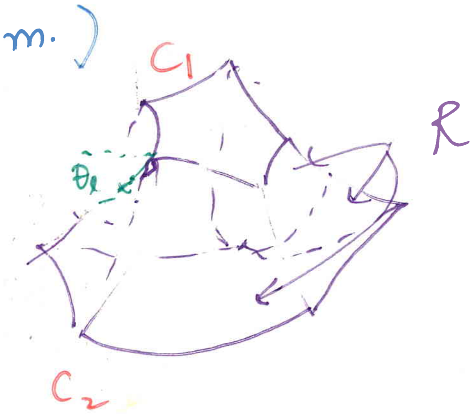
Finally, we prove

Thm// (Global Gauss-Bonnet Thm.)



Let $R \subseteq S$ be a region

and $\partial R = \underset{\uparrow}{C_1} \cup \cdots \cup \underset{\uparrow}{C_n}$,

polygons

Orient every C_i positively, Let $\theta_1, \dots, \theta_p$ be all the external angles of C_i's, then

$$\sum_{i=1}^{n} \int_{C_i} K_g \underset{\uparrow \text{arclength}}{(s)}\, ds + \iint_R K\, dA + \sum_{l=1}^{p} \theta_l = 2\pi \chi(R)$$

Pf// Triangulate R by $\mathcal{J} = \{T_1, \dots, T_F\}$ s.t.

- each T_j is contained in an isothermal coordinate
- each T_j oriented as described above (s.t. $\sum_{i=1}^{F} \int_{\partial T_i} = \sum_{j=1}^{n} \int_{C_j}$)

Summing up all the local Gauss-Bonnet,

$$\sum_{i=1}^{n} \int_{C_i} K_g(s)\, ds + \iint_R K\, dA + \sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{jk} = 2\pi F$$

where $\theta_{j1}, \theta_{j2}, \theta_{j3}$ are exterior angles of T_j.

$\Downarrow$

CONT

It remains to show that

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E - 2\pi V + \sum_{l=1}^{p}\theta_l$$

Let $\alpha_{jk} = \pi - \theta_{jk}$ be the interior angle of the kTH vertex of T_j

$$\Rightarrow \sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 3\pi F - \sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{jk}$$

write $E = \underbrace{E_i}_{\text{internal edges}} + \underbrace{E_e}_{\text{external edges}}$ ($\Sigma E_e = \Sigma C_i$)

$V = \underbrace{V_i}_{\text{internal vertices}} + \underbrace{\overset{p}{V_{ec}}}_{\text{external vertices that are vertices of } C_i} + \underbrace{V_{et}}_{\text{ext. vertices that are not}}$

and $E_e = V_e = V_{ec} + V_{et}$ we have

By induction on F,

$$3F = 2E_i + E_e$$

$$\Rightarrow \sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{j,k}$$

$$= 2\pi E_i + \pi E_e - (2\pi V_i - \pi V_{et} - \sum_{l=1}^{p}(\pi - \theta_l))$$

3 types of interior angles

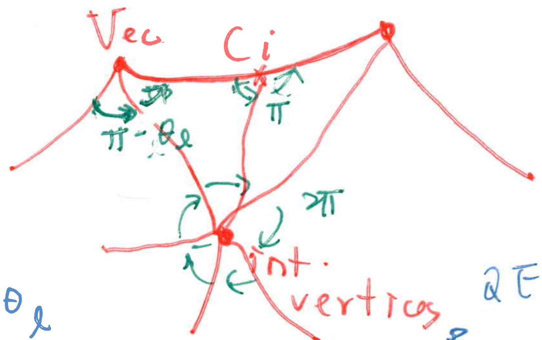
$$= 2\pi E_i + \pi(V_{ec} + V_{et}) - 2\pi V_i - \pi V_{et}$$

$$- \pi V_{ec} + \sum_{l=1}^{p}\theta_l$$

$$= 2\pi E_i - 2\pi V_i + \sum_{l=1}^{p}\theta_l$$

$E_e = V_e$ since each C_i is closed

$$= 2\pi(E_i + E_e) - 2\pi(V_i + V_e) + \sum_{l=1}^{p}\theta_l$$



QED,,

$$= 2\pi E - 2\pi V + \sum_{l=1}^{p}\theta_l$$