Homework 11.

10.1

$A_1 = \{-3, -2, 2, 3\}$ $\quad |A_1| = |A_2| = 4$

$A_2 = \{-5, -4, -3, 5\}$ $\quad |A_3| = 6$

$A_3 = \{-2, -1, 0, 1, 2, 3\}$ $\quad |A_4| = |A_5| = 3$

$A_4 = \{-1, 0, 1\}$

$A_5 = \{-4\ 0\ -4\}$

$[A_1] = \{A_1, A_2\}$ $\quad [A_4] = \{A_4. A_5\}$.

$[A_3] = \{A_3\}$

10.4

A, B denumerable

$A = \{a_1, a_2 \dots \} \subset R^+$

$B = \{b_1, b_2 \dots \} \subset R^+$. Then $A \cup C = \{ \begin{array}{cccc} a_1 & -b_1 & a_2 & -b_2 \dots \\ \| & \| & \| & \| \\ d_1 & d_2 & d_3 & d_4 \dots \end{array} \}$

$C = \{-b_1, -b_2 \dots \} \subseteq R^-$

$d_i = \begin{cases} a_i & i \text{ odd.} \\ -b_{i-1} & i \text{ even.} \end{cases}$

$A \cap C = \phi$

$A \cup C$ denumerable.

10.5

$Z - \{2\} \subset \mathbb{Z}$, $\mathbb{Z}$ denumbeable leads $Z - \{2\}$ denumberable.

$\Rightarrow |\mathbb{Z}| = |Z - \{2\}$

10.13

$f: G \to \mathbb{Z} \times \mathbb{Z}$ by $f(a+bi) = a+b$. f bijective and so

$|G| = |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$. G denumberable.

10.15

$S \subset \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N}$ denumbeable leads S denumberable.

$|S| = |\mathbb{N}|$

10.18 (a) Let $f(m,n) = f(m',n')$

Then $\underbrace{2^{m-1}}_{\text{even}}\underbrace{(2n-1)}_{\text{odd}} = \underbrace{2^{m'-1}}_{\text{even}}\underbrace{(2n'-1)}_{\text{odd}}$

Then $2^{m-1} = 2^{m'-1}$, $2n-1 = 2n'-1 \Rightarrow m = m'.\ n = n'$

$\Rightarrow f$ is 1-1.

Let $X \in \mathbb{N}$. then $\exists! m$ s.t.

$$2^{m-1} \mid X \text{ and } 2^m \nmid X$$

Then $X = 2^{m-1} \cdot y$. since $2^m \nmid X$. $2 \nmid y$

Then y is ~~even~~ odd, then $\exists! n$ s.t. $X = 2^{m-1}(2n-1)$

(m,n) is the integer pair s.t. $f(m,n) = X$.

$\Rightarrow f$ is onto.

(b)

By (a) f is bijection.

Then $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ denumberable.

10.40 (a) $\exists n, n'$ s.t. $f(n) = f(n') = (-1)^n \lfloor n/2 \rfloor = (-1)^{n'} \lfloor n'/2 \rfloor$

$\{0, 1, 2, 3, 4, 5, 6 \cdots\}$ By the mapping. f is 1-1.

$\frac{n}{2}$: $\{\frac{0}{2}\ \frac{1}{2}\ \frac{2}{2}\ \frac{3}{2} \cdots\}$ If $X \in \mathbb{Z}$,

$\lfloor \frac{n}{2} \rfloor$: $\{0\ 1\ 1\ 2\ 2\ 3\ 3 \cdots\}$ Case 1 $X = 0$, then $n = 0$

$(-1)^n \lfloor \frac{n}{2} \rfloor$: $\{0\ {-1}\ 1\ {-2}\ 2\ {-3}\ 3 \cdots\}$ Case 2 $|X| = X$, then $n = 2X$

Case 3 $|X| = -X$, then $n = |2X| - 1$

s.t. $f(n) = X$. f onto.

(b) $\mathbb{Z}$ is denumberable.

10.41

(a) $f_1 : (0,1) \to (0,\infty)$ by $f_1(x) = \dfrac{x}{1-x}$

(b) $f_2 : (0,1] \to [0,\infty)$ by $f_2(x) = \dfrac{1-x}{x}$

(c) $f_3 : [b,c) \to [a,\infty)$ by $f_3(x) = \dfrac{x-b}{c-x} + a$ //

f_1, f_2, f_3 are bijection.

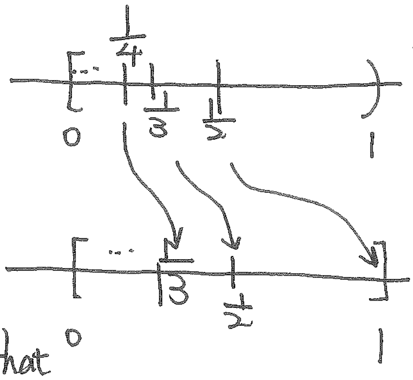
Thus, the inteval pairs are numericllay equivalent.

Homework 11 Supplymentary Problems.

1. Let the function be $f : [0,1) \to [0,1]$ by

$f(\frac{1}{2}) = 1$, $f(\frac{1}{n}) = \frac{1}{n-1}$ $\forall n \in \mathbb{N}$ $n \geq 3$, $f(x) = x$ for $x = \frac{1}{n}$

In figure, it is



It's easy to check that
f is bijection. Therefore, $|[0,1)| = |[0,1]|$

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2. $A = \{\text{algebraic number}\}$. By definition of A. $A \subseteq \{\text{roots of all integer poly.}\}$

complex root isn't in A.

Let $P_n = \{\text{integer poly of degree } n\}$

And set $f : P_n \to \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n+1}$ by $f(a_0 + \cdots a_n x^n) = (a_0 \ldots a_n)$ with $a_n \neq 0$.

It's easy to see that f is bijection.

Since $\mathbb{Z}^n$ is countable. P_n is coutable. Which means we can index the set $P_n = \{P_n^1, P_n^2, \cdots\}$

Let $E_n = \{\text{roots of poly. from } P_n\} = \bigcup_{j=1}^{\infty} (\text{roots of } P_n^j)$

For each P_n^j, there are at most n roots, i.e finite.

Then $\bigcup_{j=1}^{\infty} (\text{roots of } P_n^j)$ is countable union of finite number sets

Leads $\bigcup_{j=1}^{\infty} (\text{roots of } P_n^j)$ countable. Then $\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (\text{roots of } P_n^j) \supseteq A$

is coutable union of countable sets, still countable. Thus A countable.

3. To prove $|(A\times B)\to C| = |A\to(B\to C)|$

We can find a bijection function T s.t.

$$A\times B\to C \xrightarrow{\quad T \quad} A\to(B\to C)$$

Notice that elements in $A\times B\to C$ are functions from $A\times B$ to C.

Then $\forall f\in(A\times B\to C)$, $T(f)\in(A\to(B\to C))$

Again, notice that $T(f)$ is function send A to $(B\to C)$

Then $\forall a\in A \quad T(f)(a)\in B\to C$.

Same argument $T(f)(a)$ is function from B to C

Then $\forall b\in B$. $T(f)(a)(b)\in C$

We set the function be $T(f)(a)(b) = f(a,b)\in C$

① Injective

$$\begin{aligned}\text{If } T(f)=T(g) &\Rightarrow \forall a\in A \quad T(f)(a) = T(g)(a)\\ &\Rightarrow \forall b\in B \quad T(f)(a)(b) = T(g)(a)(b)\\ &\Rightarrow T(f)(a)(b) = f(a,b) = T(g)(a)(b) = g(a,b) \quad \forall a,b\\ &\Rightarrow f=g.\end{aligned}$$

② Surjective.

Let $h: A\to(B\to C)$. Need $f\in((A\times B)\to C)$ s.t $T(f)=h$

$T(f) = f(a,b) = h(a)(b) \Rightarrow T(f) = h$

Thus, T is bijection. $|(A\times B)\to C| = |A\to(B\to C)|$