Homework 13

The notations of the problems here refer to discussion of Lebesgue measure μ on $\mathbb{R}$, discussed in class. Throughout the homework, you may assume the following facts:

- If a set E is measurable, E^c is also measurable.
- If $\{E_i\}_{i=1}^{\infty}$ are measurable sets, so is $\cup_{i=1}^{\infty} E_i$.
- All intervals, open or closed, are measurable.
- $\emptyset$ and $\mathbb{R}$ are measurable.

1. Prove that for $\forall A, B \in \mathcal{M}$, $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
2. Finish the proof of the construction of the non-measurable set N:
   (a) $N_r \cap N_s = \emptyset \; \forall \; r, s \in R$ with $r \neq s$.
   (b) $[0, 1) = \cup_{r \in R} N_r$.
3. Prove that if a set $A \in \mathcal{M}$ has positive Lebesgue measure, it must be uncountable.
4. Prove that the Lebesgue measure μ, satisfying the three properties mentioned in class, takes any interval to its usual length. That is, $\forall a, b \in [-\infty, \infty]$ with $a \leq b$,

$$\mu((a, b)) = \mu(([a, b)) = \mu((a, b]) = \mu([a, b]) = b - a.$$

5. $\forall E, F \in \mathcal{M}$, $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.
6. Define $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ by

$$\mu(A) = \textit{number of elements of } A.$$

   (a) Explain why μ is well defined on all domain.
   (b) Show that μ satisfies properties 1,2 of Lebesgue measure, but fails property 3.
   (c) What set(s) has measure 0?

   μ is called the *counting measure.*