LOCAL EXISTENCE FOR SEMILINEAR WAVE EQUATIONS AND APPLICATIONS TO YANG-MILLS EQUATIONS

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Abstract. In this work we are concerned with a local existence of certain semilinear wave equations for which the initial data has minimal regularity. Assuming the initial data are in $H^{1+\epsilon}$ and H^{ϵ} for any $\epsilon > 0$, we prove a local result for the problem using a fixed point argument. The main ingredient is an a priori estimate for the quadratic nonlinear term uDu. They can be applied to the Yang-Mills equations in the Lorentz gauge.

0. Introduction.

In this paper, we are interested in deriving a new estimate which enables us to establish a local existence result for Yang-Mills equations with minimal assumptions on the regularity of the initial data. For this purpose, we want to study the following type of system of semilinear wave equations:

$$\begin{cases} \Box u^i = a^i_{jkl} u^l \partial_j u^k, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $\Box$ denotes the D'Alembertian $-\partial_t^2 + \Delta$ in $\mathbb{R}^{3+1}$, a^i_{jkl} are constants, i, j, k, $l = 0, 1, 2, 3$, and summation over repeating indices is implied. For equations (0.1), the energy is $E = \int |\partial_t u^i|^2 + |\nabla u^i|^2 dx$, so that the ideal minimal regularity for initial data is $f^i \in H^1(\mathbb{R}^3)$, $g^i \in L^2(\mathbb{R}^3)$. Unfortunately, this is not the case, see [13] and [14].

In 1979, Segal [17] established a local existence result of classical solution for the Cauchy problem of Yang-Mills equations in temporal gauge, with initial data in C^3. In 1981, Glassey and Strauss [7] found a class of

global weak solutions of Yang-Mills equations in the temporal gauge with initial data which are radial symmetric. In 1982, Eardley and Moncrief [3] obtained the global existence of a classical solution for the Cauchy problem of Yang-Mills equations in temporal gauge, with initial data in $H^{2+k}(\mathbb{R}^3)$ and $H^{1+k}(\mathbb{R}^3)$ for $k \geq 3$. In 1990, Schirmer [16] proved global existence of spherically symmetric solutions. In 1993, Ponce and Sideris [15] proved a local existence result for a similar equation with the initial data in $H^{2+\epsilon}(\mathbb{R}^3) \times H^{1+\epsilon}(\mathbb{R}^3)$ for $\epsilon > 0$. In 1995, Klainerman and Machedon [12] proved global existence for the Yang-Mills equations in the temporal gauge with finite energy initial data in $\mathbb{R}^3$.

We will prove a local existence result for the system (0.1) assuming that the regularity of initial data is just a little better than energy norm, namely $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$ and $g^i \in H^{\epsilon}(\mathbb{R}^3)$, where $\epsilon > 0$. This result is optimal (see [14] Lindblad). The main ingredients in the proof are the standard energy estimate and an a priori estimate given by the following theorem.

Theorem A. *(A Priori Estimate) Let u^i be a solution of the equations*

$$\begin{cases} \Box u^i = b^i, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u^i(0,x) = f^i(x), \qquad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3. \end{cases} \tag{0.2}$$

Then, for every $\frac{1}{2} > \epsilon > 0$, the following estimate holds

$$\begin{aligned} &\Big(\int_0^T \|D^\epsilon(u^l Du^k)\|^q_{L^2(\mathbb{R}^3)} dt \Big)^{\frac{1}{q}} \\ &\qquad \leqslant C(\epsilon) \Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^l\|_{L^2} dt \Big) \cdot \\ &\qquad\qquad \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^k\|_{L^2} dt \Big), \end{aligned} \tag{0.3}$$

where i, l, k = 0, 1, 2, 3, $q = \frac{2}{1-2\epsilon}$, the constant $C(\epsilon) = O(1/\sqrt{\epsilon})$, and D^ϵ is a fractional derivative defined via Fourier transform.

Next, we want to apply the above ideas to the Yang-Mills equations which can schematically be written as a system of the following form:

$$\Box A + ADA + A^3 = 0, \tag{0.4}$$

where A^a_μ are the gauge potentials, $a = 1, 2, 3$, and $\mu = 0, 1, 2, 3$, and DA represents any kind of first order derivatives of A. For the Yang-Mills equations in the Lorentz gauge, we can prove a local existence assuming that initial data is in $H^{1+\epsilon}(\mathbb{R}^3)$, for each $\frac{1}{2} > \epsilon > 0$.

1. Semilinear Wave Equation.

Consider the system equations of the following form.

$$\begin{cases} \Box u^i = a^i_{jkl} u^l \partial_j u^k, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Let us introduce some notations. The space $H^s(\mathbb{R}^n)$ is defined to be

$$H^s(\mathbb{R}^n) = \{f | (1-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R}^n)\}, \tag{1.2}$$

where $1-\Delta$ is the multiplier operator associated with $1+|\xi|^2$. The space is equipped with the norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$ defined by

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1+|\xi|^2)^{\frac{s}{2}} \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

The space $\dot{H}^s(\mathbb{R}^n)$ is defined to be

$$\dot{H}^s(\mathbb{R}^n) = \{f | (-\Delta)^{\frac{s}{2}} \hat{f} \in L^2(\mathbb{R}^n)\}, \tag{1.3}$$

with norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \||\xi|^s \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

The fractional derivative D^ϵ, via Fourier transform, is defined by

$$\widehat{D^\epsilon f}(\xi) = |\xi|^\epsilon \hat{f}(\xi). \tag{1.4}$$

We use the notation D to represent the all possible partial derivatives ∂_t and ∂_j. The case we are interested in is $n = 3$.

Theorem 1.1. *There exists a $T > 0$ and C depending only on $\|f^i\|_{H^{1+\epsilon}}$ and $\|g^i\|_{H^\epsilon}$, where $\epsilon > 0$, such that the system of equations (1.1) has solutions $u^i : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ with*

$$\sup_{[0,T]} \|u^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leqslant C, \tag{1.5}$$

where $i = 0, 1, 2, 3$, and

$$\int_0^T \|D^\epsilon(u^l Du^k)\|^q_{L^2(\mathbb{R}^3)} dt \leqslant C, \tag{1.6}$$

where $l, k = 0, 1, 2, 3$ and $q = \dfrac{2}{1-2\epsilon}$.

To prove theorem 1.1, we need the following main estimate whose proof will be given at the end.

Theorem 1.2. *Let u^i be the solutions of the system*

$$\begin{cases} \Box u^i = b^i, & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u^i(0,x) = f^i(x), \quad u^i_{,t}(0,x) = g^i(x), & x \in \mathbb{R}^3 \end{cases} \tag{1.7}$$

If $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$, $g^i \in H^\epsilon(\mathbb{R}^3)$, and $0 < \epsilon < \frac{1}{2}$, then we have

$$\begin{aligned} &\Big(\int_0^T \|D^\epsilon(u^l Du^k)\|^q_{L^2(\mathbb{R}^3)} dt\Big)^{\frac{1}{q}} \\ &\qquad \leqslant C(\epsilon)\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^l\|_{L^2} dt\Big)\cdot \\ &\qquad\qquad \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^k\|_{L^2} dt\Big), \end{aligned} \tag{1.8}$$

where $i, l, k = 0, 1, 2, 3$, $q = \frac{2}{1-2\epsilon}$ and $C(\epsilon) = O\big(1/\sqrt{\epsilon}\big)$.

We also need the standard energy estimate which states below.

Lemma 1.3. *Let u^i be the solutions of the system (1.7) and $\epsilon > 0$. If the initial data $f^i \in H^{1+\epsilon}(\mathbb{R}^3)$ and $g^i \in H^\epsilon(\mathbb{R}^3)$, then the following estimate holds*

$$\sup_{[0,T]} \|u^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leqslant C\Big(\|f^i\|_{H^{1+\epsilon}} + \|g^i\|_{H^\epsilon} + \int_0^T \|D^\epsilon b^i\|_{L^2(\mathbb{R}^3)} dt\Big). \tag{1.9}$$

Proof. We omit indices of the u^i for simplicity. It suffices to prove the followings:

$$\|Du(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|Du(0,\cdot)\|_{L^2} + \int_0^t \|b(\tau,\cdot)\|_{L^2} d\tau\Big), \tag{1.10}$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|u(0,\cdot)\|_{L^2} + \int_0^t \|Du(\tau,\cdot)\|_{L^2} d\tau\Big). \tag{1.11}$$

For the estimate (1.10), multiplying (1.7) by u_t and integrating by parts on $[0,T]\times\mathbb{R}^3$, then we have

$$\int_{t=T} e(u)dx - \int_{t=0} e(u)dx = \int_0^T \int_{\mathbb{R}^3} b(\tau,x)u_t dxdt, \tag{1.12}$$

where $e(u) = \frac{1}{2}(u_t^2 + |\nabla u|^2)$.

Let $E(t) = \int_{t=t} e(u)dx$, then we have

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}^3} bu_t dx \leq C\|b(t,\cdot)\|_{L^2} E^{\frac{1}{2}}(t). \tag{1.13}$$

Dividing both sides by $E^{\frac{1}{2}}$ and integrating them on $[0,t]$ give (1.10). In order to prove the estimate (1.11), we use the formula

$$u(t) - u(0) = \int_0^t \partial_t u d\tau. \tag{1.14}$$

Taking L^2-norm on both sides and applying Minkowski inequality, we get the desired result. □

Now we come to the stage to prove the Theorem 1.1.

Proof of theorem 1.1. The existence proof is based on a simple iteration procedure. Let $u^i_{-1} \equiv 0$. Define

$$\Box u^i_n = a^i_{jkl} u^l_{n-1} \partial_j u^k_{n-1}, \tag{1.18}$$

with initial data given by

$$u^i_n(0,x) = f^i(x), \quad \partial_t u^i_n(0,x) = g^i(x), \tag{1.19}$$

for all $n \geq 0$. Our goal is to prove that $u = \lim u_n$ exists and satisfies the equation and the desired estimates. It suffices to prove that there exists $T \leq 1, \triangle \geq 1$, such that, for all n,

$$\sup_{[0,T]} \|u_n^i(t) - u_{n-1}^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq \frac{\triangle}{2^{n+1}}, \tag{1.20}$$

$$\sup_{[0,T]} \|u_n^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C, \tag{1.21}$$

for some constant C that depends on $\triangle$ but not on n. Also

$$\int_0^T \left\|D^\epsilon[u_n^l \partial_j (u_n^k - u_{n-1}^k)]\right\|_{L^2(\mathbb{R}^3)}^q dt \leq \frac{\triangle^{2q}}{2^{nq}}, \tag{1.22}$$

$$\int_0^T \left\|D^\epsilon[(u_n^l - u_{n-1}^l)\partial_j u_{n-1}^k]\right\|_{L^2(\mathbb{R}^3)}^q dt \leq \frac{\triangle^{2q}}{2^{nq}}, \tag{1.23}$$

$$\int_0^T \left\|D^\epsilon(u_n^l \partial_j u_n^k)\right\|_{L^2(\mathbb{R}^3)}^q dt \leq \triangle^{2q}. \tag{1.24}$$

For $n = 0$, it is easy to check that (1.20)-(1.24) are satisfied.

Next, we assume they are true for some $n \geq 0$ and prove (1.20)-(1.24) are true for $n+1$ with a $T \leq 1$, T is independent of n and the same $\triangle$. $C = C(\|f^i\|_{H^{1+\epsilon}}, \|g^i\|_{H^\epsilon})$.

Using (1.24) and (1.8) give

$$\begin{aligned}
&\Big(\int_0^T \left\|D^\epsilon(u_{n+1}^l \partial_j u_{n+1}^k)\right\|_{L^2(\mathbb{R}^3)}^q dt\Big)^{1/q} \\
\leq & C\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^l\|_{L^2} dt\Big) \\
& \cdot \Big(\|f^k\|_{H^{1+\epsilon}} + \|g^k\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^k\|_{L^2} dt\Big) \\
\leq & \left(\frac{\triangle}{2} + CT^{1/p}\left(\int_0^T \left\|D^\epsilon(u_n^r Du_n^s)\right\|_{L^2}^q dt\right)^{1/q}\right)^2 \\
\leq & \Big(\frac{\triangle}{2} + CT^{1/p}\triangle \cdot \triangle\Big)^2 \leq \triangle^2,
\end{aligned}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, hence it is enough to have $CT^{1/p}\triangle \leq 1/2$.

Since

$$b_n^k - b_{n-1}^k = \big(u_n^r Du_n^s - u_{n-1}^r Du_n^s\big) + \big(u_{n-1}^r Du_n^s - u_{n-1}^r Du_{n-1}^s\big),$$

thus we can use (1.22) and (1.8) to get

$$\begin{aligned}
&\left(\int_0^T \big\|D^\epsilon[u_{n+1}^l \partial_j(u_{n+1}^k - u_n^k)]\big\|_{L^2(\mathbb{R}^3)}^q dt\right)^{\frac{1}{q}} \\
\leq& C\Big(\|f^l\|_{H^{1+\epsilon}} + \|g^l\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^l\|_{L^2} dt\Big)\Big(\int_0^T \big\|D^\epsilon(b_n^k - b_{n-1}^k)\big\|_{L^2} dt\Big) \\
\leq& C\Big(\frac{\triangle}{2} + T^{1/p}\triangle^2\Big)\cdot\Big(\int_0^T \big\|D^\epsilon\big[u_{n-1}^r D(u_n^s - u_{n-1}^s)\big]\big\|_{L^2} dt + \\
&\int_0^T \big\|D^\epsilon\big[(u_n^r - u_{n-1}^r)Du_n^s\big]\big\|_{L^2} dt\Big) \\
\leq& \triangle\Big[CT^{1/p}\big(\frac{\triangle^2}{2^n} + \frac{\triangle^2}{2^n}\big)\Big] \leq \frac{\triangle^2}{2^{n+1}}.
\end{aligned}$$

The computation for (1.23) is analogous.

Using (1.20) and (1.9) give

$$\begin{aligned}
&\sup_{[0,T]} \|u_{n+1}^i(t) - u_n^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \\
\leq& C\Big(\int_0^T \|D^\epsilon(b_n^i - b_{n-1}^i)\|_{L^2} dt\Big) \leq 2CT^{1/p}\frac{\triangle^2}{2^n} \leq \frac{\triangle}{2^{n+2}}.
\end{aligned}$$

Using (1.21) and (1.9) give

$$\begin{aligned}
&\sup_{[0,T]} \|u_{n+1}^i(t)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C\Big(\|f^i\|_{H^{1+\epsilon}} + \|g^i\|_{H^\epsilon} + \int_0^T \|D^\epsilon b_n^i\|_{L^2(\mathbb{R}^3)} dt\Big) \\
&\leq \frac{\triangle}{2} + CT^{1/p}\triangle^2 \leq \triangle.
\end{aligned}$$

Therefore the sequence of functions $\{u_n^i\}$ forms a Cauchy sequence under the norm $H^{1+\epsilon}(\mathbb{R}^3)$. □

We will also prove the following technical proposition.

Proposition 1.4. *If f has compact support in the ball B_R, then for any $\epsilon > 0$ there is a constant C such that*

$$\|f\|_{L^2(\mathbb{R}^3)} \leqslant C(R,\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}. \tag{1.25}$$

Proof. Using Plancherel theorem, we get

$$\left\|D^\epsilon f\right\|_{L^2} = \left\||\xi|^\epsilon \hat{f}\right\|_{L^2}.$$

and since the Fourier transform of $\frac{1}{|x|^{3-\epsilon}}$ is $\frac{1}{|\xi|^\epsilon}$ so that the function f can be written as follows.

$$f = \mathfrak{F}^{-1}\Big(\frac{1}{|\xi|^\epsilon}\widehat{D^\epsilon f}\Big) = \frac{1}{|x|^{3-\epsilon}} * D^\epsilon f. \tag{1.26}$$

Hence, from [18], we have

$$\|f\|_{L^{\frac{6}{3-2\epsilon}}(\mathbb{R}^3)} \leqslant C(\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}. \tag{1.27}$$

Since $\frac{6}{3-2\epsilon} > 2$ and f has compact support in B_R, we have

$$\|f\|_{L^2(\mathbb{R}^3)} \leqslant C(R,\epsilon)\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}.$$

□

Remark. Thus $\|f\|_{H^\epsilon(\mathbb{R}^3)}$ is equivalent to $\|D^\epsilon f\|_{L^2(\mathbb{R}^3)}$ if f has compact support. Therefore, the proof in this paper also works for the $\dot{H}^s(\mathbb{R}^3)$ norm .

2. An Application to Yang-Mills Equations.

In this part, we apply the estimates (1.8) and (1.9) to prove a local existence result for the Yang-Mills equations in the Lorentz gauge.

Let A_μ be the gauge potentials, where $\mu = 0,1,2,3$. The Minkowski metric is $\eta_{\mu\nu} = diag(-1,1,1,1)$ and the gauge fields are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where $[A_\mu, A_\nu]$ denotes the commutator. The Lagrangian for the gauge field $F_{\mu\nu}$ is

$$\mathfrak{L}(A) = \int F_{\mu\nu}F^{\mu\nu}dxdt.$$

Here A_μ are regarded as vector valued functions from $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$ i.e. $A_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$, where A^a_μ are functions from $\mathbb{R}^4$ to $\mathbb{R}$, and $[A_\mu, A_\nu] = A_\mu \wedge A_\nu$, $\wedge$ means the usual cross product. For computation need, let us introduce the notations, $\vec{A} = (A_1, A_2, A_3)$, $A = (A_0, \vec{A})$.

The Yang-Mills equations are the Euler-Lagrange equations of the formal Lagrangian

$$\mathcal{L} = \frac{1}{2}\int |E|^2 - |B|^2 dxdt,$$

subject to the constraints $B = -\nabla \wedge \vec{A}, E = \partial_t \vec{A} + \nabla A_0$. They can be written as

$$\begin{aligned}
&\Box A_0 + \partial_0(\partial^k A_k - \partial_0 A_0) + \Big[A_0 \wedge \partial^k A_k + \partial^k A_0 \wedge A_k + A^k \wedge \partial_0 A_k\Big] \\
&+ A^k \wedge (A_0 \wedge A_k) = 0, \\
&\Box A_j + \partial_j(\partial^k A_k - \partial_0 A_0) + \Big[\partial_0 A_0 \wedge A_j + 2A_0 \wedge \partial_0 A_j - A_0 \wedge \partial_j A_0 \\
&+ A_j \wedge \partial^k A_k + (2\partial^k A_j \wedge A_k + A^k \wedge \partial_j A_k)\Big] + \Big[A_0 \wedge (A_0 \wedge A_j) \\
&+ A^k \wedge (A_j \wedge A_k)\Big] = 0.
\end{aligned} \tag{2.1}$$

In the Lorentz gauge : $-\partial_0 A_0 + \nabla \cdot \vec{A} = 0$, the Yang-Mills equations can be rewritten as

$$\begin{aligned}
&\Box A_0 + A_0 \wedge \partial_0 A_0 + 2\partial^k A_0 \wedge A_k + A^k \wedge \partial_0 A_k + A^k \wedge (A_0 \wedge A_k) = 0, \\
&\Box A_j + \Big[2A_0 \wedge \partial_0 A_j + \partial_j A_0 \wedge A_0 + 2\partial^k A_j \wedge A_k + A^k \wedge \partial_j A_k\Big] + \\
&\Big[A_0 \wedge (A_0 \wedge A_j) + A^k \wedge (A_j \wedge A_k)\Big] = 0,
\end{aligned} \tag{2.2}$$

with initial conditions $A_\mu(0,x) = f_\mu(x)$, $\partial_t A_\mu(0,x) = g_\mu(x)$.

Theorem 2.1. *Assume that the initial data $f_\mu \in H^{1+\epsilon}(\mathbb{R}^3)$ and $g_\mu \in H^\epsilon(\mathbb{R}^3)$. Then, for any $0 < \epsilon < \frac{1}{2}$, there is a $T > 0$, $C > 0$ depends on*

the initial data such that the system (2.2) has solutions $A_\mu : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ *with*

$$\sup_{[0,T]} \|A_\mu(t,\cdot)\|_{H^{1+\epsilon}(\mathbb{R}^3)} \leq C, \tag{2.4}$$

$$\int_0^T \left\|D^\epsilon(A_\mu \wedge DA_\alpha)\right\|^q_{L^2(\mathbb{R}^3)} dt \leqslant C, \tag{2.5}$$

and

$$\sup_{[0,T]} \left\|D^\epsilon[A_\mu \wedge (A_\mu \wedge A_\alpha)](t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \leq C. \tag{2.6}$$

In order to prove the above theorem, we need the following lemma.

Lemma 2.2. *(Ponce & Sideris) If f, g* $\in \mathcal{S}(\mathbb{R}^n)$ *and* $\dfrac{1}{p_i} + \dfrac{1}{q_i} = \dfrac{1}{2}$, *i = 1, 2, with* $2 < p_i \leq +\infty$, *then ,for* $s > 0$,

$$\|fg\|_{H^s} \leq C\Big(\|f\|_{L^{p_1}} \left\|(1-\Delta)^{\frac{s}{2}} g\right\|_{L^{q_1}} + \|g\|_{L^{p_2}} \left\|(1-\Delta)^{\frac{s}{2}} f\right\|_{L^{q_2}}\Big). \tag{2.7}$$

The proof of the theorem 2.1 is similar to that of theorem 1.1. Besides those estimates (1.8), and (1.9), we also need an estimate for the cubic terms in the equations (2.2).

Theorem 2.3. *If f, g, and h are all in* $H^{1+\epsilon}(\mathbb{R}^3)$, *then for any* $\epsilon > 0$ *there is a constant* C *depends on* ϵ *such that*

$$\|D^\epsilon(fgh)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{H^{1+\epsilon}(\mathbb{R}^3)}\|g\|_{H^{1+\epsilon}(\mathbb{R}^3)}\|h\|_{H^{1+\epsilon}(\mathbb{R}^3)}. \tag{2.8}$$

Proof. Here we employ (2.7), Hölder inequality, Sobolev inequality and (1.25) to estimate $D^\epsilon(fgh)$ in L^2-norm.

$$\begin{aligned}
&\|D^\epsilon(fgh)\|_{L^2} \leq \|fgh\|_{H^\epsilon} \\
\leq &\|f\|_{L^6}\left\|(1-\Delta)^{\epsilon/2}(gh)\right\|_{L^3} + \|gh\|_{L^3}\left\|(1-\Delta)^{\epsilon/2} f\right\|_{L^6} \\
\leq &\|f\|_{H^{1+\epsilon}}\left\|(1-\Delta)^{\epsilon/2}(gh)\right\|_{H^{\frac{1}{2}}} + \|g\|_{L^6}\|h\|_{L^6}\left\|(1-\Delta)^{\epsilon/2} f\right\|_{H^1} \\
\leq &\|f\|_{H^{1+\epsilon}}\|gh\|_{H^{\frac{1}{2}+\epsilon}} + \|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}\|f\|_{H^{1+\epsilon}}.
\end{aligned}$$

In the same vein, we estimate the product gh in $H^{\frac{1}{2}+\epsilon}$-norm,

$$\begin{aligned}\|gh\|_{H^{\frac{1}{2}+\epsilon}} &\leq \|g\|_{L^6}\big\|(1-\Delta)^{\frac{1}{4}+\frac{\epsilon}{2}}h\big\|_{L^3} + \|h\|_{L^6}\big\|(1-\Delta)^{\frac{1}{4}+\frac{\epsilon}{2}}g\big\|_{L^3}\\ &\leq \|g\|_{H^{1+\epsilon}}\big\|(1-\Delta)^{\frac{1}{4}+\frac{\epsilon}{2}}h\big\|_{H^{\frac{1}{2}}} + \|h\|_{H^{1+\epsilon}}\|g\|_{H^{1+\epsilon}}\\ &\leq 2\|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}.\end{aligned}$$

Hence, we have

$$\|D^\epsilon(fgh)\|_{L^2} \leq C\|f\|_{H^{1+\epsilon}}\|g\|_{H^{1+\epsilon}}\|h\|_{H^{1+\epsilon}}.$$

□

Proof of theorem 2.1. The proof is also based on an iteration scheme like that of theorem 1.1. Define A^n_μ by setting $A^{-1}_\mu \equiv 0$ and for $n \geq 0$,

$$\Box A^n_\mu = F_\mu(A^{n-1}, \partial A^{n-1}), \tag{2.9}$$

with initial conditions

$$A^n_\mu(o,x) = f_\mu(x), \quad \partial_t A^n_\mu(0,x) = g_\mu(x). \tag{2.10}$$

It suffices to prove that there exists $T \leq 1$, $\triangle \geq 1$ such that, for all n,

$$\sup_{[0,T]}\big\|(A^n_\mu - A^{n-1}_\mu)(t,\cdot)\big\|_{H^{1+\epsilon}} \leq \frac{\triangle}{2^{n+1}}, \tag{2.11}$$

$$\sup_{[0,T]}\|A^n_\mu(t,\cdot)\|_{H^{1+\epsilon}} \leq \triangle, \tag{2.12}$$

$$\int_0^T \Big\|D^\epsilon\big[A^n_\mu \wedge D(A^n_\alpha - A^{n-1}_\alpha)\big]\Big\|^q_{L^2(\mathbb{R}^3)}dt \leq \frac{\triangle^{2q}}{2^{nq}}, \tag{2.13}$$

$$\int_0^T \Big\|D^\epsilon\big[(A^n_\mu - A^{n-1}_\mu) \wedge DA^{n-1}_\alpha\big]\Big\|^q_{L^2(\mathbb{R}^3)}dt \leq \frac{\triangle^{2q}}{2^{nq}}, \tag{2.14}$$

$$\sup_{[0,T]}\Big\|D^\epsilon\big[A^n_\mu \wedge \big(A^n_\mu \wedge (A^n_\alpha - A^{n-1}_\alpha)\big)\big](t,\cdot)\Big\|_{L^2(\mathbb{R}^3)} \leq \frac{\triangle^3}{2^n}, \tag{2.15}$$

$$\sup_{[0,T]}\Big\|D^\epsilon\big[A^n_\mu \wedge \big((A^n_\mu - A^{n-1}_\mu) \wedge A^{n-1}_\alpha\big)\big](t,\cdot)\Big\|_{L^2(\mathbb{R}^3)} \leq \frac{\triangle^3}{2^n}, \tag{2.16}$$

$$\sup_{[0,T]}\Big\|D^\epsilon\big[(A^n_\mu - A^{n-1}_\mu) \wedge (A^{n-1}_\mu \wedge A^{n-1}_\alpha)\big](t,\cdot)\Big\|_{L^2(\mathbb{R}^3)} \leq \frac{\triangle^3}{2^n}, \tag{2.17}$$

$$\left(\int_0^T \left\|D^\epsilon(A^n_\mu \wedge DA^n_\alpha)\right\|^q_{L^2(\mathbb{R}^3)} dt\right)^{1/q} \leq \triangle^2, \tag{2.18}$$

$$\sup_{[0,T]} \left\|D^\epsilon\left[A^n_\mu \wedge (A^n_\mu \wedge A^n_\alpha)\right](t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \leq \triangle^3. \tag{2.19}$$

For $n = 0$, the above inequalities follow from the energy estimate (1.9) and theorem 1.2. Next, we assume that inequalities (2.11)-(2.19) are true for some $n \geq 0$ and prove they are true for $n+1$. The calculation here is similar to those in the proof of theorem 1.1. Inequalities (2.13), (2.14) and (2.18) follow from the proof of theorem 1.2. Using theorem 2.3 gives inequalities (2.15)-(2.17), and (2.19) for all $n \geq 0$. Using (1.9), (1.8), and (2.8) gives inequalities (2.11), (2.12) for all $n \geq 0$. This completes the proof. □

3. Proof of the Estimates.

In this part, we prove theorems 1.2.

Proof of Theorem 1.2. By Duhamel's principle, it suffices to prove the following case: $\Box u^i = 0$, $u^i(0,x) = 0$, $u^i_{,t}(0,x) = f^i(x)$. For the case, $D = \partial_j$, since $\left|\frac{\eta_j}{|\eta|}\right| \leq 1$, it is enough to prove the estimate for $D = \partial_t$. Let

$$A(\xi) = 2\pi i(t|\xi| + x\cdot\xi), \quad B(\xi) = 2\pi i(-t|\xi| + x\cdot\xi).$$

For simplicity, we skip the superscripts. Then the solution and its time derivative can be written as

$$u(t,x) = \int \frac{1}{|\xi|}\left(e^{A(\xi)} - e^{B(\xi)}\right)\hat{f}(\xi)d\xi = u_+(t,x) - u_-(t,x), \tag{3.3a}$$

$$\partial_t u(t,x) = \int \left(e^{A(\eta)} + e^{B(\eta)}\right)\hat{f}(\eta)d\eta = \partial_t u_+(t,x) + \partial_t u_-(t,x). \tag{3.3b}$$

Then for $u_+\partial_t u_+$, we have

$$u_+\partial_t u_+ = \iint \frac{1}{|\xi|} e^{A(\xi)+A(\eta)}\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta. \tag{3.4}$$

Consider its inner product with an arbitrary function g,

$$\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle = \iint \frac{|\xi+\eta|^\epsilon}{|\xi|}\hat{g}_e\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta, \tag{3.5}$$

where $\hat{g}_e = \hat{g}(-|\xi|-|\eta|, -\xi-\eta)$. Taking absolute value on both sides, we can get the inequality

$$\Big|\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle\Big| \le \Big(\iint \frac{|\xi+\eta|^{2\epsilon}}{|\xi|^{2+2\epsilon}|\eta|^{2\epsilon}}|\hat{g}_e|^2 d\xi d\eta\Big)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}. \tag{3.6}$$

Let $\xi = \rho\omega$ where ω is a unit vector, $\xi+\eta = z$, and $|\xi|+|\eta| = \tau$. Then we have the formula for changes of variables

$$\rho = \frac{1}{2}\frac{\tau^2-|z|^2}{\tau - z\cdot\omega}, \tag{3.7a}$$

$$|z-\rho\omega| = \frac{1}{2}\frac{\tau^2-2\tau z\cdot\omega+|z|^2}{\tau - z\cdot\omega}, \tag{3.7b}$$

$$\frac{d\rho}{d\tau} = \frac{1}{2}\frac{\tau^2-2\tau z\cdot\omega+|z|^2}{(\tau - z\cdot\omega)^2}. \tag{3.7c}$$

Since

$$d\xi d\eta = d\xi dz = \rho^2 d\rho d\omega dz = \rho^2\frac{d\rho}{d\tau}d\omega d\tau dz,$$

we can change the variables in the integral (3.6) to get

$$\begin{aligned}&\Big|\Big\langle D^\epsilon(u_+\partial_t u_+), g\Big\rangle\Big| \\ \le&\Big(\iint\int \frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}(-\tau,-z)|^2 d\tau dz\Big)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}.\end{aligned}$$

We denote

$$E(\tau,z) = \int \frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega. \tag{3.8}$$

Employ (3.7) we can estimate it as follows.

$$\begin{aligned}E(\tau,z) &= \int \frac{|z|^{2\epsilon}}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega \le \tau^{2\epsilon}\int \frac{1}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega \\ &= \frac{\tau^{2\epsilon}}{(\tau^2-|z|^2)^{2\epsilon}}\int \frac{(\tau^2+|z|^2-2\tau z\cdot\omega)^{1-2\epsilon}}{(\tau-z\cdot\omega)^{2-4\epsilon}}d\omega \\ &\le \frac{C}{\tau^{2\epsilon}}\frac{1}{(1-\lambda)^{2\epsilon}}\int_0^\pi \frac{(1+\lambda^2-2\lambda\cos\theta)^{1-2\epsilon}}{(1-\lambda\cos\theta)^{2-4\epsilon}}\sin\theta d\theta \\ &\le \frac{C}{\tau^{2\epsilon}}\int_1^{\frac{1+\lambda}{1-\lambda}} \frac{(s-\frac{1+\lambda}{2})^{1-2\epsilon}}{s^{2-4\epsilon}}ds \le \frac{C(\epsilon)}{\tau^{2\epsilon}(1-\lambda)^{2\epsilon}} \le \frac{C(\epsilon)}{(\tau-|z|)^{2\epsilon}},\end{aligned}$$

where the used substitution $\lambda = \frac{|z|}{\tau}$, $z \cdot \omega = |z| \cos\theta$ and $s = \frac{1-\lambda\cos\theta}{1-\lambda}$. Here $\epsilon > 0$ and $\mathrm{C}(\epsilon) = \mathrm{O}(\frac{1}{\epsilon})$. Therefore, we have

$$\left|\left\langle D^{\epsilon}(u_+\partial_t u_+), g\right\rangle\right| \leq C(\epsilon)\left(\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz\right)^{\frac{1}{2}} \|f\|^2_{H^{\epsilon}}, \tag{3.9}$$

where $\mathrm{C}(\epsilon) = \mathrm{O}(\frac{1}{\sqrt{\epsilon}})$. Now we want to estimate the integral

$$\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz.$$

Since we know that if $0 < 2\epsilon < 1$, the Fourier transform of $\frac{1}{|\tau|^{2\epsilon}}$ is $\frac{C_\epsilon}{|t|^{1-2\epsilon}}$, where C_ϵ is some constant depends on ϵ (see [18] Stein). Using Hardy-Littlewood inequality, we have the following

$$\begin{aligned}
&\iint \frac{1}{|\tau-|z||^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz = \left\langle \frac{1}{|\tau-|z||^{2\epsilon}}\hat{g}, \hat{g}\right\rangle \\
=&\left\langle \mathcal{F}_t^{-1}\left(\frac{1}{|\tau-|z||^{2\epsilon}}\right) * \mathcal{F}_x(g), \mathcal{F}_x(g)\right\rangle = C_\epsilon\left\langle \frac{e^{it|z|}}{|t|^{1-2\epsilon}} * \mathcal{F}_x(g), \mathcal{F}_x(g)\right\rangle \\
=&C_\epsilon \iiint \frac{e^{i(t-s)|z|}}{|t-s|^{1-2\epsilon}}\mathcal{F}_x(g)(s,-z)\overline{\mathcal{F}_x(g)}(t,-z) ds dt dz \\
\leq&C_\epsilon \iint \frac{1}{|t-s|^{1-2\epsilon}}\|\mathcal{F}_x(g)(s,\cdot)\|_{L^2}\|\mathcal{F}_x(g)(t,\cdot)\|_{L^2} ds dt \\
\leq&C_\epsilon \left(\int \|g(t,\cdot)\|^p_{L^2} dt\right)^{\frac{2}{p}},
\end{aligned}$$

where $p = \frac{2}{1+2\epsilon}$, $\mathcal{F}_t^{-1}$ is the inverse Fourier transform over time t and $\mathcal{F}_x$ the Fourier transform over space variable x. Therefore, we have

$$\left|\left\langle D^{\epsilon}(u_+\partial_t u_+), g\right\rangle\right| \leq C\left(\int \|g(t,\cdot)\|^p_{L^2} dt\right)^{\frac{1}{p}} \|f\|^2_{H^{\epsilon}}. \tag{3.10}$$

Hence, we get

$$\left(\int \left\|D^{\epsilon}(u_+^l Du_+^k)\right\|^q_{L^2(\mathbb{R}^3)} dt\right)^{1/q} \leq C(\epsilon)\|f^l\|_{H^{\epsilon}}\|f^k\|_{H^{\epsilon}}, \tag{3.11}$$

where $\mathrm{C}(\epsilon) = \mathrm{O}(\frac{1}{\sqrt{\epsilon}})$ and $q = \frac{2}{1-2\epsilon}$. The calculation for the term $u_-\partial_t u_-$ is analogous to that of the above.

For the term, $u_+\partial_t u_-$, we have

$$\begin{aligned}
&\Big\langle D^\epsilon(u_+\partial_t u_-), g\Big\rangle \\
&= \iiiint \frac{|\xi+\eta|^\epsilon}{|\xi|} e^{A(\xi)+B(\eta)} \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta g(t,x) dt dx \\
&= \iint \frac{|\xi+\eta|^{2\epsilon}}{|\xi|} \hat{g}_h \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta
\end{aligned} \tag{3.12}$$

We can split the above integral into two parts, one is $|\xi| > |\eta|$ and another is $|\xi| < |\eta|$. Let

$$\Big\langle D^\epsilon(u_+\partial_t u_-), g\Big\rangle_+ = \iint_{\{|\xi|>|\eta|\}} \frac{|\xi+\eta|^\epsilon}{|\xi|} \hat{g}_h \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta, \tag{3.13}$$

and

$$\Big\langle D^\epsilon(u_+\partial_t u_-), g\Big\rangle_- = \iint_{\{|\xi|<|\eta|\}} \frac{|\xi+\eta|^\epsilon}{|\xi|} \hat{g}_h \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta. \tag{3.14}$$

For (4.13), using the change of variables, we can rewrite it as

$$\begin{aligned}
&\Big\langle D^\epsilon(u_+\partial_t u_-), g\Big\rangle_+ \\
&= \iint \int_\Sigma \frac{|z|^\epsilon}{\rho} \hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2 \frac{d\rho}{d\tau} d\omega \hat{g}(-\tau,-z) d\tau dz,
\end{aligned} \tag{3.15}$$

where $\Sigma = \{\omega : |\omega| = 1, |z| \geq \omega\cdot z \geq \tau > 0\}$, $\hat{g}_h = \hat{g}(-|\xi|+|\eta|, -\xi-\eta)$, $\xi = \rho\omega$, $z = \xi+\eta$, and $\tau = |\xi|-|\eta|$. It is sufficient to consider $\tau > 0$. Thus, we have the formula for changes of variables that

$$\rho = \frac{1}{2}\frac{|z|^2-\tau^2}{z\cdot\omega-\tau}, \tag{3.16a}$$

$$|z-\rho\omega| = \frac{1}{2}\frac{|z|^2-2\tau z\cdot\omega+\tau^2}{z\cdot\omega-\tau}, \tag{3.16b}$$

$$\frac{d\rho}{d\tau} = \frac{1}{2}\frac{|z|^2-2\tau z\cdot\omega+\tau^2}{(z\cdot\omega-\tau)^2}, \tag{3.16c}$$

and the Fourier transform of $\partial^\epsilon(u_+\partial_t u_-)$ is

$$\mathcal{F}(\partial^\epsilon(u_+\partial_t u_-))(-\tau,-z)=\int_\Sigma \frac{|z|^\epsilon}{|\rho|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega. \tag{3.17}$$

Let $\Sigma=\Sigma_1\cup\Sigma_2$, which

$$\Sigma_1=\{z:\ |z|\geq\omega\cdot z\geq\tau+\delta(1-\lambda)|z|\},$$

and

$$\Sigma_2=\{z:\ \tau+\delta(1-\lambda)|z|\geq\omega\cdot z\geq\tau>0\},$$

where δ is any small positive number. We will estimate the integral on Σ_1 and Σ_2 separately. Thus we can split the integral into two parts so that

$$\Big\langle D^\epsilon(u_+\partial_t u_-),g\Big\rangle_+=\iint\int_{\Sigma_1}\cdots+\iint\int_{\Sigma_2}\cdots=I_1+I_2. \tag{3.18}$$

For the term I_1, we have

$$I_1=\iint\int_{\Sigma_1}\frac{|z|^\epsilon}{|\rho|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz.$$

Then

$$|I_1|\leq\left(\iint\int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}(-\tau,-z)|^2d\tau dz\right)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}. \tag{3.19}$$

We denote the notation

$$H_{\Sigma_1}(\tau,z)=\int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega.$$

Thus, we can use (3.16) and some substitution to get

$$\begin{aligned}
H_{\Sigma_1}(\tau,z)&=\int_{\Sigma_1}\frac{|z|^{2\epsilon}}{\rho^{2+2\epsilon}|z-\rho\omega|^{2\epsilon}}\rho^2\frac{d\rho}{d\tau}d\omega=|z|^{2\epsilon}\int\frac{1}{\rho^{2\epsilon}|z-\rho\omega|^{2\epsilon}}\frac{d\rho}{d\tau}d\omega\\
&=\frac{|z|^{2\epsilon}}{(|z|^2-\tau^2)^{2\epsilon}}\int\frac{(\tau^2+|z|^2-2\tau z\cdot\omega)^{1-2\epsilon}}{(z\cdot\omega-\tau)^{2-4\epsilon}}d\omega\\
&\leq\frac{C}{|z|^{2\epsilon}(1-\lambda)^{2\epsilon}}\int_0^{\cos^{-1}(\lambda+\delta(1-\lambda))}\frac{(1+\lambda^2-2\lambda\cos\theta)^{1-2\epsilon}}{(\cos\theta-\lambda)^{2-4\epsilon}}\sin\theta d\theta\\
&\leq\frac{C}{|z|^{2\epsilon}}\int_\delta^1\frac{(1+\lambda-2\lambda s)^{1-2\epsilon}}{s^{2-4\epsilon}}ds<\frac{C(\delta)}{|z|^{2\epsilon}}<\frac{C(\delta)}{(|z|-\tau)^{2\epsilon}},
\end{aligned}$$

where we used the substitution $\lambda = \dfrac{\tau}{|z|}$, $z\cdot\omega = |z|\cos\theta$, and $s = \dfrac{\cos\theta - \lambda}{1-\lambda}$. Here $C(\delta) = \dfrac{C}{(1-4\epsilon)\delta^{1-4\epsilon}}$ and C is some constant. Therefore, we get

$$|I_1| \leq \left(\iint \frac{C(\delta)}{||z|-\tau|^{2\epsilon}}|\hat{g}(-\tau,-z)|^2 d\tau dz\right)^{\frac{1}{2}} \|f\|^2_{H^\epsilon}. \tag{3.20}$$

Now we want to estimate I_2. In order to do this we need two things. First, we note that on Σ_2 there exists constants C_1, C_2, such that

$$C_1 < \frac{|\xi|}{|\eta|} < C_2. \tag{3.21}$$

This is because

$$\frac{|\xi|}{|\eta|} = \frac{\rho}{|z-\rho\omega} = \frac{|z|^2-\tau^2}{|z|^2+\tau^2-2\tau z\cdot\omega} = \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos\theta},$$

and

$$\lambda < \cos\theta < \lambda + \delta(1-\lambda)$$

on Σ_2. Hence, if we choose $\delta = \dfrac{1}{2}$, then we get $1 < \dfrac{|\xi|}{|\eta|} < 2$. We also have $|\xi+\eta| \leq |\xi| + |\eta| \leq C|\xi|$ (or $C|\eta|$). Second, let's consider the following system of equations

$$\begin{cases} \Box\phi &= 0, \\ \phi(0) &= 0, \qquad \partial_t\phi(0) = f', \end{cases} \tag{3.22}$$

where f' satisfies $\hat{f}' = |\hat{f}|$, then we have the result

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_\pm)^2\right\|_{L^2(\mathbb{R}^4)} \leq C\left\|D^\epsilon f'\right\|^2_{L^2(\mathbb{R}^3)}, \tag{3.23}$$

for any $\epsilon \geq 0$. The proof of (3.23) can be done directly in the same manner as above and will be given outline later.

For the term I_2, employing (3.21) and (3.23), we have

$$\begin{aligned} |I_2| \leq & C\iint \frac{1}{|z|^\epsilon}\int_{\Sigma_2} \frac{|z|^\epsilon|z|^\epsilon}{|\rho|}\left|\hat{f}(\rho\omega)\right|\left|\hat{f}(z-\rho\omega)\right|\rho^2\frac{d\rho}{d\tau}d\omega\left|\hat{g}(-\tau,-z)\right|d\tau dz \\ \leq & C\iint \frac{1}{|z|^\epsilon}\int_{\Sigma_2} \frac{1}{|\rho|^{\frac{1}{2}-\epsilon}|z-\rho\omega|^{\frac{1}{2}-\epsilon}}\left|\hat{f}(\rho\omega)\right|\left|\hat{f}(z-\rho\omega)\right|\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}|d\tau dz \end{aligned}$$

The integral over Σ_2 is basically the Fourier transform of $D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-$ which can be seen as follows.

$$\begin{aligned}
&\left\langle D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-, g\right\rangle \\
&= \iint \frac{1}{|\xi|^{\frac{1}{2}-\epsilon}|\eta|^{\frac{1}{2}-\epsilon}}\hat{g}(-|\xi|+|\eta|, -\xi-\eta)\hat{f}'(\xi)\hat{f}'(\eta)d\xi d\eta \\
&= \iint \int_\Sigma \frac{1}{|\rho|^{\frac{1}{2}-\epsilon}|z-\rho\omega|^{\frac{1}{2}-\epsilon}}\hat{f}'(\rho\omega)\hat{f}'(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}\hat{g}(-\tau,-z)d\tau dz.
\end{aligned} \tag{3.24}$$

Therefore we can estimate I_2 as follows.

$$\begin{aligned}
|I_2| &\le C\iint \frac{1}{|z|^\epsilon}\mathcal{F}\Big(D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-\Big)|\hat{g}|d\tau dz \\
&\le C\left(\iint \frac{1}{|z|^{2\epsilon}}\chi_{\{0<\tau<|z|\}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}}\left\|D^{\frac{1}{2}+\epsilon}\phi_+ D^{\frac{1}{2}+\epsilon}\phi_-\right\|_{L^2(\mathbb{R}^4)} \\
&\le C\left(\iint \frac{1}{||z|-\tau|^{2\epsilon}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}}\left\|D^\epsilon f'\right\|^2_{L^2}.
\end{aligned}$$

Combining the estimates for the terms I_1 and I_2, we have

$$\left|\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle\right| \le C(\epsilon,\delta)\left(\iint \frac{1}{||z|-\tau|^{2\epsilon}}|\hat{g}|^2 d\tau dz\right)^{\frac{1}{2}}\|f\|^2_{H^\epsilon}. \tag{3.25}$$

Thus, with the same reason as in the estimate of the previous term, $D^\epsilon(u_+\partial_t u_+)$, we have

$$\left|\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle\right| \le C\left(\int \|g(t,\cdot)\|^p_{L^2}dt\right)^{\frac{1}{p}}\|f\|^2_{H^\epsilon}.$$

Hence, we get

$$\left(\int \left\|D^\epsilon(u_+^l Du_-^k)\right\|^q_{L^2(\mathbb{R}^3)}dt\right)^{1/q} \le C(\epsilon)\|f^l\|_{H^\epsilon}\|f^k\|_{H^\epsilon}, \tag{3.26}$$

where $q = \dfrac{2}{1-2\epsilon}$.

Finally, we will discuss the estimate of $\langle D^\epsilon(u_+\partial_t u_-), g\rangle_-$ briefly.

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_- = \iint_{\{|\xi|<|\eta|\}}\frac{|\xi+\eta|^\epsilon}{|\xi|}\hat{g}_h\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta. \tag{3.27}$$

Let's switch the variables ξ with η, thus we get

$$\left\langle D^\epsilon(u_+\partial_t u_-), g\right\rangle_- = \iint\int_\Sigma \frac{|z|^\epsilon}{|z-\rho\omega|}\hat{f}(\rho\omega)\hat{f}(z-\rho\omega)\rho^2\frac{d\rho}{d\tau}d\omega\hat{g}(-\tau,-z)d\tau dz.$$

For this case, we compute H_{Σ_1} as follows.

$$\begin{aligned}H_{\Sigma_1}(\tau,z) &= \int_{\Sigma_1}\frac{|z|^{2\epsilon}\rho^{2-2\epsilon}}{|z-\rho\omega|^{2+2\epsilon}}\frac{d\rho}{d\tau}d\omega\\ \leq &\frac{C}{|z|^{2\epsilon}}\int_\delta^1\frac{1}{(1+\lambda-2\lambda s)^{1+2\epsilon}s^{2-4\epsilon}}ds\\ \leq &\frac{C}{|z|^{2\epsilon}}\left(\frac{1}{2\epsilon}\frac{1}{(1-\lambda)^{2\epsilon}}+\frac{1}{1-4\epsilon}\frac{1}{\delta^{1-4\epsilon}}\right)\leq\frac{C(\epsilon,\delta)}{(|z|-\tau)^{2\epsilon}}.\end{aligned}$$

Thus, it works for the term, I_1. For another term, I_2, the proof is similar. The proof for the term $D^\epsilon(u_-\partial_t u_+)$ is similar to those of the term $D^\epsilon(u_+\partial_t u_-)$. Hence, the proof is complete. □

Proof of (3.23). We only outline the proof. Since

$$\left(D^{\frac{1}{2}+\epsilon}\phi_+\right)^2 = \iint\frac{1}{|\xi|^{\frac{1}{2}-\epsilon}+|\eta|^{\frac{1}{2}-\epsilon}}e^{A(\xi)+A(\eta)}\hat{f}'(\xi)\hat{f}'(\eta)d\xi d\eta,$$

thus we have

$$\begin{aligned}&\left|\left\langle\left(D^{\frac{1}{2}+\epsilon}\phi_+\right)^2, g\right\rangle\right|^2\\ \leq &\iint\int\frac{1}{\rho|z-\rho\omega|}\rho^2\frac{d\rho}{d\tau}d\omega|\hat{g}(-\tau,-z)|^2d\tau dz\left\|D^\epsilon f'\right\|^4_{L^2(\mathbb{R}^3)}.\end{aligned}$$

The inner-most integral called $E(\tau,z)$ can be computed as follows.

$$\begin{aligned}E(\tau,z) &= \int\frac{1}{\rho|z-\rho\omega|}\rho^2\frac{d\rho}{d\tau}d\omega\\ &= \int_0^\pi\frac{1-\lambda^2}{(1-\lambda\cos\theta)^2}\sin\theta d\theta = \frac{1+\lambda}{\lambda}\int_1^{\frac{1+\lambda}{1-\lambda}}\frac{1}{s^2}ds\end{aligned}$$

which is bounded by some constant. Hence we have

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_+)^2\right\|_{L^2(\mathbb{R}^4)}\leq C\left\|D^\epsilon f'\right\|^2_{L^2(\mathbb{R}^3)}.$$

Similar calculation gives

$$\left\|(D^{\frac{1}{2}+\epsilon}\phi_-)^2\right\|_{L^2(\mathbb{R}^4)}\leq C\left\|D^\epsilon f'\right\|^2_{L^2(\mathbb{R}^3)}.$$

□

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