac{2}{p}},
\end{aligned}
$$

where $\bar{\phi}$ can be chosen as the average of ϕ over $B(x_0, 3r)$.

Thus,

$$H_r[Q_0^7](t_0,x_0) \leq C\Big[M\Big(|D\phi(t_0,\cdot)|^p\Big)(x_0)\Big]^{\frac{2}{p}}.$$

Therefore, we have

$$\int_{R^n} H[Q_0^7](t_0,x)dx = \int \sup_{0<r\leq 1} H_r[Q_0^7]dx \leq C\int \Big|D\phi(t_0,x)\Big|^2 dx. \tag{5.19}$$

The estimate for $H_r[Q_0^8]$ is the same as (5.19).

For the terms $Q_0^{11}, Q_0^{12}, Q_0^{15}$, and Q_0^{16} in the same vein, we have

$$\int_{R^n} H[Q_0^k](t_0,x)dx \leq C\left(\int \Big|D\phi(t_0,x)\Big|^2 dx + \int \Big|\phi_t(t_0,x)\Big|^2 dx\right), \tag{5.20}$$

where k = 11, 12, 15, 16.

Now, we consider the term Q_0^1.

$$
\begin{aligned}
&H_r[Q_0^1]\\
&\leq \Big(\int \Big|\hat{\psi}_t(\xi)\hat{\psi}_t(\eta)\Big|^q d\xi d\eta\Big)^{\frac{1}{q}} \Big(\int J_e^p \Big|\hat{\varphi}\big(r(|\xi|+|\eta|), r(\xi+\eta)\big)\Big|^p d\xi d\eta\Big)^{\frac{1}{p}}\\
&\leq \Big(\int_{B(x_0,3r)} |\psi_t(t_0,\xi)|^p d\xi\Big)^{\frac{2}{p}} \Big(\int J_e^p \Big|\hat{\varphi}\big(r(|\xi|-|\eta|), r(\xi+\eta)\big)\Big|^p d\xi d\eta\Big)^{\frac{1}{p}}.
\end{aligned}
$$

To estimate the above second integral term, call it I_2, in the same vein, we change the variables by setting $\xi = \rho\omega$ where ω is a unit vector, $\xi + \eta =$ z, and $|\xi| + |\eta| = \tau$. Then we have the same formula as (4.7) and

(5.21)

$$
I_2^p = C \int_{\mathbb{R}^n \times \mathbb{R}^+} \int_{S^{n-1}} \frac{(\tau - z\cdot\omega)^{2p-n-1}(\tau^2 - |z|^2)^{n-1}}{(\tau^2 - 2\tau z\cdot\omega + |z|^2)p-1} d\omega |\hat{\varphi}(r\tau, rz)|^p d\tau dz.
$$

For the above integral over S^{n-1}, call it I_3, we set $\lambda = \frac{|z|}{\tau}$ and $z\cdot\omega = |z|\cos\theta$ to get

$$
I_3 = c\tau^{n-1}(1-\lambda^2)^{n-1}\int_0^\pi \frac{(1-\lambda\cos\theta)^{2p-n-1}}{(1-2\lambda\cos\theta+\lambda^2)^{p-1}} \sin^{n-2}\theta d\theta. \tag{5.22}
$$

When λ is close to 0, it is easy to bound the term, I_3, by $C(|\tau|+|z|)^{n-1}$. Thus, we mainly concern when λ is close to 1 and change the variable, $s = \frac{1-\lambda\cos\theta}{1-\lambda}$ to get

(5.23)

$$
\begin{aligned}
I_3 =& C\big(|\tau|+|z|\big)^{n-1}(1-\lambda)^{p+\frac{n-3}{2}}\\
&\cdot \int_1^{\frac{1+\lambda}{1-\lambda}} \frac{s^{2p-n-1}(s-1)^{\frac{n-3}{2}}\big(1+\lambda-(1-\lambda)s\big)^{\frac{n-3}{2}}}{(2s-1-\lambda)^{p-1}} ds.
\end{aligned}
$$

To estimate the integral in (5.23), we need to treat it differently in n = 2 and n = 3. For n = 2, we split the integral, $\int_1^{\frac{1+\lambda}{1-\lambda}}$, into three parts, $\int_1^2 + \int_2^{\frac{1}{1-\lambda}}$

$+ \int_{\frac{1}{1-\lambda}}^{\frac{1+\lambda}{1-\lambda}}$ and estimate them separately, then I_3 can be bounded by $C(|\tau| + |z|)$ for $1 < p < 2$. For n = 3, the case is simpler, we split the integral, $\int_1^{\frac{1+\lambda}{1-\lambda}}$, into two parts, $\int_1^2 + \int_2^{\frac{1+\lambda}{1-\lambda}}$ and estimate them separately, then I_3 can be bounded by $C(|\tau| + |z|)^2$.

For I_2^p, hence, we have

$$
\begin{aligned}
I_2^p &\leq C \int_{\mathbb{R}^n \times \mathbb{R}^+} (|\tau| + |z|)^{n-1} |\hat{\varphi}(r\tau, rz)|^p d\tau dz \\
&\leq C \frac{1}{r^{2n}} C(\varphi)^p.
\end{aligned}
$$

Thus, for $H_r[Q_0^1]$, we have

$$
H_r[Q_0^1](t_0, x_0) \leq CM\Big(|\phi_t(t_0, \cdot)|^p\Big)^{\frac{2}{p}}(x_0).
$$

Therefore, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} H[Q_0^1](t_0, x)dx &= \int \sup_{0<r\leq 1} H_r[Q_0^1](t_0, x)dx \\
&\leq C \int M\Big(|\phi_t(t_0, \cdot)|^p\Big)^{\frac{2}{p}}(x)dx \leq C\|\phi_t(t_0, \cdot)\|_{L^2}^2
\end{aligned}
$$

Similarly, we get

$$
\int_{\mathbb{R}^n} H[Q_0^k](t_0, x)dx \leq C\Big(\|D\phi(t_0, \cdot)\|_{L^2}^2 + \|\phi_t(t_0, \cdot)\|_{L^2}^2\Big),
$$

where k = 2, 5, 6, 9, 10, 13, 14.

Finally, we have

$$
\|Q_0(\phi, \phi)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\Big(\|\phi_0\|_{H^1(\mathbb{R}^n)}^2 + \|\phi_1\|_{L^2(\mathbb{R}^n)}^2\Big).
$$

This completes the proof. □

Remark. For $n \geq 4$, the estimate (5.5) fails.

5.3. Estimate of another Null Form.

Next we consider null forms of the system

$$\begin{cases} \Box\phi^i = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ \phi^i(0,x) = \phi_0^i(x), \quad \phi_t^i(0,x) = \phi_1^i(x), & x \in \mathbb{R}, \end{cases} \tag{5.24}$$

where $i = 1, 2$. The null forms are

$$Q_{jk}(\phi^1,\phi^2)(t,x) = \partial_j\phi^1\partial_k\phi^2(t,x) - \partial_j\phi^2\partial_k\phi^1(t,x), \tag{5.25}$$

where $j, k = 1, 2, 3$.

As in Theorem 5.1 for the null form Q_0, we can show a similar result for Q_{jk} in the same vein.

Theorem 5.2. *Let the initial data, ϕ_0^i and ϕ_1^i, of the system (5.24) be in $H^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ respectively. Let ϕ^i be the solutions to system (5.24). Then, there is a constant C such that*

$$\left\|D_x Q_{jk}(\phi^1,\phi^2)\right\|_{\mathcal{H}^1(\mathbb{R}^3)} \le C\Big(\|\partial^2\phi_0^1\|_{L^2}^2 + \|\partial\phi_1^1\|_{L^2}^2 + \|\partial\phi_0^2\|_{L^2}^2 + \|\phi_1^2\|_{L^2}^2\Big). \tag{5.26}$$

REFERENCES

1. R. Coifman and Y. Meyer, *Au-delà des Opérateurs Pseudo-différentiels*, Astériques **57** (1978), Société de Mathématique de France.
2. F. M. Christ and M. I. Weinstein, *Dispersion of Small Amplitude Solutions of the Generalized Korteweg-de Vries Equation*, J. Funct. Anal. **100** (1991), 87-109.
3. D. Eardley and V. Moncrief, *The Global Existence of Yang-Mills-Higgs Fields in 4-Dimensional Minkowski Space*, Comm. Math. Phys. **83** (1982), 171-212.
4. D. Gilbarg and N. Trudinger, *Elliptic P.D.E. of Second Order*, Springer Verlag, Germany, 1983.
5. J. Ginibre and G. Velo, *The Cauchy Problem for Couple Yang-Mills and Scalar Fields in the Temporal Gauge*, Comm. Math. Phys. **82** (1981), 1–28.
6. R.T. Glassey and W. Strauss, *Decay of Classical Yang-Mills Fields*, Comm. Math. Phys. **65** (1979), 1–13.
7. ______, *Some Global Solutions of the Yang-Mills Equations in Minkowski Space*, Comm. Math. Phys. **81** (1981), 171–187.
8. T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), 891-907.
9. S. Klainerman, *The Null Conditions and Global Existence to Nonlinear Wave Equations*, Lectures in Applied Math. Ed. B. Nicolaenko **23** (1986).
10. S. Klainerman and M. Machedon, *On the Maxwell-Klein-Gordon Equations with Finite Energy*, Duke Math. J. **74** (1994), 19-44.
11. ______, *Space-time Estimates for Null Forms and the Local Existence Theorem*, Comm. Pure Appl. Math. **46** (1993), 1221-1268.
12. ______, *Finite Energy Solutions of the Yang-Mills Equations in $\mathbb{R}^{3+1}$*, preprint.

13. H. Lindblad, *A Sharp Counter Example to Local Existence of Low Regularity Solutions to Nonlinear Wave Equations*, Duke Math. J. **72** (1993), 503-539.
14. Gustavo Ponce and T. C. Sideris, *Local Regularity of Nonlinear Wave Equation in Three Space Dimension*, Comm. PDE **18** (1993), 169-177.
15. Pedro Paulo S. Schirmer, *Global Existence for Spherically Symmetric Yang-Mills Fields on 3+1 Space-Time Dimensions*, PhD dissertation (1990), NYU.
16. I. Segal, *The Cauchy Problem for the Yang-Mills Equations*, J. Funct Anal. **33** (1979), 175-194.
17. E. M. Stein, *Singular Integral and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.
18. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory integrals*, Princeton Univ. Press, Princeton, New Jersey, 1993.
19. W.A. Strauss, *Nonlinear Wave Equations*, vol. 73, CBMS AMS, Providence, Rhode Island, 1989.