Homework 10.

9.23 The function $f: A \to P(A)$ defined by $f(a) = \{a\}$ has the desired property. The statement is true.

9.26 (a) $f(n) = n \quad \forall n \in \mathbb{N}$.

(b) $f(n) = n+1 \quad \forall n \in \mathbb{N}$ (not onto)

(c) $f(n) = \begin{cases} n, & n \text{ even} \\ n-1, & n \text{ odd} \end{cases}$ (not 1-1)

(d) $f(n) = \begin{cases} n+1, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$ (not 1-1, not onto)

9.29. $\boxed{\subseteq}$ If $x \in C \cap D$, $x \in C$ and $x \in D$

Then $f(x) \in f(C \cap D)$, then $f(x) \in f(C)$ and $f(D)$

That is, $f(C \cap D) \subseteq f(C) \cap f(D)$

$\boxed{\supseteq}$ Assume $y \in f(C) \cap f(D)$, $y \in f(C)$ and $y \in f(D)$

Then $\exists x, z \in C, D$ s.t. $f(x) = y$ and $f(z) = y$

Since f is 1-1, $x = z \in C \cap D$.

That is, $f(C) \cap f(D) \subseteq f(C \cap D)$

Therefore, if f is 1-1, $f(C \cap D) = f(C) \cap f(D)$

9.32 $y = \frac{5x+1}{x-2} \Rightarrow xy - 2y = 5x+1 \Rightarrow (y-5)x = 2y+1$

$\Rightarrow x = \frac{2y+1}{y-5}$

If $y_1 = y_2$, then $x_1 = \frac{2y_1+1}{y_1+5} = \frac{2y_2+1}{y_2+5} = x_2$. f is injective.

$\forall y \in \mathbb{R} - \{5\}$ $\exists x = \frac{2y+1}{y-5}$. f is surjective.

$f(x)$ is bijection.

9.41.

Let $a, b \in A$ s.t. $f(a) = f(b)$. Then $a = i_A(a) = (f \circ f)(a) = f(f(a))$

$= f(f(b)) = (f \circ f)(b) = i_A(b) = b$.

Thus f is 1-1.

Let $c \in A$, Suppose that $f(c) = d \in A$.

Then $f(d) = f(f(c)) = f \circ f(c) = i_A(c) = c$.

f is onto. Therefore, f is bijection.

9.45 (a) $(g \circ f)(18, 11) = g(29, 18) = (47, 29)$.

(b) Assume $g(f(a,b)) = g(f(c,d))$, $(a,b), (c,d) \in A \times B$.

Then $g(a+b, a) = g(c+d, c)$. Then $(2a+b, a+b) = (2c+d, c+d)$

$\Rightarrow \begin{cases} 2a+b = 2c+d \\ a+b = c+d \end{cases} \Rightarrow \begin{cases} a = c \\ b = d \end{cases}$. Thus $g \circ f$ is 1-1

(c)
Let $(m, n) \in B \times B$ m, n are odd integers. Then $a = m-n \in A$

$b = 2n - m \in B$. s.t. $g(f(a,b)) = g(f(m-n, 2n-m)) = g(n, m-n)$

$= (m, n)$.

$g \circ f$ is onto. Therefore, $g \circ f$ is bijection.

9.48 $g(2x-1) = \frac{4x-1}{2\sqrt{x-x^2}}$

Let $2x-1=y$, then $x=\frac{y+1}{2}$. $g(y) = \frac{2y+1}{\sqrt{2y+2-y^2-2y-1}}$

$$= \frac{2y+1}{\sqrt{-y^2+1}} \quad /\!\!/$$

9.52

$(f\circ g^{-1})\circ(g\circ f^{-1}) = i_R$

9.53

$y = g(x) = 3x-5 \Rightarrow x = \frac{y+5}{3}$

$3! = 6$ fns $/\!\!/$

$f\circ g^{-1}(y) = f\left(\frac{y+5}{3}\right) = \frac{2y+10}{3}+1 \quad /\!\!/$

9.57.

(a) Assume $f(a)=f(b)$

Case 1 $f(a)=f(b)\geq 0$. Then $\sqrt{a-1}=\sqrt{b-1} \Rightarrow a-1=b-1$
$\Rightarrow a=b$

Case 2 $f(a)=f(b)<0$. Then $\frac{1}{a-1}=\frac{1}{b-1} \Rightarrow a=b$.

f is 1-1.

$\forall r\in\mathbb{R}$.

Case 1

$r\geq 0$. $f(r^2+1)=\sqrt{(r^2+1)-1}=r$

Case 2

$r<0$, $f\left(\frac{r+1}{r}\right)=\frac{1}{\frac{r+1}{r}-1}=r$. f is onto.

Thus f is bijection.

(b)

$$f^{-1}(x) = \begin{cases} \frac{x+1}{x}, & x<0 \\ x^2+1, & x\geq 0 \end{cases} \quad /\!\!/$$

9.81

(a) If $[a]=[b]$ in $\mathbb{Z}_{16}$, then $a=16n+b$ $n\in\mathbb{Z}$.

Then $3a = 48n+3b = 24\cdot(2n)+3b$

Then $[3a]=[3b]$ in $\mathbb{Z}_{24}$. #

(b) $\mathbb{Z}_{16} \xrightarrow{h} \mathbb{Z}_{24}$

$\mathbb{Z}_{16}$	$\mathbb{Z}_{24}$
[0]	[0]
[3]	[9]
[6]	[18]
[8]	[24]=[0]
[9]	[27]=[3]
[12]	[36]=[12]
[15]	[21]=[45]

$h(A)=\{[0],[9],[18],[3],[12],[21]\}$

$h(B)=\{[0]\}$.

(c) $\mathbb{Z}_{24} \xrightarrow{h^{-1}} \mathbb{Z}_{16}$

$\mathbb{Z}_{24}$	$\mathbb{Z}_{16}$
[0]	[0] [8]
[4]	×
[6]	[2] [10]
[8]	×
[16]	×
[18]	[6] [14]

$h^{-1}(A)=\{[0]\,[8]\,[2]\,[10]\,[6]\,[14]\}$

$h^{-1}(B)=\emptyset$.

Homework 10 Supplymentary Problems.

1.

(a) $\Phi(x,y,z) = \frac{(x,y)}{1-z} = \frac{\sqrt{x^2+y^2}}{1-z}\,\frac{(x,y)}{\sqrt{x^2+y^2}} = \frac{\sqrt{1-z^2}}{\sqrt{1-z}}\cdot\frac{(x,y)}{\sqrt{x^2+y^2}} = \underbrace{\frac{\sqrt{1+z}}{\sqrt{1-z}}}_{\text{length}}\cdot\underbrace{\frac{(x,y)}{\sqrt{x^2+y^2}}}_{\text{unit vector}}$

Φ is well-defined since domain avoid $1-z=0$

(b),(c) $(a,b)\in\mathbb{R}^2$, then $(a,b) = \underbrace{\sqrt{a^2+b^2}}_{\text{length}}\,\frac{(a,b)}{\sqrt{a^2+b^2}} = \underbrace{\frac{\sqrt{1+z}}{\sqrt{1-z}}}_{\text{length}}\,\frac{(x,y)}{\sqrt{x^2+y^2}}$

Then $\sqrt{a^2+b^2} = \frac{\sqrt{1+z}}{\sqrt{1-z}} \Rightarrow (a^2+b^2) - z(a^2+b^2) = 1+z$

$\Rightarrow \frac{a^2+b^2-1}{a^2+b^2+1} = z$

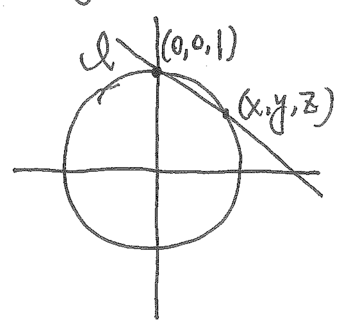
Also, $\frac{(x,y)}{1-z} = (a,b) \Rightarrow \begin{cases} x = (1-z)a = \frac{2}{a^2+b^2+1}\cdot a \\ y = (1-z)b = \frac{2}{a^2+b^2+1}\cdot b \end{cases}$

Thus, $\Phi^{-1}(a,b) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right)$

Since Φ^{-1} exists, Φ is bijective.

(d)

$$l : \begin{cases} a = 0 + tx \\ b = 0 + ty \\ c = 1 + t(z-1) \end{cases}$$



(e) $c = 0 = 1 + t(z-1) \Rightarrow t = \frac{1}{1-z}$

$$\Rightarrow \begin{cases} a = \frac{x}{1-z} \\ b = \frac{y}{1-z} \\ c = 0 \end{cases} = \frac{(x,y)}{1-z}$$

Stereographic projection between $S^n\setminus\{N\}$ and $\mathbb{R}^n$

$$\Phi(x_0,\dots x_n) = \frac{(x_0, x_1, \dots, x_{n-1})}{1-x_n}$$

2. $X \sim_A X$ since $X = X$ (Reflexive)

(a) If $X \sim_A y$ then $y \sim_A X$ since $y = \pm X$ (Symmetric)

If $X \sim_A y$, $y \sim_A Z$, then $X \sim_A Z$ since $X = \pm y = \mp Z$ (Transitive)

Take $\lambda = 1$, then $w = w \cdot 1$, we have $w \sim_B w$ (Reflexive)

If $Z \sim_B W$, then $Z = \lambda W$, since $\lambda \neq 0$.

$W = \frac{1}{\lambda} Z$. we have $W \sim_B Z$. (Symmetric)

If $Z \sim_B W$ and $W \sim_B V$, then $\exists \lambda_1, \lambda_2 \neq 0$ s.t.

$Z = \lambda_1 W = \lambda_1 \lambda_2 V$. then $Z \sim_B V$ (Transitive).

Therefore, $\sim_A$, $\sim_B$ are both equivalence relations.

(b) Let $X = (X_0 \dots X_n) \in A$, $X' = (X_0', \dots X_n') \in A$, and $F([X]_A) = [X]_B$

If $F([X]_A) = F([X']_A)$, then $[X]_B = [X']_B$ which means $X = \lambda X'$, $\lambda \neq 0$

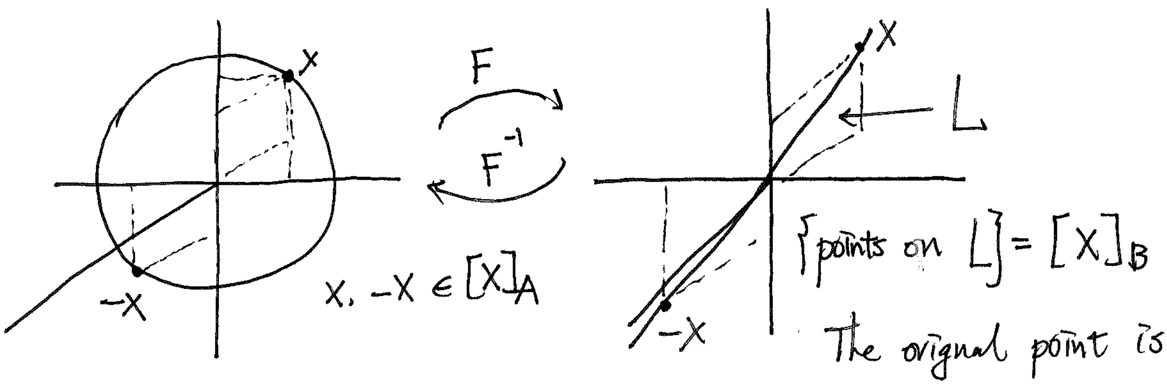
And also since $X, X' \in A = S^n$, leads two possible value of λ, $\lambda = \pm 1$

Then $[X]_A = [X']_A$ Then F is injection.

For any $w \in B$ $F^{-1}\left([(W_0, W_1 \dots W_n)]_B\right) = \left[\frac{(W_0, W_1 \dots, W_n)}{\sqrt{W_0^2 + W_1^2 \dots + W_n^2}}\right]_A$

Then F is surjective. Then $F: A/{\sim_A} \to B/{\sim_B}$ is the correspondence function we desired.

Note: Take $n=3$, then the figure would be:



The orignal point is not in line.