5.3 If $n=3$, then $2n^2+1=19$, $3\nmid 19$ (a counterexample)

5.9. Let $x=3, n=2$, then $x^n+(x+1)^n=3^2+4^2=25=5^2=(x+2)^n$ (a counterexample)

5.10 If exists the largest negative rational number a.

~~then a~~ ~~If a~~

Then $-1\leq a<0$ since a is largest.

Then $a-(\frac{a+0}{2})>a$. Contradiction. ※

$\in \mathbb{Q}$

5.14. If $a\nmid b$, then we finish the proof.

Assume $a|b$, then $na=b$ for some n.

$\Rightarrow na+1=b+1 \Rightarrow a(n+\frac{1}{a})=b+1$, $\frac{1}{a}$ isn't integer since $a\geq 2$.

Then $a\nmid b+1$ ※

5.18 Let $a=\sqrt{3}$, $r=2$, $s=6$

$2\sqrt{3}+6=A_1$ $2\sqrt{3}-6=A_2$, A_1 A_2 are both irrational.

~~題目~~命題有誤!!

5.24 Since x is positive integer. $2x<x^2 \Leftrightarrow 2<x$

$x^2<3x \Leftrightarrow x<3$

There is no such integer exist. ※

5.32. Let $n>m$, then $\exists k\in\mathbb{N}$ s.t. $m+k=n$

$m^2+m+1=n^2$, ~~then~~ also $n^2=m^2+2mk+k^2$

$\Rightarrow \cancel{m^2}+m+1=\cancel{m^2}+2mk+k^2 \Rightarrow m(1-2k)=k^2+1$

Since $k\geq 1$, then $(1-2k)\in\mathbb{Z}^-$ $(k^2+1)\in\mathbb{Z}^+ \Rightarrow m\in\mathbb{Z}^-$ ※

※

5.40 Let $a=1$ and $b=\sqrt{2}$, then $1^{\sqrt{2}}=1 \in$ rational.

5.41 $a=2$, $b=\frac{1}{2\sqrt{2}}$, then assume $2^{\frac{1}{2\sqrt{2}}}$ is rational.

(Other wise, $2^{\frac{1}{2\sqrt{2}}}$ is the example we want)

$$\left(2^{\frac{1}{2\sqrt{2}}}\right)^{\sqrt{2}} = 2^{\frac{\sqrt{2}}{2\sqrt{2}}} = 2^{\frac{1}{2}} = \sqrt{2} \text{ is irrational.}$$

So $a=2^{\frac{1}{2\sqrt{2}}}$, $b=\sqrt{2}$ ~~the~~ is the example we want.

~~5.43~~

~~Since $x<$~~ ~~when $x \to -\infty$~~

When ~~$x<-1$, x is decreasing and $(-1)^3+(-1)^2-1<0$~~

~~$x>1$, x is increasing and $1^3+1^2-1>0$~~

~~$\left(\frac{2}{3}\right)^3+\left(\ \right)>$~~

5.44 x is increasing in $\left[\frac{2}{3}, 1\right]$ and $\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2-1<0$

and $1^3+1^2-1>0$

$\Rightarrow \exists$ a unique real number solution to $x^3+x^2-1=0$.

5.51 Case1

n is even, Then $n=2a$ for some integer a.

Then $n^4+n^3+n^2+n = n(n+1)(n^2+1) = \boxed{2}a(n+1)(n^2+1)$

even #

Case2

n is odd Then $n=2a+1$ for some integer a.

Then $n^4+n^3+n^2+n = n(n+1)(n^2+1)$

$= (2a+1)\boxed{2}(n+1)(n^2+1)$ even #

5,61

$\boxed{\text{Lemma } \sqrt{2}+\sqrt{3} \text{ is irrational.}}$

Assume $\sqrt{2}+\sqrt{3} = Q_1$ for some rational $Q_1 \in \mathbb{Q}$

then $2+3+2\sqrt{6} = Q_1^2 = Q_1'$ rational.

then $2\sqrt{6} = Q_1''$ rational, then $\sqrt{6} = Q_1'''$ rational $\times$

Now, $\underbrace{\sqrt{2}+\sqrt{3}}_{\text{Assume.}}+\sqrt{5} = Q_2$ (Assume $Q_2 \in \mathbb{Q}$)

Then $\sqrt{2}+\sqrt{3} = Q_2 - \sqrt{5}$

Then $\cancel{2+3}+2\sqrt{6} = Q_2^2 - 2Q_2\sqrt{5} \cancel{+5}$

$\Rightarrow \quad 2\sqrt{6} = Q_2^2 - 2Q_2\sqrt{5}$

$\Rightarrow \quad 2(\sqrt{6}+Q_2\sqrt{5}) = Q_2^2 \Rightarrow \sqrt{6}+Q_2\sqrt{5} = Q_2'$ rational.

$\Rightarrow \quad 6 + Q_2^2 5 + 2\sqrt{5}\sqrt{6} = (Q_2')^2$

$\Rightarrow \quad 2\sqrt{30} = Q_2''$ rational.

$\sqrt{30} = Q_2'''$ rational $\times$

Thus, $\sqrt{2}+\sqrt{3}+\sqrt{5}$ irrational.

6.4 (1) Let $n=1$, then $1=1^2$ (True)

Assume $n=k$ is true for $1+3+5+\cdots+(2k-1)=k^2$

Then $1+3+5+\cdots+(2k-1)+(2k+1)=k^2+2k+1=(k+1)^2$

By mathematical induction, the statement is true.

(2)

$$\begin{array}{rl} 1+3+5+\cdots+\cdots+2n-1 & =A \\ +)\ (2n-1)+(2n-3)+\cdots\cdots+1 & =A \\ \hline \underbrace{2n+2n+2n\cdots+2n} = 2n\cdot n = 2n^2 & =2A \end{array}$$

$A=n^2$ //

6.9.

Let $n=1$, then $\frac{1(1+1)(2+7)}{6}=3$ (true)

Assume $1\cdot3+2\cdot4+3\cdot5+\cdots+k(k+2)=\frac{k(k+1)(2k+7)}{6}$

Then $1\cdot3+2\cdot4+\cdots+k(k+2)+(k+1)(k+3)=\frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)$

$=(k+1)\left[\frac{k(2k+7)}{6}+\frac{6k+18}{6}\right]=\left[\frac{2k^2+13k+18}{6}\right](k+1)$

$=\frac{(k+2)(2k+9)(k+1)}{6}=\frac{([k+1])([k+1]+1)(2(k+1)+7)}{6}$

By mathematical induction, it is true.

6.13 Let $n=1$, $1\cdot1!=2!-1$ (True)

Assume $1\cdot1!+2\cdot2!+\cdots+k\cdot k!=(k+1)!-1$

Then $1\cdot1!+2\cdot2!+\cdots+(k+1)(k+1)!=(k+1)!-1+(k+1)(k+1)!$

$=[(k+1)!](k+2)-1$

$=(k+2)!-1$

By mathematical induction, it is true.

6.16 $7 \mid [3^{4n+1} - 5^{2n-1}]$

When $n=1$ $7 \mid (3^5 - 5) = 243 = 7 \cdot 34$ (True)

Assume $7 \mid (3^{4k+1} - 5^{2k-1})$

Then $3^{4k+1} - 5^{2k-1} = 7m$ for some $m \in \mathbb{N}$.

Then $3^{4(k+1)+1} - 5^{2(k+1)-1} = 3^{4k+5} - 5^{2k+1}$

$$= 3^4 \cdot 3^{4k+1} - 5^2 \cdot 5^{2k-1}$$

$$= 3^4 \cdot 3^{4k+1} + (-(5^{2k-1}) \cdot 3^4) - (-(5^{2k-1}) \cdot 3^4) - 5^2 \cdot 5^{2k-1}$$

$$= 3^4 (3^{4k+1} - 5^{2k-1}) + [(5^{2k-1})(3^4 - 5^2)]$$

$$= 3^4 \cdot 7m + 5^{2k-1} \cdot \underline{\underline{56}} = 7 \cdot 8$$

$$= 7(3^4 \cdot m + 5^{2k-1} \cdot 8)$$

Thus, $7 \mid 3^{4(k+1)+1} - 5^{2(k+1)-1}$

By mathematical induction, it is true!