On a System of Dirac-Klein-Gordon Type in 1+1 Dimensions

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1 Introduction

In the present work, we want to consider the Cauchy problem for the type of Dirac-Klein-Gordon equations

\[
\begin{align*}
\mathcal{D}\psi &= \phi\psi; \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\
\Box \phi &= \overline{\psi}\gamma^5\psi; \\
\psi(0) &= \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1,
\end{align*}
\]

(1.1)

where the vector function \( \psi \) takes values in \( \mathbb{C}^4 \), the scalar function \( \phi \) takes values in \( \mathbb{R}^1 \), the Dirac operator \( \mathcal{D} := -i\gamma^\mu \partial_\mu, \mu = 0, 1, 2, 3, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \), and \( \gamma^\mu \) are the Dirac matrices

\[
\gamma^0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\gamma^2 = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \gamma^5 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]
the wave operator $\Box = -\partial_{tt} + \partial_{xx}$, and $\overline{\psi} = \psi^\dagger \gamma^0$, where $\dagger$ is the complex conjugate transpose.

Chadam showed that the Cauchy problem for DKG equations has a global unique solution if $\psi_0 \in H^1$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$ in 1973, see [4]. In 1993, Zheng proved that there exists a global weak solution with $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, see [9]. In 2000, Bournaveas gave a new proof of a global existence for the DKG equations, by using a null form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, see [3]. In 2003, Fang obtained a simple direct proof for the problem and the result is parallel to that of Bournaveas, see [5].

Our purpose of this work is to demonstrate a proof of a variant null form estimate, see [5]. The motivation of studying this type of nonlinearity, $\overline{\psi} \gamma^5 \psi$, comes from the fact that it is a bilinear covariant of Lorentz transformation, see [2]. Notice that the quadratic term can be rewritten in the following form:

$$\overline{\psi} \gamma^5 \psi = 2 \Im (\psi_1 \overline{\psi}_3 + \psi_2 \overline{\psi}_4), \quad (1.2)$$

i.e. the imaginary part of $2(\psi_1 \overline{\psi}_3 + \psi_2 \overline{\psi}_4)$, where the $\psi_j$, $j = 1, 2, 3, 4$, are the components of the vector $\psi$. We give an interpretation of the null form structure different from that in [3]. The nonlinear term has the null form structure, see [6, 7]. Notice that the Dirac-Klein-Gordon equations in one space dimension can be decoupled into two similar subsystems, in other words, $\psi$ can be taken as 2-spinors, instead of 4-spinors, see [4, 9].

We adopt the approach and ideas in [3, 5] and make necessary modification. First, we derive the conservation law of charge,

$$\int |\psi(t)|^2 dx = \text{constant}, \quad (1.3)$$

which can be applied to derive the global solution existence for the DKG-type equations. Next, we write down the direct solution representation and use it to estimate the nonlinear form $\overline{\psi} \gamma^5 \psi$, and the derivations of some necessary estimates become straight forward. Finally we can prove the local and global existence results of DKG-type equations with data $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, which are called charge class solutions.

**Theorem 1 (Global Existence)**

*If the initial data of (1.1), $\psi_0 \in L^2, \phi_0 \in H^1, \phi_1 \in L^2$, then there is a unique*
global solution \((\psi, \phi)\) for (1.1) and \((\psi, \phi) \in C([0, \infty) \times L^2) \times \left(C^1([0, \infty) \times H^1) \times C^0([0, \infty) \times L^2)\right)\).

2 Solution Representation

Consider the Dirac equation,
\[
\begin{cases}
  D\psi = G; & (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\
  \psi(0) = \psi_0.
\end{cases}
\]
(2.4)

Using the equation
\[I_{4 \times 4} \Box \psi = D D \psi = D G,\]
the solution is
\[
2\psi(t, x) = \left[\psi_0(x + t) + \psi_0(x - t)\right] + \int_{x-t}^{x+t} \gamma^1 \gamma^0 \partial_x \psi_0(y) dy
\]
\[
+ i \gamma^0 \int_0^t G(s, x + t - s) + G(s, x - t + s) ds + i \gamma^1 \int_0^t \int_{x-t+s}^{x+t-s} \partial_x G(s, y) dy ds
\]
\[
= (\gamma^0 + \gamma^1) \gamma^0 \psi_0(x + t) + (\gamma^0 - \gamma^1) \gamma^0 \psi_0(x - t)
\]
\[
+ i \int_0^t (\gamma^0 + \gamma^1) G(s, x + t - s) ds + i \int_0^t (\gamma^0 - \gamma^1) G(s, x - t + s) ds.
\]
(2.5)

Recall that, for the wave equation
\[
\begin{cases}
  \Box \phi = F; & (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\
  \phi(0) = \phi_0, & \partial_t(0) = \phi_1,
\end{cases}
\]
(2.6)

the solution of the equation is
\[
2\phi(t, x) = \phi_0(x - t) + \phi_0(x + t) + \int_{x-t}^{x+t} \phi_1(y) dy + \int_0^t \int_{x-t+s}^{x+t-s} F(s, y) dy ds.
\]
(2.7)

3 Estimates

Lemma 1 We have the law of conservation of charge, i.e.
\[
\int |\psi(t)|^2 dt = \text{constant}.
\]
Proof. The equation $D \psi = \phi \psi$ implies
\begin{equation}
\psi_t + \gamma^0 \gamma^1 \psi_x = i \gamma^0 \phi \psi \tag{3.8}
\end{equation}
Multiplying (1.3) by $\psi^\dagger$ and its complex conjugate transpose by $\psi$, and then summing up, we get
\begin{equation}
\partial_t(|\psi|^2) + \partial_x(\psi^\dagger \gamma^0 \gamma^1 \psi) = 0.
\end{equation}
This completes the proof. $\square$

Lemma 2
For the solution of the Dirac equation, we have
\begin{equation}
||\psi(t)||_{L^2} \leq C \left(||\psi_0(t)||_{L^2} + \int_0^T ||G(s)||_{L^2}ds\right). \tag{3.9}
\end{equation}
This can be shown straightforward, using the solution representation (2.3), so that we skip the proof.

Consider the Dirac equations
\begin{equation}
\begin{cases}
D \psi_j = G_j, & j = 1, 2 \\
\psi_j(0) = \psi_{0j}.
\end{cases}
\end{equation}

Lemma 3
(Null Form Estimate)
\begin{equation}
||\overline{\psi_1} \gamma^5 \psi_2||_{L^2([0,T],L^2)} \leq C \left(||\psi_{01}||_{L^2} + \int_0^T ||G_1(s)||_{L^2}ds\right) \left(||\psi_{02}||_{L^2} + \int_0^T ||G_2(s)||_{L^2}ds\right) \tag{3.10}
\end{equation}
Proof: For simplicity, we prove a special case when $\psi_1 = \psi_2$, and then the general case will follow. Consider the linear Dirac equation and write its solution as
\begin{equation}
2\psi(t,x) = U_+ + U_- + iV_+ + iV_-, \tag{3.11}
\end{equation}
where
\begin{align}
U_\pm(t,x) &= (\gamma^0 \pm \gamma^1)\gamma^0 \psi_0(x \pm t), \tag{3.12} \\
V_\pm(t,x) &= \int_0^t (\gamma^0 \pm \gamma^1)G(s,x \pm (t-s))ds. \tag{3.13}
\end{align}
Throughout some elementary calculations, we get

\[ U_{\pm} \gamma^5 U_{\pm} = V_{\pm} \gamma^5 V_{\pm} = U_{\pm} \gamma^5 V_{\pm} = V_{\pm} \gamma^5 U_{\pm} = 0, \]  

(3.14)

thus

\[ ||U_{\pm} \gamma^5 U_{\pm}||_{L^2([0,T],L^2)} \leq C ||\psi_0||_{L^2}^2, \]  

(3.15)

\[ ||V_{\pm} \gamma^5 U_{\pm}||_{L^2([0,T],L^2)} \leq C ||\psi_0||_{L^2} \int_0^T ||G(s)||_{L^2} ds, \]  

(3.16)

\[ ||U_{\pm} \gamma^5 V_{\pm}||_{L^2([0,T],L^2)} \leq C ||\psi_0||_{L^2} \int_0^T ||G(s)||_{L^2} ds, \]  

(3.17)

\[ ||V_{\pm} \gamma^5 V_{\pm}||_{L^2([0,T],L^2)} \leq C \left( \int_0^T ||G(s)||_{L^2} ds \right)^2. \]  

(3.18)

The calculations for these cases are analogous. Among these cases, we only demonstrate the case of \( U_{\pm} \gamma^5 U_{\pm}, \) \( V_{\pm} \gamma^5 U_{\pm}, \) and \( V_{\pm} \gamma^5 V_{\pm}. \) For convenience, we denote \( \gamma = (\gamma^0 - \gamma^1) \gamma^0 \gamma^5 (\gamma^0 - \gamma^1). \) Since

\[ ||U_{\pm} \gamma^5 U_{\pm}||_{L^2([0,T],L^2)} = \left( \int_0^T \int |\psi_0(x + t)\gamma^0 \gamma^5 \psi_0(x - t)|^2 dx dt \right)^{\frac{1}{2}} \leq C \left( \int_0^T \int |\psi_0(x + t)\gamma^0 \gamma^5 \psi_0(x - t)|^2 dx dt \right)^{\frac{1}{2}} = C ||\psi_0||_{L^2}^2. \]

If we use Minkovski inequality, we get

\[ ||V_{\pm} \gamma^5 U_{\pm}||_{L^2([0,T],L^2)} \]

\[ = \left( \int_0^T \int \left| \int_0^t G(s, x + t - s) \gamma^0 \gamma^5 \psi_0(x - t) ds \right|^2 dx dt \right)^{\frac{1}{2}} \]

\[ \leq C \int_0^T \left( \int_0^T \left| \int_0^t |G(s, x + t - s)|^2 |\psi_0(x - t)|^2 dx dt \right|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq C ||\psi_0||_{L^2} \int_0^T ||G(s)||_{L^2} ds. \]
Finally, for the nonhomogeneous term, we have
\[
\| V + \gamma V_\gamma \|_{L^2(0,T),L^2}
\]
\[
= \left( \int_0^T \int_0^T \left( \int_0^t \int_0^t |G(s,x+t-s)\gamma G(r,x-t+r)|drds \right)^2 dxdt \right)^{\frac{1}{2}}
\]
\[
\leq C \int_0^T \int_0^T \left( \int_0^T \int_0^T |G(s,x+t-s)|^2 |G(r,x-t+r)|^2 dxdt \right)^{\frac{1}{2}} drds
\]
\[
\leq C \int_0^T \int_0^T \| G(s) \|_{L^2} \| G(r) \|_{L^2} drds
\]
\[
\leq C \left( \int_0^T \| G(s) \|_{L^2} ds \right)^2 .
\]
This complete the proof of the lemma. 

\[\square\]

\textbf{Lemma 4} For the wave equation, we have the energy estimate

\[
\| \phi(t) \|_{H^1} + \| \phi_t(t) \|_{L^2} \leq C(T) \left( \| \phi_0 \|_{H^1} + \| \phi_1 \|_{L^2} + \int_0^T \| F(s) \|_{L^2} ds \right).
\]

(3.19)

\textbf{Proof.} For the solution of wave equation (1.9), we have

\[
2\phi(t,x) = \phi_0(x+t) + \phi_0(x-t) + \int_{x-t}^{x+t} \phi_1(y) dy + \int_{x-t+s}^{x+t-s} F(s,y) dy ds.
\]

Differentiating \( \phi(t,x) \) with respect to \( t \) and \( x \), respectively, give

\[
2\partial_t \phi(t,x) = \partial_t \phi_0(x-t) + \partial_t \phi_0(x+t) + (\phi_1(x+t) - \phi_1(x-t)) + \int_0^t F(s,x+t-s) + F(s,x-t+s) ds
\]

\[
2\partial_x \phi(t,x) = \partial_x \phi_0(x-t) + \partial_x \phi_0(x+t) + (\phi_1(x+t) + \phi_1(x-t)) + \int_0^t F(s,x+t-s) + F(s,x-t+s) ds.
\]

The above equations imply (3.19). 

\[\square\]
4 Existence.

Let \((\psi, \phi)\) and \((\psi', \phi')\) be two charge class solutions of the DKG-type equations. We define the following quantities:

\[
J(0) = ||\psi_0||_{L^2} + ||\phi_0||_{H^1} + ||\phi_1||_{L^2} \tag{4.20}
\]

\[
J'(0) = ||\psi'_0||_{L^2} + ||\phi'_0||_{H^1} + ||\phi'_1||_{L^2} \tag{4.21}
\]

\[
J(T) = \sup_{[0,T]}(||\psi(t)||_{L^2} + ||\phi(t)||_{H^1} + ||\phi(t)||_{L^2}) \tag{4.22}
\]

\[
J'(T) = \sup_{[0,T]}(||\psi'(t)||_{L^2} + ||\phi'(t)||_{H^1} + ||\phi'(t)||_{L^2}) \tag{4.23}
\]

\[
\Delta(0) = ||\psi_0 - \psi'_0||_{L^2} + ||\phi_0 - \phi'_0||_{H^1} + ||\phi_1 - \phi'_1||_{L^2} \tag{4.24}
\]

\[
\Delta(T) = \sup_{[0,T]}(||\psi(t) - \psi'(t)||_{L^2} + ||\phi(t) - \phi'(t)||_{H^1} + ||\phi(t) - \phi'(t)||_{L^2}) \tag{4.25}
\]

**Lemma 5** For the equations (1.1), we have

\[
||\phi(t)||_{L^\infty(\mathbb{R})} \leq C(T, J(0)). \tag{4.26}
\]

**Proof.** Write \(\phi = \phi_L + \phi_N\) where \(\phi_L\) is the solution of

\[
\Box \phi_L = 0, \ \phi_L(0, x) = \phi_0, \ \partial_t \phi_L(0, x) = \phi_1,
\]

and \(\phi_N\) is a solution of

\[
\Box \phi_N = \overline{\psi}\gamma^5\psi, \ \phi_N(0, x) = 0, \ \partial_t \phi_N(0, x) = 0.
\]

Apply the standard energy estimate and the Sobolev inequality to get

\[
||\phi_L(t)||_{L^\infty(\mathbb{R})} \leq C||\phi_L(t)||_{H^1(\mathbb{R})} \leq C(T)(||\phi_0||_{H^1(\mathbb{R})} + ||\phi_1||_{L^2(\mathbb{R})}) \leq C(T)J(0).
\]

We use the law of conservation of charge here to get

\[
|\phi_N(t, x)| \leq C \left| \int_0^t \int_{x-t+s}^{x+t-s} \overline{\psi}(s, y)\gamma^5\psi(s, y)dyds \right|
\]

\[
\leq C \int_0^t \int_{-\infty}^{\infty} |\psi(s, y)|^2dyds
\]

\[
\leq C \int_0^T ||\psi(s)||^2_{L^2(\mathbb{R})}ds \leq CT||\psi_0||^2_{L^2(\mathbb{R})}.
\]
Since $\phi_N + \phi_L = \phi$, we have

$$||\phi(t)||_{L^\infty(\mathbb{R})} \leq C(T, J(0)).$$

□

**Lemma 6** Let $T > 0$ and let $(\psi, \phi)$ be a charge class solution of the DKG equations. Then there exists a constant $C > 0$, depending only on $T$ and $J(0)$, such that $J(T) \leq C(T, J(0))$.

**Proof.** Since $||\psi(t, x)||_{L^2} = ||\psi(0, x)||_{L^2}$ and

$$||\phi(t)||_{H^1} + ||\phi_t(t)||_{L^2} \leq C(T) \left(||\phi_0||_{H^1} + ||\phi_1||_{L^2} + \int_0^T ||F(s)||_{L^2} ds\right),$$

we compute

$$\int_0^T ||F(s)||_{L^2} ds = \int_0^T ||\psi^5\psi(s)||_{L^2} ds \leq T^2 \left(||\psi^5\psi||_{L^2([0, T], L^2)} \right)$$

$$\leq CT^2 \left(||\psi_0||_{L^2} + \int_0^T ||\phi(s)\psi(s)||_{L^2} ds\right)^2$$

$$\leq T^2 \left(J(0) + \int_0^T ||\phi(s)||_{L^\infty} ||\psi(s)||_{L^2} ds\right)^2$$

$$\leq C(T, J(0))T^{-\frac{1}{2}}.$$  \hspace{1cm} (4.27)

Now we can get

$$||\phi(t)||_{H^1} + ||\phi_t(t)||_{L^2} \leq C(T) \left(J(0) + \int_0^T ||F(s)||_{L^2} ds\right)$$

$$\leq C(T, J(0)).$$ \hspace{1cm} (4.28)

This completes the proof of the lemma. □

**Lemma 7** Let $T > 0$ and $(\psi, \phi)$ and $(\psi', \phi')$ be two charge class solutions of the DKG equations. Then there exists a constant $\epsilon > 0, C > 0$, depending only on $T$ and $J(0)$ and $J'(0)$, such that if $T \leq \epsilon$, then

$$\Delta(T) \leq C\Delta(0).$$ \hspace{1cm} (4.29)

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**Proof** Consider the difference of the two solutions, we have

\[
\mathcal{D}(\psi - \psi') = (\phi - \phi')\psi + \phi'(\psi - \psi')
\]

\[
\Box(\phi - \phi') = \overline{\psi} - \overline{\psi}'\gamma^5\psi + \overline{\psi}'\gamma^5(\psi - \psi').
\] (4.30)

Recall that the quantity

\[
\Delta(T) = \sup_{[0,T]}(||\psi(t) - \psi'(t)||_{L^2} + ||\phi(t) - \phi'(t)||_{H^1} + ||\phi_t(t) - \phi_t'||_{L^2})
\]

We compute the first term of \(\Delta(T)\). Since

\[
||\mathcal{D}(\psi - \psi')(s)||_{L^2}
\]

\[
\leq ||\phi(s) - \phi'(s)||_{L^\infty}||\psi(s)||_{L^2} + ||\phi'(s)||_{L^\infty}||\psi(s) - \psi'(s)||_{L^2}
\] (4.31)

\[
\leq ||\phi(s) - \phi'(s)||_{H^1}||\psi_0||_{L^2} + ||\phi'(s)||_{L^\infty}||\psi(s) - \psi'(s)||_{H^1}
\] (4.32)

\[
\leq C(T, J(0), J'(0)) \Delta(T),
\] (4.33)

thus we can get

\[
||\psi(t) - \psi'(t)||_{L^2} \leq C\left(||\psi_0 - \psi_0'||_{L^2} + \int_0^T ||\mathcal{D}(\psi - \psi')(s)||_{L^2} ds\right)
\]

\[
\leq C(\Delta(0) + C(T, J(0), J'(0)) T \Delta(T)).
\] (4.35)

For the other two terms, by invoking Lemma 3, we first calculate

\[
||\overline{\psi - \psi}'\gamma^5\psi||_{L^2} \leq C\left(||\psi_0 - \psi_0'||_{L^2} + \int_0^T ||\mathcal{D}(\psi - \psi')(s)||_{L^2} ds\right).
\]

\[
\left(||\psi_0||_{L^2} + \int_0^T ||\mathcal{D}(\psi)(s)||_{L^2} ds\right)
\] (4.36)

and the calculation for \(||\overline{\psi}'\gamma^5(\psi - \psi')||_{L^2}\) is the same. Then we compute

\[
\int_0^T ||\Box(\phi - \phi')(s)||_{L^2} ds
\]

\[
\leq \int_0^T ||\overline{\psi - \psi}'\gamma^5\psi(s)||_{L^2} + ||\overline{\psi}'\gamma^5(\psi - \psi')(s)||_{L^2} ds
\]

\[
\leq T^{\frac{1}{2}}\left(||\overline{\psi - \psi}'\gamma^5\psi||_{L^2([0,T];L^2)} + ||\overline{\psi}'\gamma^5(\psi - \psi')||_{L^2([0,T];L^2)}\right).
\] (4.37)
Therefore, applying (3.19), and we get
\[
||\phi(t) - \phi'(t)||_{H^1} + ||\phi_t(t) - \phi'_t(t)||_{L^2} \\
\leq C(T)(||\phi_0 - \phi'_0||_{H^1} + ||\phi_1 - \phi'_1||_{L^2} + \int_0^T ||\Box(\phi - \phi'(s))||_{L^2} ds) \\
\leq C(T)(\Delta(0) + T^{\frac{5}{2}} C(T, J(0), J'(0))(\Delta(0) + T\Delta(T)) \\
\leq C(T, J(0), J'(0))(\Delta(0) + T\Delta(T)).
\] (4.38)

Now we assume \( T < 1 \), thus we can get
\[
C(T, J(0), J'(0)) \leq C(J(0), J'(0)) = C(1, J(0), J'(0)) \quad (4.39)
\]
and
\[
\Delta(T) \leq C(J(0), J'(0))(\Delta(0) + T\Delta(T)). \quad (4.40)
\]
This concludes the proof. \( \square \)

**Theorem 2 Local Existence**

Let \( \psi_0 \in L^2(\mathbb{R}), \phi_0 \in H^1(\mathbb{R}), \phi_1 \in L^2(\mathbb{R}) \) then there exists a \( T > 0 \), depending only on \( J(0) \) and a unique charge class solution of DKG-type equations defined on \([0, T) \times \mathbb{R}\)

**proof:** Let \((\psi^0, \phi^0)\) be the solution of
\[
\begin{cases}
\mathcal{D}\psi = 0; \quad (t, x) \in \mathbb{R} \times \mathbb{R}^1 \\
\Box\phi = 0; \\
\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1,
\end{cases}
\] (4.41)

\((\psi^1, \phi^1)\) be the solution of
\[
\begin{cases}
\mathcal{D}\psi = \delta^0\psi^0; \quad (t, x) \in \mathbb{R} \times \mathbb{R}^1 \\
\Box\phi = \overline{\psi^0}^5\psi^0; \\
\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1,
\end{cases}
\] (4.42)

and \((\psi^{k+1}, \phi^{k+1})\) be the solution of
\[
\begin{cases}
\mathcal{D}\psi = \delta^k\psi^k; \quad (t, x) \in \mathbb{R} \times \mathbb{R}^1 \\
\Box\phi = \overline{\psi^k}^5\psi^k; \\
\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1,
\end{cases}
\] (4.43)
where \( k = 1, 2, 3, \cdots \). Thus we have
\[
||\psi_k(t) - \psi_{k-1}(t)||_{L^2} \leq T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N},
\]
\[
||\phi_k(t) - \phi_{k-1}(t)||_{H^1} \leq T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N},
\]
\[
||\partial_t \phi_k(t) - \partial_t \phi_{k-1}(t)||_{L^2} \leq T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N}.
\]
So we can get, for \( m > n \)
\[
||\psi^m(t) - \psi^n(t)||_{L^2} + ||\phi^m(t) - \phi^n(t)||_{H^1} + ||\partial_t \phi^m(t) - \partial_t \phi^n(t)||_{L^2} \\
\leq \frac{T^{m-1}(1 - T^{m-n})}{1 - T}(C(T, J(0)))
\]
Since \( T < 1 \), we get \( ||\psi^m - \psi^n||_{L^2} \to 0 \) as \( m, n \to \infty \). We obtain that \( \{\psi^k\} \) is a Cauchy sequence in \( L^2 \), thus its limiting function \( \psi \) is the solution. Similarly we can get \( \{\phi^k\} \) be a Cauchy sequence in \( H^1 \) and its limiting function \( \phi \) is the solution. Thus if \( 0 < T < 1 \), then there exists a solution \( (\psi, \phi) \) for (1.1).

Finally Lemma 7 gives us the uniqueness of the solution. \( \square \)

The existence of the global solution is ensured by the law of the conservation of the charge.

References


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