6,24 Proof $n! > 2^n$ $\forall n \geq 4$ $n \in \mathbb{N}$

When $n=4$, $4! = 24 > 16 = 2^4$ (True)

Assume $k! > 2^k$. $k \geq 4$ $k \in \mathbb{N}$

Then $(k+1)! = (k+1)k! > (k+1)\cdot 2^k \geq (4+1)2^k = 5\cdot 2^k > 2^{k+1}$.

Thus, $(k+1)! > 2^{k+1}$. By mathemeitical iduction.

6,31 $\forall n \geq 1$. $\forall a_1 a_2 \ldots a_n > 0$

When $n=1$, $a \cdot \frac{1}{a} = 1^2$. Assume $a_1, a_2 \ldots a_k > 0$, s.t. $\left(\sum_{i=1}^{k+1} a_i\right)\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \geq k^2$

Then, let $b_1, b_2, \ldots b_{k+1} > 0$ ~~s.t.~~, ~~s.t.~~

$$\left(\sum_{i=1}^{k+1} b_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) = \left(\sum_{i=1}^{k} b_i\right)\left(\sum_{i=1}^{k} \frac{1}{b_i}\right) + b_{k+1}\left(\sum_{i=1}^{k} \frac{1}{b_i}\right) + \frac{1}{b+1}\left(\sum_{i=1}^{k} b_i\right)$$

$$+ b_{k+1} \cdot \frac{1}{b_{k+1}} \geq k^2 + \sum_{i=1}^{k}\left(\frac{b_{k+1}}{b_i} + \frac{b_i}{b_{k+1}}\right) + 1 \quad \text{— ①}$$

$\boxed{\text{Lemma}}$ $\frac{a}{b} + \frac{b}{a} \geq 2$ $a, b > 0$.

Since $(a-b)^2 = a^2 + b^2 - 2ab \geq 0 \Rightarrow a^2 + b^2 \geq 2ab \Rightarrow \frac{a^2+b^2}{ab} \geq 2$

$\Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2$

Then $\forall \left(\frac{b_{k+1}}{b_i} + \frac{b_i}{b_{k+1}}\right) \geq 2$.

Then ① $\geq k^2 + k(2) + 1 = (k+1)^2$

By mathematical induction, $\left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \geq k^2$. $\forall a_1 \ldots a_n > 0$.

(6.34) Assume exist a number $n \in \mathbb{N}$. s.t. $3 \nmid 2^{2n}-1$.

Then exist a minmum number $m \in \mathbb{N}$ s.t. $3 \nmid 2^{2m}-1$

~~Since $3 \nmid 2$~~ Then for $k=m-1$ $3 \mid 2^{2k}-1$, assume $2^{2k}-1 = 3q$

Then $2^{2(k+1)}-1 = [2^2 \cdot 2^{2k} - 2^{2k}] + [2^{2k}-1] = 2^{2k}(3) + 3q = 3(2^{2k}+q)$

A contradiction to ~~minimum~~ assumption, n.

Thus, $3 \mid 2^{2n}-1$ $\forall n>0$, $n \in \mathbb{N}$. #

(6.40) Assume $\underbrace{S_n \subseteq S}_{\exists n \in \mathbb{N}}$ s.t. $S_n \subseteq S$ and $\sum_{i \in S_n} \neq n$

Then $\exists m \in \mathbb{N}$, m is minimal number s.t. $S_m \subseteq S$ and $\sum_{i \in S_n} \neq n$

Then $\exists S_{m-1} \subseteq S$ s.t. $\sum_{i \in S_n} i = m+1$

Then there is a minimal number q, $0 \leq q \leq m$ s.t.

$2^{q+1} \notin S_{m-1}$ ~~and~~ $2^0, 2^1, 2^2, \dots, 2^q \in S_{m-1}$, where $2^0+2^1+\dots+2^q = 2^{q+1}-1$

Set $S_m = (S_{m-1} - \{2^0, 2^1, \dots 2^q\}) \cup \{2^{q+1}\}$

Then $\sum_{i \in S_m} i = \left(\sum_{j \in S_{m-1}} j\right) - (2^0+2^1+\dots+2^q) + 2^{q+1}$

$= \left(\sum_{j \in S_{m-1}} j\right) - (2^{q+1}-1) + 2^{q+1} = \sum_{j \in S_{m-1}} j + 1 = m$

Contradict with assumption n #.

(6.41) Conjecture(推测) $a_n = 2^{n-1}$ for all $n \geq 1$

~~Proof:~~ $a_1 = 2^{1-1} = 2^0 = 1$ (True)

Assume $a_k = 2^k$, then $a_{k+1} = 2 \cdot 2a_k = 2 \cdot 2^{k-1} = 2^k$.

By mathematic induction, it's true for $k \geq 1$.

(6.45) Question: $\forall n \geq 12.\ n \in \mathbb{N}.\ \exists a \geq 0\ b \geq 0\ a, b \in \mathbb{N}$ s.t. $n = 3a + 7b$

If $n = 12$, $12 = 3 \cdot 4 + 7 \cdot 0$

Assume ~~$\exists k \in \mathbb{N}$ s.t~~ $\exists k \in \mathbb{N}$ s.t. $\forall i \in \mathbb{N}$, $12 \leq i \leq k$

There exist nonnegative integers a and b s.t. $i = 3a + 7b$

Since $12 \leq k-2 \leq k$, $(13 = 3 \cdot 2 + 7,\ 14 = 3 \cdot 0 + 7 \cdot 2)$

$\exists c, d \in \mathbb{N}$, $c, d \geq 0$ s.t. $k - 2 = 3c + 7d$

then $k+1 = 3(c+1) + 7d$.

By strong ~~φ~~ mathematical Induction, the assumption is true.

(6.57) $P(1)$: $1 \leq 1 \leq 300$, exists a subset $S_1 = \{1\} \subseteq S$ s.t $\sum_{i \in S_1} i = 1$ (True)

If $P(k)$, (i.e. $1 \leq k \leq 300$, exists a subset $S_k \subseteq S$ s.t $\sum_{i = S_k} i = k$)

then we need to show that $1 \leq k+1 \leq 300$, exists a subset $S_{k+1} \subseteq S$

s.t. $\sum_{j = S_{k+1}} j = k+1$. Assume there exist a smallest integer $1 \leq m \leq 24$ s.t. $\{24, 23 \cdots m\} \subseteq S_k$ and $m-1 \notin S_k$ and a largest integer $0 \leq r \leq m$ ($r = 0$ or $r \in S_k$)

Then set $S_{k+1} = (S_k - \{r\}) \cup \{r+1\}$

Then $\sum_{j \in S_{k+1}} j = \sum_{i \in S_k} i - r + r + 1 = k + 1$ for $1 \leq k \leq 300$.

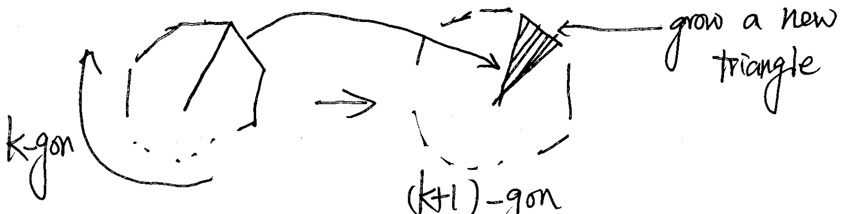
By finite mathematical induction, the statement is true.

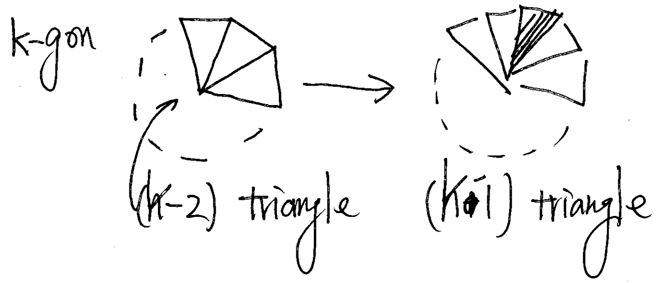
6.61 $n=3$, a triangle, sum of the interior ~~angles~~ angles of it is $(3-2)\cdot 180^\circ = 180^\circ$ (True)

Assume ~~k-gon~~, the sum of the interior angles of k-gon is

$\boxed{180^\circ (k-2)}$

Then for $(k+1)$-gon, as figure shows.





there are $(k-2)+1$ triangle in $(k+1)$-gon.

then the sum of the interior angles of $(k+1)$-gon is

$(k-1)\cdot 180^\circ$.

By mathematical induction, the ~~assumption~~ statement is true.