Chapter 8

$\boxed{\#5}$ $R^{-1} = \{(1,1), (2,1), (3,1), (2,2), (3,2), (3,3)\}$

$\boxed{\#11}$ (1) $(a,a), (b,b), (c,c), (d,d) \in R$, then R is reflexive.

(2) Since $(a,b) \in R$, and $(b,a) \notin R$, R is not symmetric.

(3) There are $\begin{matrix}(a,b)\\(a,c)\\(a,d)\end{matrix}$ and $\begin{matrix}(b,c)\\(b,d)\end{matrix}$, (c,d).

Then $(a,b), (b,c) \in R$, (a,c) must in R. $\boxed{\text{True}}$

$(a,c), (c,d) \in R$, (a,d) must in R. $\boxed{\text{True}}$

$(b,c), (c,d) \in R$, (b,d) must in R $\boxed{\text{True}}$

Thus, R is transitive.

$\boxed{\#14}$ $R = \{(a,a), (b,c), (a,b)\}$

(1) Since $(b,b) \notin R$. R fails to be reflexive.

(2) Since $(b,c) \in R$, and $(c,d) \notin R$. R fails to be symmetric.

(3) Since $(a,b), (b,c) \in R$ and $(a,c) \notin R$. R fails to be transitive.

$\boxed{\#19}$ (a) $\forall x \in \mathbb{Z}$, xRx means $x \cdot x = x^2 \geq 0$. True for all x.

Then R is reflexive.

(b) If $(x,y) \in R$, then $x \cdot y \geq 0 \Leftrightarrow y \cdot x \geq 0$ i.e. ~~yRx~~ $(y,x) \in R$

Then R is symmetric.

(c) $(-1) \cdot 0 \geq 0$ and $0 \cdot 2 \geq 0$. But $(-1) \cdot 2 < 0$.

R is fails to transitive.

#22

(a) If $p(x) \in S$ and $p(x)$ has a root. then $p(x) R p(x)$ for sure. Sharing same root(s). Thus R is reflexive.

(b) If $p(x), q(x) \in S$ and $p(x) R q(x)$. Then $q(x) R p(x)$ since it tells same thing as $p(x) R q(x)$ — which is $p(x), q(x)$ share a common root.

(c) If $p(x), q(x), r(x) \in S$ and $p(x) R q(x)$, $q(x) R r(x)$, then $p(x), q(x)$ share a common root. and $q(x), r(x)$ share a common root. Yet it may not be the same common root.

$\boxed{\text{Counter-example}}$

$p(x) = (x-1)^2(x-2)$, $q(x) = (x-2)^2(x-3)$, $r(x) = (x-3)^2(x-4)$

$p(x) R q(x)$ since 2 is common root of $p(x), q(x)$

$q(x) R r(x)$ since 3 is common root of $q(x), r(x)$

Yet $p(x) \not R r(x)$ since there is no common root.

Thus, R is not transitive.

#26 Let us start frome the simplest class [3]. Since R is an equivalence relation. $(3,3) \in R$. $[3] = \{3\}$ and if $a \neq 3$, $a \in A$. $(a,3) \notin R$ since $[3] = \{3\}$. Then $[2] = \{2,6\}$ since $(2,2) \in R$, then we have $(6,2) \in R$ which leads us to $(2,6) \in R$ and if $a \neq 2,6$ $a \in A$ $(a,2), (a,6) \notin R$, otherwise it will contradict with assumption of the class $\{2,6\}$. And we can see that $[2] = [6] = \{2,6\}$. With similar argument, since the equivalence relation R, $[1] = [4] = [5] = \{1,4,5\}$. Therefore,

$$R = \left\{ (3,3),\ \begin{matrix}(2,2) & (2,6) \\ (6,6) & (6,2)\end{matrix},\ \begin{matrix}(1,1) & (1,4) & (1,5) & (4,5) \\ (4,4) & (4,1) & (5,1) & (5,4) \\ (5,5) & & & \end{matrix} \right\}$$

$\boxed{\#28}$ (1) If aRb, then $a+b=b+a$ is even, then bRa (symmetric)

(a) (2) $(a,a) \Rightarrow a+a=2a \ \forall a\in\mathbb{Z}$, then $a+a$ even, aRa (reflexive)

(3) If aRb and bRc, then $(a+b)+(b+c)$ is even,

Let $a+c+2b=2k \Rightarrow a+c=2(k-b)$ is even.

Thus aRc, R is transitive. (transitive). Thus R is an equivalence relation on $\mathbb{Z}$

$\forall$ even number $b=2k$ $(k\in\mathbb{Z})$

$bR0$ since $b+0=2k$ is even, then $[0]=\{2k \mid k\in\mathbb{Z}\}$

$\forall$ odd number $a=2m+1$ $(m\in\mathbb{Z})$

$aR1$ since $a+1=2m+1+1=2(m+1)$, then $[1]=\{2m+1 \mid m\in\mathbb{Z}\}$

There are two distinct eqivalence classes $[0]$ and $[1]$.

(b) (1) $a \not R a$ since $a+a=2a \ \forall a\in\mathbb{Z}$, which is even. (No reflexive)

(2) aRb leads bRa since $a+b=b+a$ is odd. (symmetric)

(3) $1R2$ and $2R3$ since $1+2=3$, $2+3=5$ are both odd.

Yet $1+3=4$ is even, then $1\not R 3$. (No transitive)

$\boxed{\#30}$ (1) aRa since $a/a=1=2^0$ (reflexive)

(2) If aRb, then $a/b=2^m$, then $b/a=2^{-m}$. i.e. bRa (symmetric)

(3) If aRb and bRa, then $a/b=2^m$, $b/c=2^n$, for some $m,n\in\mathbb{Z}$

Then $a/b \cdot b/c = a/c = 2^{m+n}$. i.e. aRc (transitive)

R is equivalence relation.

Find $[3]=\{a \mid aR3 \text{ and } a\in\mathbb{Q}\}$, then $a/3=2^m$, for some m. $a=2^m\cdot 3$

$[3]=\{a \mid a=2^m\cdot 3,\ \forall m\in\mathbb{Z}\}$

$\boxed{\#35}$ Assume $\exists R_1 . R_2$ fit the assumption.

$R_1 R_2$ are reflexive, $(a,a), (b,b), (c,c) \in R_1 \cap R_2$

Since $R_1 \not\subseteq R_2$, $\exists$ some element in R_1 that is not in R_2, say $(a,b) \in R_1 - R_2$,

since the symmetric of R_1, R_2, $(b,a) \in R_1 - R_2$

Because $R_2 \not\subseteq R_1$ $\exists$ some element in R_2 that is not in R_1, say $(b,c) \in R_2 - R_1$

same reason, $(c,b) \in R_2 - R_1$

Since $R_2 \cup R_1 = S \times S$, $(a,c) \in R_1 \cup R_2$

If $(a,c) \in R_1$, then transitive of R_1 tell us $(b,c) \in R_1$ (contradiction)

Same contradiction will happen if you put (a,c) in R_2.

Therefore, there ~~is~~ are no such R_1 R_2 fit the assumption.

$\boxed{\#37}$ (1) ~~the~~ Since $a^2+a^2 = 2a^2$ is even, aRa $\forall a \in \mathbb{N}$. (reflexive)

(2) If aRb, then $a^2+b^2 = b^2+a^2$ is even. i.e. bRa (symmetric)

(3) If $x = 2n$ (even), then $x^2 = 4n^2$, and if xRy, then $4n^2+y^2$ is even,
$(n \in \mathbb{N}$

then $y^2 = 2k - 4n^2 (k \in \mathbb{N})$ is even, then y is even.

Same as x is odd, then if xRy, y^2 is odd, implies y odd.

Finally, if aRb and bRc, it means a, b, c are either even or odd.

If a even, c even, a^2+c^2 even. If a odd, c odd, a^2+c^2 even.

Then we have aRc (Transitive).

$[1] = \{x \in \mathbb{N} : x^2+1^2 \text{ is even}\} = \{x \in \mathbb{N} : x^2 \text{ is odd}\} = \{x \in \mathbb{N} : x \text{ is odd}\}$

$[2] = \{x \in \mathbb{N} : x^2+2^2 \text{ is even}\} = \{x \in \mathbb{N} : x^2 \text{ is even}\} = \{x \in \mathbb{N} : x \text{ is even}\}$

$\boxed{\#40}$

(1) $\forall x \in \mathbb{Z}$, $3x - 7x = -4x = 2(-2x)$ even, then xRx (Reflexive)

(2) If xRy, then $3x-7y$ is even, let $3x-7y = 2n$, $n \in \mathbb{Z}$.

then $(3y-7x)+(3x-7y) = -4x-4y = 2(-2x-2y)$ even.

then $(3y-7x) = 2(-2x-2y)-2n$ even. That is, yRx. (Symmetric)

(3) If aRb and bRc, then $\underbrace{(3a-7b)}_{\text{even}}+\underbrace{(3b-7c)}_{\text{even}} = \underline{3a-7c} - 4b$ is even

Then $3a-7c$ is even. i.e. aRc. (Transitive).

$[0] = \{x \mid x \in \mathbb{Z}\ \ 3x-7\cdot 0 \text{ is even}\} = \{x \mid x \in \mathbb{Z},\ 3x \text{ even}\} = \{x \mid x \text{ even } x \in \mathbb{Z}\}$

$[1] = \{x \mid x \in \mathbb{Z}\ \ 3x-7 \text{ is even}\} = \{x \mid x \in \mathbb{Z},\ 3x \text{ odd}\} = \{x \mid x \text{ odd } x \in \mathbb{Z}\}$

$\boxed{\#42}$ If $a \in S \neq \phi$, then aR_1a and aR_2a, then $a(R_1 \cup R_2)a$. (Reflexive)

If $a(R_1 \cup R_2)b$, then aR_1b or $aR_2b \Rightarrow bR_1a$ or bR_2a (Symmetric)

$\Rightarrow b(R_1 \cup R_2)a$ (Symmetric)

If $a(R_1 \cup R_2)b$ and $b(R_1 \cup R_2)c$, then $\begin{pmatrix} aR_1b \\ \text{or} \\ aR_2b \end{pmatrix}$ and $\begin{pmatrix} bR_2c \\ \\ bR_2c \end{pmatrix}$

aR_1b	aR_2b	bR_1c	bR_2c	$a(R_1 \cup R_2)c$
T	F	F	T	F

That is, union of two equivalence relations may not be a equivalence relation.

$\boxed{\#68}$ If $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$, then $b = a+2k = a+3m$ for $m, k \in \mathbb{Z}$.

Since $(2,3)=1$, $b = a+6n$ $(n \in \mathbb{Z})$

Then aRb means $a \equiv b \pmod 6$. Which is a equivlence relation. on $\mathbb{Z}$.

$\boxed{\#75}$

(a) Since $|a|+|b| = |a|+|b|\ \forall a,b \in \mathbb{R}$, $(a,b)R(a,b)$. (Reflexive).

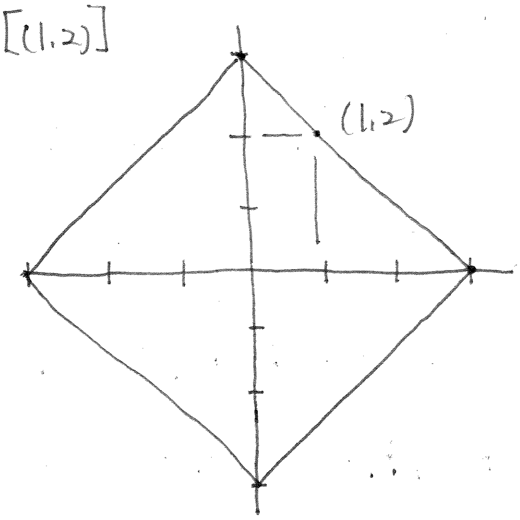
(b) If $(a,b)R(c,d)$, then $|a|+|b| = |c|+|d| \Leftrightarrow |c|+|d| = |a|+|b|$

i.e. $(c,d)R(a,b)$ (Symmetric)

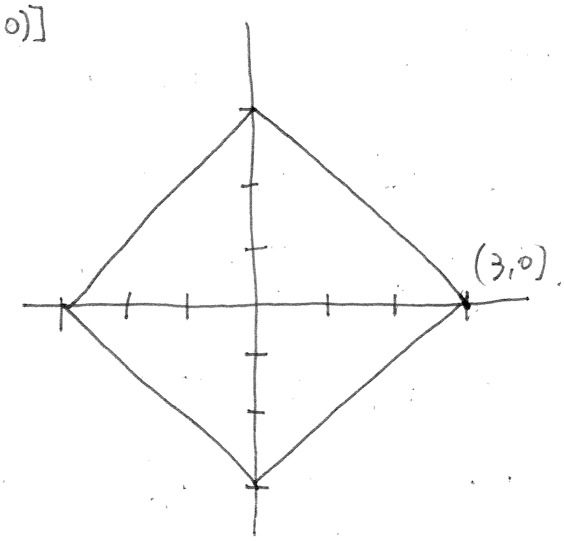
(c) If $(a,b)R(c,d)$; $(c,d)R(e,f)$, then $|a|+|b| = |c|+|d| = |e|+|f|$

i.e. $(a,b)R(e,f)$ (Transitive).

$[(1,2)]$



$[(3,0)]$



A Refinement Proof of $\boxed{6.40}$

For $n=0$, set $S_0 = \phi \subset S$ s.t. $\sum_{i \in S_0} i = 0$.

Assume for $n=k$. $\exists S_n \subset S$ s.t $\sum_{i \in S_k} i = k$

Then for $n=k+1$, find $a = \min(S - S_k)$, since $a \in S$, $a = 2^p$ for some $p \geq 0, p \in \mathbb{Z}$.

If $p=0$, then $S_k \cup \{2^0\} = S_{k+1}$, where $\sum_{i \in S_{k+1}} i = \sum_{j \in S_k} j + 1 = k+1$

If $p \geq 1$, then set $S_{k+1} = \left(S_k - \{2^j \mid 0 \leq j \leq p-1\}\right) \cup \{2^p\}$ s.t.

$$\sum_{i \in S_{k+1}} i = \sum_{j \in S_k} j - (2^0 + 2^1 + \cdots 2^{p-1}) + 2^p = k - (2^p - 1) + 2^p = k+1$$

Therefore, by mathematical induction, the statement is true for every $n \in \mathbb{N}$

Note: $A = 2^0 + 2^1 + \cdots + 2^{p-1}$, then $2A - A = 2^p - 2^0 = 2^p - 1$

$2A = \quad 2^1 + \cdots + 2^{p-1} + 2^p$

A Refinement Proof of $\boxed{6.57}$

(a) $P(1)$: Let $S_1 = \{1\} \subset S$, then $\sum_{i \in S_1} i = 1$ (True)

(b) If $1 \leq k < 300$, $\exists S_k \subset S$ s.t. $\sum_{i \in S_k} i = k$, then find $a = \min(S - S_k)$.

$\boxed{\text{Case 1}}$ If $a = 1$, then set $S_{k+1} = S_k \cup \{1\}$ s.t. $\sum_{j \in S_{k+1}} j = \sum_{i \in S_k} i + 1 = k+1$

$\boxed{\text{Case 2}}$ If $2 \leq a \leq 24$, then set $S_{k+1} = (S_k - \{a-1\}) \cup \{a\}$ s.t.

$$\sum_{j \in S_{k+1}} j = \sum_{i \in S_k} i - (a-1) + a = \sum_{i \in S_k} i + 1 = k+1$$

Therefore, by ~~finite of induction of mathematical~~ finite mathematical induction, the statement is true.