EXISTENCE AND UNIQUENESS FOR BOUSSINESQ TYPE EQUATIONS ON A CIRCLE

Yung-fu Fang[1] & Manoussos G. Grillakis[2]

Abstract. We establish local and global existence results for Boussinesq type equations on a circle, employing Fourier series and a fixed point argument.

0. Introduction and Main Results.

In the present work, we want to consider the question of existence and uniqueness of solutions for Boussinesq type equations

$$u_{tt} - u_{xx} + u_{xxxx} + \partial_x^2 f(u) = 0, \qquad x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{0.1}$$

where $\mathbb{T}$ is the unit circle and $f(u)$ is a polynomial of u and $|u|$, under minimal regularity assumptions on the initial data prescribed at time $t = 0$,

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x). \tag{0.2}$$

Equations of this type, but with the opposite sign in the fourth derivative, were originally derived by Boussinesq [Bo] in the context of water waves. Zakharov [Z] proposed equation (0.1) as a model of a nonlinear string. Falk et al derived an equation which is equivalent to (0.1) in their study of shape-memory alloys, see [FLS]. In fact, the equation studied in [FLS] is of the following type

$$e_{tt} - g e_{xx} + e_{xxxx} + \partial_x^2 f(e) = 0, \tag{0.3}$$

[1]The first author was partially supported by Sloan Fellowship.
[2]The second author is supported by a PYI, DMS-9157512, and a Sloan Fellowship.

where g is a constant, $e = u_x$ is the strain and $f(e) = 4e^3 - 6e^5$. In general however $f(e)$ contains a term of the form $\exp(\gamma e^2)$. McKean studied the complete integrability of the good Boussinesq equation on a circle, see [M]. An interesting observation connecting the Kadomstev-Petviashvili equation with the Boussinesq equation is the following. For the KP equation

$$(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0, \tag{0.4}$$

consider waves that move in the x direction, i.e. $u(t, x, y) = v(x - ct, y)$ and denote $\xi = x - ct$, thus the KP equation is reduced to

$$v_{yy} - cv_{\xi\xi} + v_{\xi\xi\xi\xi} + \partial_\xi^2(u^2/2) = 0, \tag{0.5}$$

which is (0.1) with $f(u) = u^2/2$, and the time variable is now played by the y direction, see [HP].

Equation (0.1) has certain features that are interesting, the linear equation

$$u_{tt} - u_{xx} + u_{xxxx} = 0 \tag{0.6}$$

has solutions that are periodic in space but only aperiodic in time. By this we mean that the function is a linear combination of functions with different non integer periods. Also in contrast to the equation on the real line, *i.e.* $x \in \mathbb{R}$, there is no dispersion and no decay in the time variable.

On the other hand, equation (0.1) can be written as a Hamiltonian system as follows

$$\begin{cases} u_t = v_x, \\ v_t = u_x - u_{xxx} - \partial_x f(u). \end{cases} \tag{0.7}$$

The above equation conserves the energy, namely the integral

$$E = \frac{1}{2}\int_{\mathbb{T}} [v^2 + u^2 + u_x^2 - 2F(u)]dx, \tag{0.8}$$

where $F' = f$ and $F(0) = 0$, does not depend on the time t. Another conserved quantity is the momentum

$$I = \int_{\mathbb{T}} uvdx \tag{0.9}$$

which turns out to be a relevant quantity in the investigation of stability properties of traveling waves. The conservation of energy can lead to global existence if it is positive definite. However if it is not positive definite, then it is possible to show blow-up in finite time, see [S] and [KL].

Sachs in [S] proved that the "good" Boussinesq equation, which is the equation (0.1) with $f(u) = u^2$, (the energy E is indefinite in this case), has solutions that can only exist for finite time for certain initial data (also see [KL]). The same method applies to equation (0.1) to show that solutions blow-up if the energy is indefinite. Liu extended and refined the blow-up results in [S], see [L]. Some local and global results of the Boussinesq type equation on the real line were shown by F. Linares [Ln].

It was shown by Zakharov, [Z], that the Boussinesq equation, which agrees with (0.1) for $f(u) = u^2$ and with the opposite sign in the fourth derivative term has infinitely many conservation laws and is formally completely integrable. Using the same method, one can show that equation (0.1) is also formally completely integrable. McKean developed a rigorous theory of complete integrability for the good Boussinesq equation on a circle.

The fact that Boussinesq type equations have solitary wave solutions has been studied by many authors, see Bona & Sachs [BS], Alexander & Sachs [AS] and Liu [L]. Solitary waves of (0.1) are traveling wave solutions of the form

$$u(x,t) = \varphi(\xi) = \varphi(x - ct), \tag{0.10}$$

where c is the speed of the wave and satisfy the ordinary differential equation

$$\varphi'' = (1 - c^2)\varphi - f(\varphi); \qquad ' = \frac{d}{d\xi}, \tag{0.11}$$

with appropriate boundary conditions. The quantity

$$(\varphi')^2 - (1 - c^2)\varphi^2 + 2F(\varphi) \tag{0.12}$$

is a quadrature and enables us to determine the conditions on $F(\cdot)$ so that equation (0.1) possesses solitary waves, e.g. $f(u) = \pm u^2$, $f(u) = |u|^{p-1}u$ if

$|c| < 1$ and $p > 1$, $f(u) = \lambda|u|^{q-1}u - |u|^{p-1}u$ for certain values of $\lambda \in \mathbb{R}^+$ and $1 < q < p$, ... etc.

The outline of the paper is as follows, we first establish the local existence of solutions. The main ingredient in the proof is an a priori estimate inspired by recent work of J. Bourgain, see [B1] and [B2], and it can be understood as a multiplier estimate on the set $\mathbb{R} \times \mathbb{Z}$, (dual variables in the Fourier transform), where $\mathbb{Z}$ is the one dimensional lattice. The proof which we present in chapter 2 is somewhat different from the one in [B1] and we believe it is more transparent. The proof relies on an idea of Zygmund [Zy] and reduces to a counting argument. Related previous arguments in the continuous case can be found in [Fe] and [CS].

Once the local existence is proved, the time interval of existence and the size of the initial data are reciprocal, so that the maximal existence time, T_{max}, can be finite. On the other hand, to prove global existence, we can use the conservation of energy. This is one of the motivations for the local existence under minimal regularity assumptions on the initial data. The other motivation is related to the construction of invariant measures in the space of solutions.

The main theorems proved in this paper are stated below.

Theorem 0.1. *(Local Existence and Uniqueness) Assume that the initial data (0.2) satisfy $u_0 \in H^s$, $u_1 \in H^{-2+s}$ with $0 \le s \le 1$. Assume also that $|f(u)| \le C|u|^p$ and the p and the s satisfy*

$$\begin{cases} p & \le p(s) = \frac{3-2s}{1-2s}, \quad \text{if} \quad 0 \le s < \frac{1}{2}; \\ p & < +\infty \qquad \text{if} \quad \frac{1}{2} \le s. \end{cases} \tag{0.13}$$

Then equation (0.1) has a local unique solution.

Theorem 0.2. *(Global Existence) Assume that the initial data of problem (0.1) satisfy $u(0,x) \in H^1$, $u_t(0,x) \in H^{-1}$. Let $f(u) = \lambda|u|^{q-1}u - |u|^{p-1}u$ and $1 < q < p$, for any $\lambda \in \mathbb{R}$. Then a unique solution of (0.1) exists for all time. The solution has the same regularity as the local solution and belongs to $L^4(\mathbb{R}_{\text{loc}} \times \mathbb{T})$.*

In general we would like to consider (0.1) with different periods. If $u(t,x)$

is periodic in x with period L, then calling $\mu = 2\pi/L$ and rescaling in space-time, $t \to \mu t$ and $x \to \mu x$, we obtain the equation

$$u_{tt} - u_{xx} + \mu^2 u_{xxxx} + \partial_x^2 f(u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{0.14}$$

The methods and the corresponding results for equation (0.14) are of course completely analogous to those of equation (0.1). It is interesting however to see how the a priori estimate and the time of existence are influenced by $\mu = 2\pi/L$.

1. Local and Global Existence.

Throughout the rest of this paper, we will consider functions of two variables, $g(t,x)$ with $t \in \mathbb{R}$ the time variable and $x \in \mathbb{T}$ the space variable. We will denote by $\tilde{g}$ the Fourier transform of the function g with respect to the space variable and by $\hat{g}$ the Fourier transform of the function g with respect to both the space variable and the time variable, i.e.

$$\tilde{g}(t,\xi) = \int_{\mathbb{T}} e^{-ix\xi} g(t,x)dx \quad \text{and} \quad \hat{g}(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{T}} e^{-i(x\xi+t\tau)} g(t,x)dxdt.$$

We will also use the following notation

$$\begin{cases} \omega(\xi) & = \sqrt{\xi^2+\xi^4}, \\ S & = ||\tau| - \omega| + 1 \end{cases} \tag{1.1}$$

and the negative Sobolev space $H^{-k}(\mathbb{T})$ with norm defined as follows.

Definition. $H^{-k}(\mathbb{T})$ is the space of periodic functions $u(x)$ with norm

$$\|u\|_{H^{-k}} = \left(\sum_{\xi \in \mathbb{Z}} \frac{|\hat{u}(\xi)|^2}{(1+|\xi|^2)^k} \right)^{\frac{1}{2}}.$$

Remark. Notice that the space $L^1(\mathbb{T})$ is contained in the space $H^{-k}(\mathbb{T})$ for all $k > 0$.

Our first step is to write the solution of equation,

$$u_{tt} - u_{xx} + u_{xxxx} = g(t,x), \tag{1.2a}$$

in integral form using Fourier series. The solution can be written as follows.

(1.2b)

$$\begin{aligned} u(t,x) = t\hat{u}_1(0) + \hat{u}_0(0) + \sum_{\xi\neq 0} e^{ix\xi}\left(\frac{\sin t\omega}{\omega}\hat{u}_1(\xi) + \cos t\omega \hat{u}_0(\xi)\right) \\ + \sum_{\xi\neq 0} e^{ix\xi}\int_0^t \Big[\frac{\sin(t-s)\omega}{\omega}\tilde{g}(s,\xi)\Big]ds, \end{aligned}$$

where $g(t,x) = -\partial_x^2 f(u)$ corresponding to equation (0.1). Observe that $g(t,x)$ has average zero, i.e. $\tilde{g}(t,0) = 0$. The solution (1.2) can be split into the linear and nonlinear parts

$$\begin{cases} U(t,x) &= \sum_{\xi\neq 0} e^{ix\xi}\left(\frac{\sin t\omega}{\omega}\hat{u}_1(\xi) + \cos t\omega\hat{u}_0(\xi)\right) + t\hat{u}_1(0) + \hat{u}_0(0), \\ V(t,x) &= \sum_{\xi\neq 0} e^{ix\xi}\int_0^t \Big[\frac{\sin(t-s)\omega}{\omega}\tilde{g}(s,\xi)\Big]ds. \end{cases} \tag{1.3}$$

The idea of the proof of local existence is to consider the nonlinear map (1.2) and prove that it is a contraction in the appropriate space. The right space in this case is dictated by the equation and it is expressed in the dual variables with the norm

$$N(u) = \|S^{\frac{1}{2}}\hat{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})}, \tag{1.4}$$

see (1.2). The idea to use the contraction principle with norms like (1.4) is due to Bourgain. However this is not essential for the construction here, alternatively one can use the norm $\|u\|_{L^4(\mathbb{R}_{\text{loc}}\times\mathbb{T})}$, instead of $N(u)$. The heuristic idea of the norm is that one can formally takes Fourier transform over the space and time variables on the both sides of (1.2a) to have

$$(\tau^2 - \omega^2)\hat{u}(\tau,\xi) = \hat{g}(\tau,\xi).$$

Then one can get

$$(|\tau| - \omega)^{\frac{1}{2}}\hat{u}(\tau,\xi) = \frac{\hat{g}(\tau,\xi)}{(|\tau| - \omega)^{\frac{1}{2}}(|\tau| + \omega)}$$

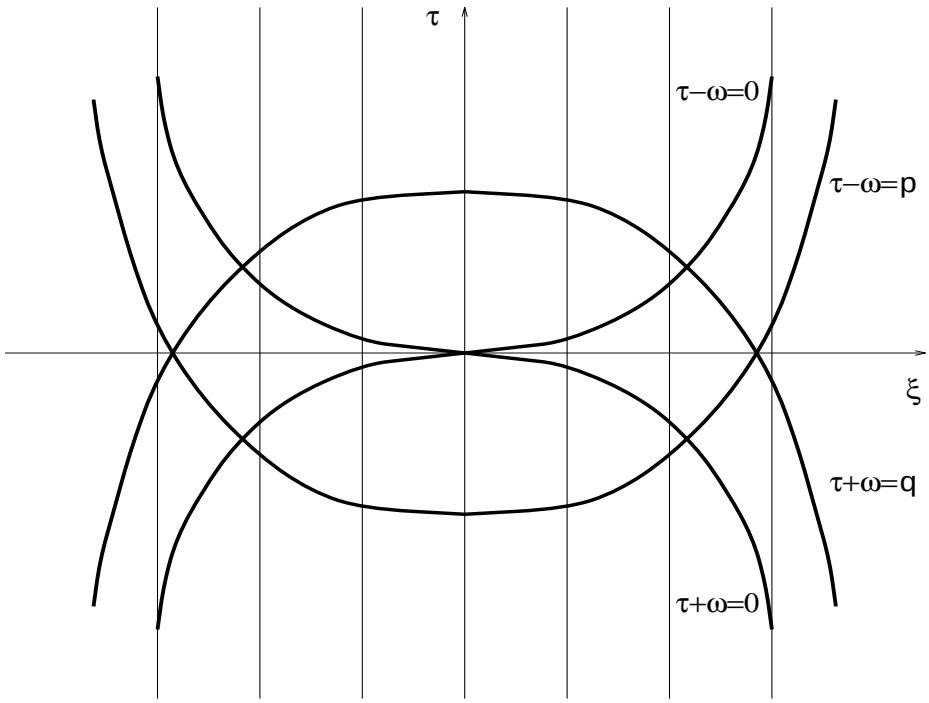

FIG 1. The level curves of $|\tau| \pm \omega$.

and take the L^2 norm on both sides.

Observe that the linear part of the solution, $U(t,x)$, is only aperiodic in time, for this reason we have to localize it in time using a cutoff function $\psi(t)$ which is identically one if $|t| \leq 1$ and identically zero if $|t| > 2$. Denote by $\psi_\delta(t) = \psi(t/\delta)$ its dilation.

In order to handle the term V, see (1.3), consider first the linear equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = g, \\ u(0,x) = 0, \quad u_t(0,x) = 0. \end{cases} \tag{1.5}$$

Assume that $\tilde{g}(t,0) = 0$ for all t. The solution of (1.5), compare with the expression for $V(t,x)$ in (1.3), can be rewritten as follows

$$\begin{aligned} -2u(t,x) = \sum_{\xi \neq 0} \Big(& \frac{e^{i(x\xi + t\omega)}}{\omega} \int_{\mathbb{R}} \frac{e^{it(\tau-\omega)} - 1}{\tau - \omega} \hat{g}(\tau,\xi) d\tau \\ & - \frac{e^{i(x\xi - t\omega)}}{\omega} \int_{\mathbb{R}} \frac{e^{it(\tau+\omega)} - 1}{\tau + \omega} \hat{g}(\tau,\xi) d\tau \Big). \end{aligned} \tag{1.6}$$

We want to use cutoff functions to decompose the integrals into parts near and far off the level curves of $\tau \pm \omega$, see figure 1. For this reason, let us introduce a smooth function $\hat{a}$ and denote $\hat{b} = 1 - \hat{a}$. Assume that $\hat{a}$ has support in $|\tau| < 2R$ and is identically 1 for $|\tau| < R$. The solution of (1.5) can be decomposed in the following manner

$$u(t,x) = \Psi(t,x) + F(t,x), \tag{1.7}$$

where

$$\hat{F} = \left(\frac{\hat{b}(\tau - \omega)}{\tau - \omega} - \frac{\hat{b}(\tau + \omega)}{\tau + \omega} \right) \frac{\hat{g}}{\omega} \tag{1.8}$$

and

$$\Psi = \Psi_1 + \Psi_2, \tag{1.9}$$

with Ψ_1 and Ψ_2 given by the expressions

(1.10a)

$$\hat{\Psi}_1(\tau,\xi) = \frac{-\delta(\tau-\omega)}{\omega} \int \frac{\hat{b}(\lambda-\omega)}{\lambda-\omega} \hat{g}(\lambda,\xi) d\lambda + \frac{\delta(\tau+\omega)}{\omega} \int \frac{\hat{b}(\lambda+\omega)}{\lambda+\omega} \hat{g}(\lambda,\xi) d\lambda$$

and

$$\hat{\Psi}_2(\tau,\xi) = \sum_{k=1}^{\infty} [\delta^{(k)}(\tau-\omega)\hat{G}_k^-(\xi) + \delta^{(k)}(\tau+\omega)\hat{G}_k^+(\xi)]. \tag{1.10b}$$

The quantities $\hat{G}_k^{\pm}(\xi)$ are

$$\hat{G}_k^{\pm}(\xi) = \frac{i^k (2R)^{k-1}}{\omega k!} \int (\frac{\tau \pm \omega}{2R})^{k-1} \hat{a}(\tau \pm \omega) \hat{g}(\tau,\xi) d\tau, \tag{1.11}$$

where the expressions $\hat{G}_k^{\pm}(\xi)$ are obtained by expanding in power series the expression $e^{i(\tau \pm \omega)t} - 1$, see (1.6). Call

$$u_\delta(t,x) = \psi_\delta(t)\Psi(t,x) + F(t,x) \tag{1.12}$$

and we have the following theorem.

Theorem 1.1. *For u_δ given in (1.12) and for $0 < \epsilon < 1$, the following estimate holds*

$$\|S^{\frac{1}{2}}\hat{u}_\delta\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\delta^{\frac{\epsilon}{2}} \left\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \right\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{1.13}$$

Proof. Straightforward calculations and Hölder's inequality give the following bound for the term F

$$\|S^{\frac{1}{2}}\hat{F}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq \frac{C}{R^{\frac{\epsilon}{2}}} \left\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \right\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{1.14}$$

Since $\Psi_{1,2}$ consists of the delta function and its derivatives, we need to localize them in the t variable. Thus consider the convolution of $S^{\frac{1}{2}}\hat{\Psi}$ and the Fourier transform of a smooth cutoff function $\psi_\delta(t) = \psi(t/\delta)$, and observe first that

$$\hat{\psi}_\delta * (S^{\frac{1}{2}}\hat{\Psi}) = \hat{\psi}_\delta * \hat{\Psi}.$$

Now for the expression $S^{\frac{1}{2}}(\hat{\psi}_\delta * \hat{\Psi})$ we have

$$\begin{aligned}
&\|S^{\frac{1}{2}}(\hat{\psi}_\delta * \hat{\Psi}_1)\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq \Big(\int (|\tau|+1) \Big|\hat{\psi}_\delta(\tau)\Big|^2 d\tau \Big)^{\frac{1}{2}} \cdot \\
&\quad \left(\Big\| \int \frac{\hat{b}(\lambda-\omega)}{\lambda-\omega} \frac{\hat{g}}{\omega} d\lambda \Big\|_{L^2(\mathbb{Z})} + \Big\| \int \frac{\hat{b}(\lambda+\omega)}{\lambda+\omega} \frac{\hat{g}}{\omega} d\lambda \Big\|_{L^2(\mathbb{Z})} \right) \\
&\quad \leq C(\psi) \Big\| \frac{\hat{b}(\lambda)(|\lambda|+1)^{\frac{1-\epsilon}{2}}}{\lambda} \Big\|_{L^2(\mathbb{R})} \Big\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \Big\|_{L^2} \leq \frac{C(\psi)}{R^{\frac{\epsilon}{2}}} \Big\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned}$$

For the term Ψ_2, since

$$\|(|\tau|+1)^{\frac{1}{2}} \widehat{t^k \psi_\delta}\|_{L^2(\mathbb{R})} \leq C(\psi)(2\delta)^k$$

and

$$\|\hat{G}_k^{\pm}\|_{L^2(\mathbb{Z})} \leq C \frac{(2R)^{k-1}}{k!} R^{1-\frac{\epsilon}{2}} \Big\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})},$$

we get

$$\begin{aligned}
\|S^{\frac{1}{2}}\hat{\psi}_\delta * \hat{\Psi}_2\|_{L^2(\mathbb{R}\times\mathbb{Z})} &\leq \sum_k \|(|\tau|+1)^{\frac{1}{2}} \widehat{t^k \psi_\delta}\|_{L^2(\mathbb{R})} (\|\hat{G}_k^-\|_{L^2(\mathbb{Z})} + \|\hat{G}_k^+\|_{L^2(\mathbb{Z})}) \\
&\leq C(\psi) \frac{e^{4R\delta} R\delta}{R^{\frac{\epsilon}{2}}} \Big\| \frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}} \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned}$$

Combining the estimates for Ψ_1, Ψ_2, we have the estimate for Ψ

$$\|S^{\frac{1}{2}}\hat{\psi}_\delta * \hat{\Psi}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\frac{e^{4R\delta}}{R^{\frac{\epsilon}{2}}}\Big\|\frac{\hat{g}}{\omega S^{\frac{1-\epsilon}{2}}}\Big\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{1.15}$$

Now choose $\delta = \frac{1}{R}$ and this completes the proof. □

Now we can state the local existence theorem. For simplicity, assume first that $u_0 \in L^2$, $u_1 \in H^{-2}$ and then we will describe the modification needed in the proof of Theorem 0.1.

Theorem 1.2. *Consider the problem*

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + \partial_x^2 f(u) = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases} \tag{1.16}$$

Assume that the initial data u_0 and u_1 satisfy $u_0 \in L^2(\mathbb{T})$ and $u_1 \in H^{-2}(\mathbb{T})$. Let $|f(u)| \leq C(|u|^q + |u|^p)$, $1 < q < p \leq 3$. Then equation (1.16) has a unique weak solution for $t \in [-\delta, \delta]$, where δ depends on the initial data. The solution for each fixed time t, $0 < t < \delta$, has same regularity as initial data and belongs to $L^4(\mathbb{R}_{(0,\delta)} \times \mathbb{T})$.

The proof consists of a fixed point argument and an a priori estimate involving Fourier multipliers on the set $\mathbb{R} \times \mathbb{Z}$. The a priori estimate is stated in the next Theorem.

Theorem 1.3. *Let $f(t,x)$ be a function with $(t,x) \in \mathbb{R} \times \mathbb{T}$ and denote by $\hat{f}(\tau,\xi)$ its Fourier transform, with $(\tau,\xi) \in \mathbb{R} \times \mathbb{Z}$. The following estimates hold*

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\|(||\tau| - \omega(\xi)| + 1)^{\frac{3}{8}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \tag{1.17a}$$

and its dual

$$\Big\|\frac{\hat{f}}{(||\tau| - \omega(\xi)| + 1)^{\frac{3}{8}}}\Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \tag{1.17b}$$

The proof of Theorem 1.3 will be given in the next section.

Proof of Theorem 1.2. The linear part of (1.16), i.e. $U(t,x)$ can be written

$$t\hat{u}_1(0)+\hat{u}_0(0)+\sum_{\xi\neq 0}\Big(\frac{e^{i(x\xi+t\omega)}-e^{i(x\xi-t\omega)}}{2i\omega}\hat{u}_1(\xi)+\frac{e^{i(x\xi+t\omega)}+e^{i(x\xi-t\omega)}}{2}\hat{u}_0(\xi)\Big),$$

from which we can obtain the estimate

$$\|S^{\frac{1}{2}}\hat{\psi}_\delta * \hat{U}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C(\psi)\Big(\|u_1\|_{H^{-2}}+\|u_0\|_{L^2}\Big). \tag{1.18}$$

Call

$$D = C(\psi)\Big(\|u_1\|_{H^{-2}}+\|u_0\|_{L^2}\Big),$$

which is a constant depending only on the initial data.

Now consider the map T defined by

$$Tu(t,x) = \psi_\delta(t)U(t,x)+\psi_\delta(t)\Psi(t,x)+F(t,x). \tag{1.19}$$

Notice that $(Tu)_x = Tu_x$. We want to show that T is a contraction under the norm

$$N(u) = \|S^{\frac{1}{2}}\hat{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})}.$$

Without loss of generality, we may assume that $f(u)=|u|^{p-1}u$. Combining Theorems 1.1 and 1.3 and the estimate for U, we have, for $\epsilon=\frac{1}{4}$,

(1.20)

$$\begin{aligned} N(Tu) &\leq C(D)+\frac{C}{R^{\frac{\epsilon}{2}}}\Big\|\frac{\hat{f}}{S^{\frac{1-\epsilon}{2}}}\Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C(D)+\frac{C}{R^{\frac{\epsilon}{2}}}\||u|^p\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \\ &\leq C(D)+\frac{C}{R^{\frac{\epsilon}{2}}}\|u\|^p_{L^4(\mathbb{R}\times\mathbb{T})} \leq C(D)+\frac{C}{R^{\frac{1}{8}}}N(u)^p. \end{aligned}$$

By choosing sufficiently large M, we have, for suitable δ and R,

$$N(u)\leq M \quad\Longrightarrow\quad N(Tu)\leq M, \tag{1.21}$$

provided that the following condition holds

$$C(D)+\frac{C}{R^{\frac{1}{8}}}M^p \leq M.$$

Notice that in the estimate (1.15) we make the choice $\delta \sim \frac{1}{R}$. This implies the time interval of existence δ is small if the quantity R is large.

Next, consider the difference $Tu - Tv$ and denote

$$u = \Phi + \Psi_u + F_u, \qquad v = \Phi + \Psi_v + F_v. \tag{1.22}$$

Elementary calculations and the inequality

$$\Big| |u|^{p-1}u - |v|^{p-1}v \Big| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|, \tag{1.23}$$

give

$$N(Tu - Tv) \leq \frac{C}{R^{\frac{1}{8}}} \left(N(u)^{p-1} + N(v)^{p-1} \right) N(u - v). \tag{1.24}$$

Therefore, again for suitable δ and R, we obtain

$$N(Tu - Tv) \leq \frac{1}{2} N(u - v),$$

provided that

$$\frac{C}{R^{\frac{1}{8}}} \left(M^{p-1} + M^{p-1} \right) \leq \frac{1}{2}$$

which can be satisfied by choosing R large for given M. This proves that the map T is a contraction with respect to the norm $N(u)$, hence it has a unique fixed point. □

The above theorem proves that a unique solution exists for finite time. In order to prove that the solution persists for all time, it is necessary to control the L^2-norm over the space variable for each fixed time. The norm

$$Q(u) = \sup_t \|u(t, \cdot)\|_{L^2} \tag{1.26}$$

can be estimated as follows. Assume for simplicity that $f(u) = |u|^{p-1}u$ and $\epsilon = \frac{1}{4}$, the term F in (1.8) can be estimated as follows

(1.27a)

$$\begin{aligned}
\|F(t, \cdot)\|^2_{L^2(\mathbb{T})} = \|\tilde{F}(t, \cdot)\|^2_{L^2(\mathbb{Z})} &\leq \sum_{\xi} \left(\int \Big| \frac{\hat{b}(\tau - \omega)}{\tau - \omega} - \frac{\hat{b}(\tau + \omega)}{\tau + \omega} \Big| \frac{\xi^2 |\hat{f}|}{\omega} d\tau \right)^2 \\
&\leq \sum \int \Big| \frac{\hat{b}(\tau - \omega)}{\tau - \omega} - \frac{\hat{b}(\tau + \omega)}{\tau + \omega} \Big|^2 S^{1-\epsilon} d\tau \int \frac{\xi^2 |\hat{f}|^2}{\omega^2 S^{1-\epsilon}} d\tau \\
&\leq \frac{C}{R^{\epsilon}} \Big\| \frac{\hat{f}}{S^{\frac{1-\epsilon}{2}}} \Big\|^2_{L^2(\mathbb{R} \times \mathbb{Z})}.
\end{aligned}$$

For Ψ in (1.10), we have

(1.27b)

$$\begin{aligned}
&\|\Psi_1(t,\cdot)\|^2_{L^2(\mathbb{T})} = \|\tilde{\Psi}_1(t,\cdot)\|^2_{L^2(\mathbb{Z})} \\
&\le C\sum_{\xi}\left(\int \frac{\hat{b}^2(\tau-\omega)}{(\tau-\omega)^2}S^{1-\epsilon}d\tau + \int \frac{\hat{b}^2(\tau+\omega)}{(\tau+\omega)^2}S^{1-\epsilon}d\tau\right)\int \frac{|\xi^2\hat{f}|^2}{\omega^2 S^{1-\epsilon}}d\tau \\
&\le \frac{C}{R^{\epsilon}}\left\|\frac{\hat{f}}{S^{\frac{1-\epsilon}{2}}}\right\|^2_{L^2(\mathbb{R}\times\mathbb{Z})}
\end{aligned}$$

and

(1.27c)

$$\begin{aligned}
&\|\Psi_2(t,\cdot)\|_{L^2(\mathbb{T})} = \|\tilde{\Psi}_2(t,\cdot)\|_{L^2(\mathbb{Z})} \le \sum_k t^k(\|\hat{G}^-_k\|_{L^2(\mathbb{Z})} + \|\hat{G}^+_k\|_{L^2(\mathbb{Z})}) \\
&\le C\sum \frac{(2Rt)^k}{k!2R}R^{1-\frac{\epsilon}{2}}\left\|\frac{\xi^2\hat{f}}{\omega S^{\frac{1-\epsilon}{2}}}\right\|_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\frac{e^{2Rt}}{R^{\frac{\epsilon}{2}}}\left\|\frac{\hat{f}}{S^{\frac{1-\epsilon}{2}}}\right\|_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned}$$

For U, see (1.3), we have

(1.27d)

$$\|U(t,\cdot)\|_{L^2(\mathbb{T})} = \|\tilde{U}(t,\cdot)\|_{L^2(\mathbb{Z})} \le C(\|u_1\|_{H^{-2}} + \|u_0\|_{L^2}).$$

Combining the above results we get the estimate

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{T})} &\le D + C\frac{e^{2Rt}}{R^{\frac{\epsilon}{2}}}\left\|\frac{\hat{f}}{S^{\frac{1-\epsilon}{2}}}\right\|_{L^2(\mathbb{R}\times\mathbb{Z})} \\
&\le D + C\frac{e^{2Rt}}{R^{\frac{1}{8}}}N(u)^p.
\end{aligned} \tag{1.28}$$

However, the L^2-estimate we have for $u(t,\cdot)$ in (1.28) is not sufficient to show global existence. Choosing R large, $\delta \sim 1/R$ and $\frac{1}{2}M \equiv D$ so that $N(u) \le M$, we obtain

$$\|u(t,\cdot)\|_{L^2(\mathbb{T})} \le \frac{1}{2}M + \frac{Ce^{2Rt}}{R^{1/8}}M^p. \tag{1.29}$$

This only leads to finite time of existence. At each step we argue the local existence, the time period of existence δ_n is of the order of $1/R_n$ and $R_n \ge (2CM_n^{p-1})^8$. Using (1.29), we find that the bound on the norm $N(u)$ increases exponentially, i.e.

$$\frac{1}{2}M_{n+1} \le (\frac{1}{2} + \frac{Ce^{2R_n\delta_n}M_n^{p-1}}{R_n^{\frac{1}{8}}})M_n \le 2^n M_0. \tag{1.30}$$

Hence the maximum time period of existence can be estimated by

$$T_{max} = \sum_n \delta_n \sim \sum \frac{1}{R_n} \sim \frac{C}{M_0^{8(p-1)}} \sum \frac{1}{[2^{8(p-1)}]^n}. \tag{1.31}$$

In fact equation (1.16) can blow up in finite time, see [KL], [S] and [L2]. However, if the energy is positive or semipositive definite, one expects a global solution. This implies that we need the initial data in H^1 to have a solution for which the energy is well-defined. On the other hand, in order to be able to prove an analog of Theorem 1.2 for general $p > 1$, it is also necessary to raise the regularity of initial data.

Before proving a global result, we prove the following theorem.

Theorem 1.4. *Assume that the initial data in (1.16) satisfy $u_0 \in H^s$, $u_1 \in H^{-2+s}$ with $0 \le s \le 1$. If $|f(u)| \le C|u|^p$ and p is an integer, then equation (1.16) has a local unique solution for*

$$\begin{cases} p & \le p(s) = \frac{3-2s}{1-2s}, \quad \text{if} \quad 0 \le s < \frac{1}{2}; \\ p & < +\infty \qquad \text{if} \quad \frac{1}{2} \le s. \end{cases} \tag{1.32}$$

Remark. Notice that $s = \frac{1}{4}$ gives $p(s) = 5$, see [FLS], where $f(u) = 4u^3 - 6u^5$; on the other hand, there is no restriction on p if $s \ge 1/2$. This is to be expected since $H^{\frac{1}{2}}$ can be embedded into L^p for any $p < +\infty$. Notice that $u_0 \in H^{\frac{1}{2}}$ and $u_1 \in H^{-\frac{3}{2}}$ will give a local weak solution such that the momentum

$$I = \int_{\mathbb{T}} uv dx \tag{1.33}$$

is well defined and conserved for all time.

To prove the theorem we need a lemma for the chain rule and Leibniz's rule for fractional derivatives. In particular, we need the following proposition from [CW].

Proposition 1.5. *(Christ & Weinstein) Suppose that $F \in C^1(\mathbb{C})$, $s \in (0,1)$, $1 < p, q, r < \infty$, and $r^{-1} = p^{-1} + q^{-1}$. If $u \in L^\infty(\mathbb{R})$, $D^s u \in L^q$, and $F'(u) \in L^p$, then $D^s(F(u)) \in L^r$ and*

$$\|D^s F(u)\|_r \le C\|F'(u)\|_p \|D^s u\|_q. \tag{1.34}$$

Let $s \in (0,1)$, $1 < r, p_1, p_2, q_1, q_2 < \infty$, and suppose $r^{-1} = p_i^{-1} + q_i^{-1}$, for $i = 1, 2$. Suppose that $f \in L^{p_1}$, $D^s f \in L^{p_2}$, $g \in L^{q_2}$, $D^s g \in L^{q_1}$. Then $D^s(fg) \in L^r$ and

$$\|D^s(fg)\|_r \le C\|f\|_{p_1}\|D^s g\|_{q_1} + \|g\|_{q_2}\|D^s f\|_{p_2}. \tag{1.35}$$

Proof of Theorem 1.4. We will only give an outline of the proof. Let

$$\partial^s u(t,x) = \mathfrak{F}^{-1}\{|\xi|^s \hat{u}(\tau,\xi)\}, \tag{1.36}$$

where $\mathfrak{F}^{-1}$ is the inverse Fourier transform in the t, x variables. We want to estimate the nonlinear terms using the norms

$$N(u) = \|S^{\frac{1}{2}}\hat{u}\|_{L^2} \quad \text{and} \quad Q(u) = \sup_t \|\mathfrak{F}^{-1}(|\hat{u}|)(t,\cdot)\|_{L^2}. \tag{1.37}$$

Estimate (1.17) in Theorem 1.3 implies that $\|u\|_{L^4} \le CN(u)$. On the other hand, Sobolev's inequalities give

$$\Big(\int \Big(\int |u|^{\frac{4}{1-4s}} dx\Big)^{1-4s} dt\Big)^{\frac{1}{4}} \le CN(\partial^s u), \tag{1.38}$$

$$\sup_t \Big(\int |u|^{\frac{2}{1-2s}} dx\Big) \le C\Big(\sup_t \|\partial^s u(t,\cdot)\|_{L^2}\Big)^{\frac{2}{1-2s}}. \tag{1.39}$$

For the linear Boussinesq equation

$$(\partial^s u)_{tt} - (\partial^s u)_{xx} + (\partial^s u)_{xxxx} = h, \tag{1.40}$$

we have estimates like (1.13) and (1.28) combined with (1.17b) for $N(T\partial^s u)$ and $Q(T\partial^s u)$, these imply

$$N(T\partial^s u) + Q(T\partial^s u) \le C + \frac{C}{R^{\frac{1}{8}}}\|\frac{\hat{h}}{\omega S^{\frac{3}{8}}}\|_{L^2} \le C + \frac{C}{R^{\frac{1}{8}}}\|\mathfrak{F}^{-1}\Big(\frac{\hat{h}}{\omega}\Big)\|_{L^{\frac{4}{3}}}, \tag{1.41}$$

where R can be arbitrarily large.

In the point of view of (1.34), the nonlinear term of (1.16) after taking s derivatives is essentially like

$$h = \partial_{xx}(u^{p-1}\partial^s u) \tag{1.42}$$

(see [KPV] or [CW]).

Using the fact that $N(\mathfrak{F}^{-1}(|\hat{u}|)) = N(u)$ and Hölder's inequality, the $L^{\frac{4}{3}}$ norm of $u^{p-1}\partial^s u$ can be estimated as follows.

$$\begin{aligned}&\iint |\mathfrak{F}^{-1}(|\hat{u}|)|^{(p-1)\frac{4}{3}}|v|^{\frac{4}{3}}dxdt\\ \leq C&\Big(\int\int |\mathfrak{F}^{-1}(|\hat{u}|)|^{2p-2}dxdt\Big)^{\frac{2}{3}}\Big(\int\int |v|^4dxdt\Big)^{\frac{1}{3}}.\end{aligned}$$

The second term on the right hand side is bounded by $\Big[N(\partial^s u)\Big]^{\frac{4}{3}}$. For the first term on the right hand side, we consider the integral over the x variable first, then write $2p-2 = 4+2(p-3)$ and use Hölder's inequality to get

(1.43)
$$\begin{aligned}&\Big(\int |\mathfrak{F}^{-1}(|\hat{u}|)|^4|\mathfrak{F}^{-1}(|\hat{u}|)|^{2(p-3)}dx\Big)\\ \leq&\Big(\int |\mathfrak{F}^{-1}(|\hat{u}|)|^{\frac{4}{1-4s}}dx\Big)^{(1-4s)}\cdot\Big(\int |\mathfrak{F}^{-1}(|\hat{u}|)|^{\frac{2(p-3)}{4s}}dx\Big)^{4s},\end{aligned}$$

where the inequality exponents are $r = \frac{1}{1-4s}$, $r' = \frac{1}{4s}$ for $s < \frac{1}{4}$.

Using (1.39) to control the right-most term in (1.43), we need the condition $\frac{2(p-3)}{4s} \leq \frac{2}{1-2s}$ which implies that

$$p \leq 3 + \frac{4s}{1-2s} = \frac{3-2s}{1-2s} = p(s), \tag{1.44}$$

so that it can be bounded by $Q(\partial^s u)^{2(p-3)}$. Using (1.38), the middle term in (1.43) can be bounded by $N(\partial^s u)^4$.

Hence, we have the following inequality

$$\iint |\mathfrak{F}^{-1}(|\hat{u}|)|^{(p-1)\frac{4}{3}}|\partial^s u|^{\frac{4}{3}}dxdt \leq Q(\partial^s u)^{\frac{4}{3}(p-3)}N(\partial^s u)^4, \tag{1.45}$$

Combine (1.41) and (1.45) we have

$$N(T\partial^s u) + Q(T\partial^s u) \leq C + \frac{C}{R^{\frac{1}{8}}}Q(\partial^s u)^{p-3}N(\partial^s u)^3; \tag{1.46}$$

and similar calculations give

$$N(Tu) + Q(Tu) \leq C + \frac{C}{R^{\frac{1}{8}}}Q(\partial^s u)^{p-1}N(u). \tag{1.47}$$

At this stage it is natural to define the norm

$$N_1(u) = N(u) + Q(u) + N(\partial^s u) + Q(\partial^s u). \tag{1.48}$$

Combining (1.46) and (1.47) we get the inequality

$$N_1(Tu) \leq C(D) + \frac{C}{R^{\frac{1}{8}}} N_1(u)^p. \tag{1.49}$$

To estimate the difference $Tu - Tv$, *i.e.* to estimate

$$|\partial^s(|u|^{p-1}u) - \partial^s(|v|^{p-1}v)|,$$

we use (1.35) which is a version of Leibniz's rule (see [CW]) to get

$$N_1(Tu - Tv) \leq \frac{C}{R^{\frac{1}{8}}} \Big(N_1(u)^{p-1} + N_1(v)^{p-1} \Big) N_1(u - v). \tag{1.50}$$

Therefore we can choose suitable δ and R so that T is a contraction. □

Corollary 1.5. *Assume that $u_0 \in H^1$ and $u_1 \in H^{-1}$. Let $|f(u)| \leq C(|u|^q + |u|^p)$, where p and q are two numbers greater than 1. Then equation (1.16) has a unique weak solution for $t \in [-\delta, \delta]$, where δ depends on the initial data.*

The global existence of a solution can be obtained from the conservation of the energy. Since the energy

$$E = \frac{1}{2} \int_{\mathbb{T}} [v^2 + u^2 + u_x^2 - 2F(u)] dx. \tag{1.51}$$

is conserved we can continue the solution for all time using Corollary 1.5, provided that $f(u) = \lambda |u|^{q-1} u - |u|^{p-1} u$, $q < p$, i.e.

$$F(u) = \frac{\lambda}{q+1} |u|^{q+1} - \frac{1}{p+1} |u|^{p+1}, \tag{1.52}$$

so that $F(u) < 0$ for $|u|^{p-q} > \frac{\lambda(p+1)}{q+1}$. The energy gives the estimate

$$\frac{1}{2} \int_{\mathbb{T}} v^2 + u^2 + u_x^2 dx \leq E + \int_{|u|^{p-q} < \frac{\lambda(p+1)}{q+1}} F(u) dx \leq E + C(\lambda, p, q). \tag{1.53}$$

The above discussion proves the following global existence theorem.

Theorem 1.6. *Assume that in equation (1.16) the initial data satisfy that* $u_0 \in H^1$, $u_1 \in H^{-1}$. *If* $f(u) = \lambda|u|^{q-1}u - |u|^{p-1}u$ *for* $1 < q < p$, $\lambda \in \mathbb{R}$, *then the problem (1.16) has a global unique solution* u *such that* u *and* $\partial_x u$ *are in the space* $L^4(\mathbb{R}_{\rm loc} \times \mathbb{T})$.

2. A Priori Estimates.

This part of the paper is devoted to the proof of Theorem 1.3 which we restate below. The method of proof is actually quite general and depends only on the geometric properties of the level curves given by $\omega(\xi_1)+\omega(\xi_2) =$ constant, see (2.1b).

Theorem 2.1. *Let* $f(t,x)$ *be a function with* $(t,x) \in \mathbb{R} \times \mathbb{T}$ *and denote by* $\hat{f}(\tau,\xi)$ *its Fourier transform, with* $(\tau,\xi) \in \mathbb{R} \times \mathbb{Z}$. *The following estimate holds*

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \le C\|S^{\frac{3}{8}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{Z})}, \tag{2.1a}$$

where

$$S = (||\tau| - \omega(\xi)| + 1) \quad \textit{and} \quad \omega(\xi) = \sqrt{\xi^2 + \xi^4}. \tag{2.1b}$$

Remark. The dual of (2.1a) is

$$\Big\|\frac{\hat{f}}{S^{\frac{3}{8}}}\Big\|_{L^2} \le \; C\|f\|_{L^{\frac{4}{3}}} \tag{2.1c}$$

and can be proved by a standard duality argument.

In the general case, considering the Boussinesq equation with different period will require to rescale space and time and can be reduced to an equation of the form

$$u_{tt} - c^2 u_{xx} + \mu^2 u_{xxxx} + g = 0.$$

Obvious modifications in the proof of Theorem 2.1 yield the following corollary.

Corollary 2.2. *Let $f(t,x)$ be a function with $(t,x) \in \mathbb{R} \times \mathbb{T}$ and denote by $\hat{f}(\tau,\xi)$ its Fourier transform, with $(\tau,\xi) \in \mathbb{R} \times \mathbb{Z}$. Now call*

$$\omega_\mu = \sqrt{c^2\xi^2 + \mu^2\xi^4} \quad \textit{and} \quad S_\mu = ||\tau| - \omega_\mu| + 1.$$

The following estimate holds

$$\|f\|_{L^4} \leq C\|S_\mu^{\frac{1}{4}}\hat{f}\|_{L^2} + \frac{C}{\sqrt{\mu}}\|S_\mu^{\frac{3}{8}}\hat{f}\|_{L^2}. \tag{2.2}$$

Proof of Theorem 2.1. Without loss of generality we can assume that the support of $\hat{f}(\tau,\xi)$ is inside the set

$$\{(\tau,\xi): \quad \tau \geq 0, \quad \tau - \omega(\xi) \geq 0\},$$

because otherwise we can split the function $\hat{f}(\tau,\xi)$ into a finite sum of functions, each supported in one set of the above type and the proof is similar for each set. Next we want to make a dyadic decomposition of $\hat{f}(\tau,\xi)$ along the variable $p = \tau - \omega(\xi)$, in order to achieve this, consider a smooth function $\hat{a}(p)$ with support in the interval $[\frac{1}{2}, 2]$ such that

$$\sum_{j=-\infty}^{\infty} \hat{a}(\frac{p}{2^j}) = 1 \qquad \text{if} \quad p \neq 0.$$

Call $\hat{a}_j(p) = \hat{a}(\frac{p}{2^j})$ and

$$\hat{a}_0(p) = 1 - \sum_{j=1}^{\infty} \hat{a}_j(p).$$

Denote

$$\triangle_j = [2^{j-1}, 2^{j+1}]$$

a dyadic interval, and write

$$\hat{f}_j(\tau,\xi) = \hat{a}_j(\tau - \omega(\xi))\hat{f}(\tau,\xi).$$

Now we can write

$$f = \sum_{j=0}^{\infty} f_j$$

and the estimate (2.1) will follow from the inequality

$$\|f_j f_k\|_{L^2(\mathbb{R}\times\mathbb{T})} \leq \frac{C}{2^{\frac{1}{8}|j-k|}} \|S^{\frac{3}{8}} \hat{f}_j\|_{L^2} \|S^{\frac{3}{8}} \hat{f}_k\|_{L^2}, \tag{2.3}$$

by squaring f, taking the L^2 norm and using the triangle inequality and Cauchy-Schwartz. In order to show (2.3) we compute the Fourier transform of $f_j f_k$

$$(f_j f_k)(t,x) = \int d\tau_1 d\tau_2 \sum_{\xi_1 \xi_2} e^{i\Omega} \hat{f}_j(\tau_1,\xi_1) \hat{f}_k(\tau_2,\xi_2), \tag{2.4a}$$

where

$$\Omega = t(\tau_1+\tau_2) + x(\xi_1+\xi_2). \tag{2.4b}$$

Without loss of generality, we can assume that $k \leq j$. Make the change of variables

$$\begin{cases} \tau &= \tau_1 + \tau_2, \quad \xi = \xi_1 + \xi_2; \\ p_i &= \tau_i - \omega(\xi_i) \qquad i = 1,2. \end{cases} \tag{2.5a}$$

and call

$$p = p_1 + p_2, \quad q = p_2. \tag{2.5b}$$

Formula (2.4) can be written as

$$(f_j f_k)(t,x) = \mathfrak{F}\{\hat{G}_{jk}(\tau,\xi)\}, \tag{2.6}$$

where

$$\hat{G}_{jk}(\tau,\xi) = \int_{\triangle_k} dq \sum_{p\in\Lambda_j} (\hat{f}_j \hat{f}_k)(\tau,\xi,p,q) \tag{2.7a}$$

and the set $\Lambda_j(\tau,\xi,q)$ is a discrete set defined by

$$\Lambda_j(\tau,\xi,q) = \{p \in \triangle_j + q : \xi_{1,2}(\tau,\xi,p,q) \in \mathbb{Z}\}. \tag{2.7b}$$

Remark. Notice that $\xi_{1,2}$ are computed by solving the system of equations,

$$\begin{cases} \omega(\xi_1) + \omega(\xi_2) = \tau - p, \\ \xi_1 + \xi_2 = \xi, \end{cases}$$

with respect to ξ_1, ξ_2. If we call $X = \xi_1 - \xi_2$, the above system can be rewritten as

$$(\tau-p)^4+X^2\xi^2\Big(1+\frac{X^2+\xi^2}{2}\Big)^2-(\tau-p)^2\Big[X^2+\xi^2+\frac{(X^2+\xi^2)}{4}+X^2\xi^2\Big]=0.$$

Solving the above equation gives a root $X(\tau-p,\xi)$ and we require that

$$\xi \pm X \in 2\mathbb{Z},$$

which forces p to take discrete values.

Plancherel's theorem in equation (2.6) gives

$$\|f_j f_k\|_{L^2} = \|\hat{G}_{jk}\|_{L^2}. \tag{2.8}$$

To estimate the right hand side of the above equation, observe first that

$$|\hat{G}_{jk}|^2 \le 2^k|\Lambda_j(\tau,\xi,q)|\int_{\triangle_k} dq \sum_{p\in\Lambda_j}|\hat{f}_j|^2|\hat{f}_k|^2. \tag{2.9}$$

The crucial observation here is that the size of Λ_j is much better than what one should normally expect.

Claim. *There exists a constant C such that*

$$\sup_{\tau,\xi,q}|\Lambda_j(\tau,\xi,q)| \le C2^{\frac{j}{2}}. \tag{2.10}$$

The proof of the claim will be given at the end.

Assuming the claim, we have

$$\int d\tau \sum_{\xi}|\hat{G}_{jk}|^2 \le C2^k 2^{\frac{j}{2}}\int d\tau dq \sum_{\xi,p}|\hat{f}_j|^2|\hat{f}_k|^2$$

and the right hand side of the above can be rewritten as

$$C\frac{1}{2^{\frac{1}{4}(j-k)}}2^{\frac{3}{4}j}2^{\frac{3}{4}k}\|\hat{f}_j\|^2_{L^2}\|\hat{f}_k\|^2_{L^2},$$

in view of (2.9) and (2.10), which is exactly the right hand side of (2.2). This proves the theorem. □

Proof of Claim. In order to prove the claim, notice first that Λ_j depends only on ξ and $\tau - q$ hence

(2.11)
$$\Lambda_j(\xi, A) = \{(\xi_1, \xi_2) \in \mathbb{Z}^2 : \xi_1 + \xi_2 = \xi, \quad A \le \omega(\xi_1) + \omega(\xi_2) \le A + 3 \cdot 2^{j-1}\},$$

where $A = \tau - q - 2^{j+1}$ and consider the level curves of $\omega(\xi_1) + \omega(\xi_2)$ in $\mathbb{R}^2$ given by

$$\mathcal{K}(A) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \quad \omega(\xi_1) + \omega(\xi_2) = A\}. \tag{2.12}$$

We assert that the radius of the inscribed and circumscribed circles to the above curve $\mathcal{K}(A)$ are given by

$$\begin{cases} r^2_{max} = (-1 + \sqrt{1 + 4A^2})/2 \\ r^2_{min} = -1 + \sqrt{1 + A^2}. \end{cases} \tag{2.13}$$

To see this use a parametric representation of the curve

$$\sqrt{\xi_1^2 + \xi_1^4} = A\cos^2\theta \quad , \quad \xi_1^2 = \frac{1}{2}(-1 + \sqrt{1 + 4A^2\cos^4\theta});$$
$$\sqrt{\xi_2^2 + \xi_2^4} = A\sin^2\theta \quad , \quad \xi_2^2 = \frac{1}{2}(-1 + \sqrt{1 + 4A^2\sin^4\theta}).$$

Let $r(\theta)$ denote the distance between the point (ξ_1, ξ_2) and the origin, thus

$$r^2(\theta) = \xi_1^2 + \xi_2^2 = -1 + \frac{1}{2}\left(\sqrt{1 + 4A^2\cos^4\theta} + \sqrt{1 + 4A^2\sin^4\theta}\right) \tag{2.14}$$

and its derivative is

$$\frac{dr^2(\theta)}{d\theta} = 4A^2\sin\theta\cos\theta\left\{\frac{\sin^2\theta}{\sqrt{1 + 4A^2\sin^4\theta}} - \frac{\cos^2\theta}{\sqrt{1 + 4A^2\cos^4\theta}}\right\} \le 0 \tag{2.15}$$

with equality only if $\theta = 0$ or $\theta = \frac{\pi}{4}$. Because of the symmetries of the curve $\mathcal{K}(A)$ it is enough to consider $\theta \in [0, \frac{\pi}{4}]$. Notice that $\theta = 0$ implies $\xi_2 = 0$ while $\theta = \frac{\pi}{4}$ implies $\xi_1 = \xi_2$. See figure 2.

Call $L(\xi)$ the line perpendicular to the line $\xi_1 = \xi_2$ at the point $\xi = \xi_1 + \xi_2$, see figure 2, then $\Lambda_j(\xi)$ is the number of lattice points on $L(\xi)$ between the curves $\mathcal{K}(A)$ and $\mathcal{K}(A + d)$ with $d = \frac{3}{2}2^j$. Observe that for fixed A

$$\Lambda_j(\xi) \subset \tilde{\Lambda}_j(\xi), \tag{2.16}$$

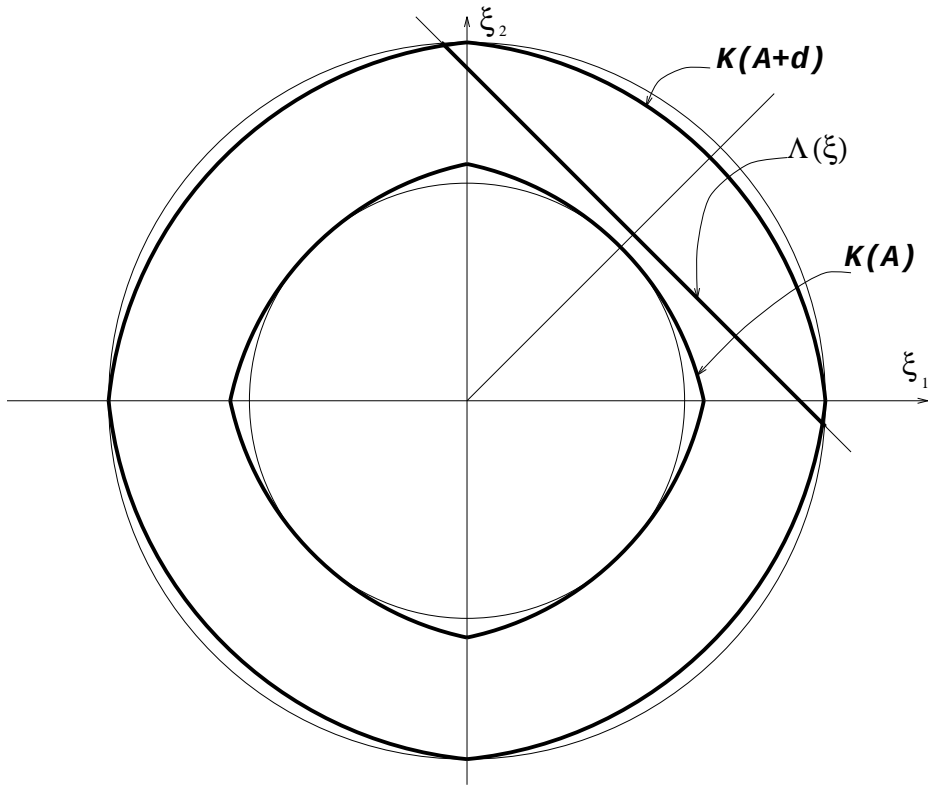

FIG 2. The level curves of $\mathcal{K}(A)$.

where $\tilde{\Lambda}_j$ is the lattice points belonging to the intersection of $L(\xi)$ with the annulus

$$\Big\{(\xi_1,\xi_2)\in\mathbb{Z}^2 : -1+\sqrt{1+A^2}\le \xi_1^2+\xi_2^2 \le \frac{-1+\sqrt{1+4(A+d)^2}}{2}\Big\}. \tag{2.17}$$

Call $P(\xi, A)$ the length of the intersection of $L(\xi)$ with the annulus in (2.17). The maximum length of $P(\xi, A)$ as ξ varies, is achieved when $\xi^2/2 = -1 + \sqrt{1+A^2}$, see figure 2. Now $P(\xi, A)$ can be estimated

$$P^2(\xi, A) \le 2(1+\sqrt{1+4(A+d)^2} - 2\sqrt{1+A^2}), \tag{2.18}$$

from which it follows that

$$\sup_\xi P(\xi, A) \sim \sup_\xi |\Lambda_j(\xi, A)| \sim C2^{\frac{j}{2}}. \tag{2.19}$$

The claim is proved. □

Remark. Notice that if A is large, namely $A \gg d$, where $A = \tau - q - 2^{j-1}$, then the thickness of the annular region defined in (2.17) can be estimated by

$$\triangle R \sim \frac{2^j}{A} \tag{2.20}$$

which means that for τ large the annular region is very narrow. On the other hand, $P(\xi, A)$ can be estimated for different ξ as follows:

If

$$\sqrt{-1+\sqrt{1+A^2}} \leq |\xi| \leq \sqrt{-1+\sqrt{1+A^2}} + \triangle R,$$

then

$$|P(\xi, A)| \sim 2^{\frac{j}{2}}. \tag{2.21}$$

If

$$|\xi| \leq \sqrt{-1+\sqrt{1+A^2}},$$

then after some straightforward calculation we obtain

$$|P(\xi, A)| \leq C\frac{2^j}{A^{\frac{3}{4}}} \tag{2.22}$$

which implies $|\Lambda_j(\xi)| \leq 1$, provided that $A > 2^{\frac{4}{3}j}$. This indicates that the estimate in Theorem 2.1 could be extended for L^p with $4 \leq p < 6$ and S^α with $\frac{3}{8} \leq \alpha < \frac{1}{2}$. This is a similar conjecture to the one which made by Bourgain in [B1].

Interpolating between (2.1a) and Plancherel's formula, one can prove the following.

Corollary 2.3. *With the same assumptions and notations as in Theorem 2.1 the following estimate holds*

$$\|f\|_{L^p} \leq C\|S^\alpha \hat{f}\|_{L^2}, \tag{2.23}$$

where $2 \leq p \leq 4$ *and* $\alpha = \frac{3}{4} - \frac{3}{2p}$.

Remark. The above inequality must hold for $2 \leq p < 6$.

References

[AS] J. C. Alexander & R. Sachs, *Linear instability of solitary waves of Boussinesq-type equation: A computer assisted computation* (preprint).

[B1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part I Schrödinger equations*, Geometric and Functional Anal. **3** (1993), 107-155.

[B2] _____, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part II The KdV equations*, Geometric and Functional Anal. **3** (1993), 209-262.

[Bo] M. J. Boussinesq, *Thèory des ondes et des remous quise propagent* ..., J. Math. Pure Appli. sect 2 **117** (1872), 55-108.

[BP] E. Bombieri & J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J. **59** (1989), 337-357.

[BS] J. Bona & R. Sachs, *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation*, Comm. Math. Phys. **118** (1988), 15-29.

[CS] Lennart Carleson & Per Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287-299.

[CW] F. M. Christ & M. I. Weinstein, *Dispersion of Small Amplitude Solutions of the Generalized Korteweg-de Vries Equation*, J. Funct. Anal. **100** (1991), 87-109.

[F] F. Falk, *Ginzburg-Landau Theory and Solitary Waves in Shape-Memory Alloys*, Z. Phys. B-Condensed Matter **54** (1984), 159-167.

[Fe] Charles Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9-36.

[FLS] F. Falk & E. W. Laedkl & K. H. Spatschek, *Stability of solitary-wave pulses in shape-memory alloys*, Physical Review B **36** (1987), 3031-3041.

[HP] M. Haragus & R. Pego, in preparation.

[KL] V. K. Kalantarov & O. A. Ladyzhenskaya, *The occurrence of collapse for quasi-linear equation of parabolic and hyperbolic types*, J. Sov. Math. **10** (1978), 53-70.

[KPV1] Carlos Kenig & Gustavo Ponce & Luis Vega, *Small Solutions to nonlinear Schrödinger Equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire **10** (1993), 255-288.

[KPV2] _____, *Well-posedness and scattering results for the generalized Korteweg de Vries equations via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527-620.

[L1] Y. Liu, *Instability of solitary waves for generalized Boussinesq equations*, J. Dyn. Diff. Eq. **5** (1993), 537-558.

[L2] _____, *Instability and blow-up of solutions to a generalized Boussinesq equation* (preprint).

[Ln] Felipe Linares, *Global Existence of small solutions for a generalized Boussinesq Equation*, J. Diff. Eq. **106** (1993), 257-293.

[LS] E. W. Laedkl & K. H. Spatschek, *Thresholds and growth times of some nonlinear field equations*, Physica 5D (1982), 227-242.

[M] H. P. McKean, *Boussinesq's equation on the circle*, Comm. Pure Applied Math. **34** (1981), 599-691.

[S] R. L. Sachs, *On the blow-up of certain solutions of the "good" Boussinesq equation*, Applicable Anal. **34** (1990), 145-152.

[Z] V. E. Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillators*, Sov. Phys. JETP **38** (1974), 108-110.

[Zy] A. Zygmund, *On Fourier Coefficients and Transforms of Functions of Two Variables*, Studia Math. **50** (1974), 189-201.

Department of Mathematics, University of Maryland College Park, MD 20742

E-mail address: fang@math.umd.edu; mng@math.umd.edu