Name and Student ID: ______________________________

Homework 10 Supplementary Problems

1. Consider the unit sphere

$$\mathbb{S}^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

and the "north pole" $N = (0, 0, 1) \in \mathbb{S}^2$. Consider the mapping

$$\Phi : \mathbb{S}^2 \backslash \{N\} \to \mathbb{R}^2$$

defined by

$$\Phi(x, y, z) = \frac{(x, y)}{1 - z}.$$

(a) Is Φ well defined on its domain?

(b) Prove the Φ is bijective.

(c) Find Φ^{-1}.

The map Φ can be understood in the following ways.

(d) Place the center of $\mathbb{S}^2$ at the origin $(0, 0, 0)$. For every $(x, y, z) \in \mathbb{S}^2 \backslash N$, write down the parametric equation of the line l going through N and (x, y, z).

(e) What is the point of intersection between l and xy plane (ie. $z = 0$)? Any relation to $\Phi(x, y, z)$?

This map Φ is known as the "*stereographic projection*" of $\mathbb{S}^2$ onto $\mathbb{R}^2$. The map can be easily generalized to any dimension. Given n-sphere

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \mid \sum_i x_i^2 = 1\} \subset \mathbb{R}^{n+1}$$

and the north pole $N = (0, \ldots, 0, 1)$, write down the stereographic projection between $\mathbb{S}^n \backslash \{N\}$ and $\mathbb{R}^n$.

2. Let $A = \mathbb{S}^n$ and $B = \mathbb{R}^{n+1} \backslash \{(0, \ldots, 0)\}$. Define relations on each set by

$$\forall \mathbf{x} := (x_0, \ldots, x_n), \mathbf{y} := (y_0, \ldots, y_n) \in A, \quad \mathbf{x} \sim_A \mathbf{y} \Longleftrightarrow (\mathbf{x} = \mathbf{y} \vee \mathbf{x} = -\mathbf{y}),$$

and

$$\forall \mathbf{z} := (z_0, \ldots, z_n), \mathbf{w} := (w_0, \ldots, w_n) \in B, \quad \mathbf{z} \sim_B \mathbf{w} \Longleftrightarrow \exists \lambda \neq 0, \ \mathbf{z} = \lambda \mathbf{w}.$$

(a) Prove that $\sim_A$ and $\sim_B$ are both equivalence relations.

(b) Prove that $A/\sim_A$ is in one-to-one correspondence with $B/\sim_B$. Recall that $B/\sim_B$ is also known as the real projective space of dimension n, or $\mathbb{RP}^n$.