系所組別： 數學系應用數學
考試科目 高等微積分

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Instructions:
R: the set of all real numbers.
N: the set of all positive integers.
You can use any common notations, such as $\overline{\lim}_{n\to\infty}$ or $\limsup_{n\to\infty}$, $\sup S$, etc.

1. (20%)

(a) Find the *limit inferior* and *limit superior* of the sequences $\{a_n\}$ and $\{b_n\}$, where $a_n = \dfrac{1-3(-1)^n n}{4n+2}$ and $b_n = [1+(-1)^n]\sin\dfrac{n\pi}{4}$, for $n \in \mathbf{N}$.

(b) Find the *infimum* and *supremum* of the sets of real numbers S and T, where $S = \{x > 0 : 3x^2 - 8x - 3 \le 0\}$ and $T = \{\dfrac{\sin x}{x} : 0 < x \le \dfrac{\pi}{2}\}$.

2. (15%) A sequence $\{a_n\}$ is called *contractive* if there exists $r, 0 < r < 1$, such that $|a_{n+1} - a_n| \le r|a_n - a_{n-1}|$, for all $n \ge 2$. Let $\{b_n\}$ be a sequence satisfying

$$b_{n+1} = \frac{1}{3}(b_n^2 + 1), \text{ for } n \ge 1,$$

and $0 < b_1 < 1$.

(a) Prove that the sequence $\{b_n\}$ is *contractive.*

(b) Show that $\lim_{n\to\infty} b_n$ exists (this is difficult) and find the limit b (this is easy).

(c) Take $b_1 = \frac{1}{2}$. Determine the minimum n such that $|b - b_n| < 10^{-3}$.

3. (14%) Prove or disprove that the following functions are uniformly continuous via $\varepsilon - \delta$ argument.

(a) $f(x) = x^2$, on **R**,

(b) $g(x) = \frac{1}{x}$, on $[\frac{1}{2}, 1]$.

4. (14%) Prove or disprove that the following sequences of functions are uniformly convergent.

(a) For $n \in \mathbf{N}, f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [2, \infty)$.

(b) For $n \in \mathbf{N}, g_n(x)$ are defined on $[0, 1]$ by

$$g_n(x) = \begin{cases} n^2 x, & 0 < x < 1/n \\ 2n - n^2 x, & 1/n \le x < 2/n \\ 0, & 2/n \le x < 1. \end{cases}$$

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5. (15%)

   (a) Evaluate the integrals $\int_0^\infty \int_0^1 e^{-xy} - xye^{-xy}\, dy\, dx$
   and $\int_0^1 \int_0^\infty e^{-xy} - xye^{-xy}\, dx\, dy$.

   (b) Why are the above two integrals different?

6. (10%)

   (a) Let $\mathbf{f} : S \to \mathbf{R}^m$ be a function defined on an open set $S \subset \mathbf{R}^n$ with vector-values in $\mathbf{R}^m$. What is the definition that the function $\mathbf{f}$ is differentiable at $\mathbf{c} \in S$ and denote the total derivative by $D\mathbf{f}(\mathbf{c})$.

   (b) Suppose $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by

   $$\mathbf{f}(x, y) = (\sin x \cos y, \sin(x+y), \cos xy).$$

   Determine the total derivative $D\mathbf{f}(x, y)$.

7. (12%) Let $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s},\ 0 < a \leq 1, s > 1$.

   (a) Show that the series converges absolutely and prove that

   $$\sum_{h=1}^{k} \zeta(s, \frac{h}{k}) = k^s \zeta(s), \quad \text{if } k = 1, 2, \ldots,$$

   where $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function.

   (b) For $s > 1$, prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$.