1. If A is any subset of $\mathbb{R}^p$, let A° be the interior of A and A^- be the closure of of A. Suppose $B \subset \mathbb{R}^p$. Prove or disprove the following statements:
   (a) (5 points) A and A^- have the same interiors.
   (b) (5 points) A and A° have the same closure.
   (c) (5 points) $(A \cap B)^\circ = A^\circ \cap B^\circ$.
   (d) (5 points) $(A \cap B)^- = A^- \cap B^-$.
   (e) (5 points) A° can be written as the union of a countable collection of open balls.

2. Let $\{f_n\}$ be a sequence of continuous functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that $\{f_n\}$ converges uniformly to f on D, and let $\{x_n\} \subset \mathbb{R}^p$ be a sequence converges to $x \in D$.
   (a) (8 points) Show that $\{f_n(x_n)\}$ converges to $f(x)$.
   (b) (7 points) Let $p = q = 1$ and $f_n(x) = x^n$, $x_n = 1 - \frac{1}{n}$, $D = [0, 1]$. Show that $\{f_n(x_n)\}$ does not converge to $f(x)$. Why?

3. (a) (10 points) Show that the maximum of $f(x_1, \ldots, x_p) = (x_1 x_2 \cdots x_p)^2$ subject to the constraint $x_1^2 + \cdots + x_p^2 = 1$ is equal to $1/p^p$.
   (b) (5 points) Use (a) to prove that

$$|y_1 y_2 \cdots y_p| \leq \frac{\|y\|^p}{p^{p/2}}, \quad y \in \mathbb{R}^p.$$

4. Let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by

$$F(u, v, w, x, y) = (uy + vx + w + x^2, uvw + x + y),$$

   and $P = (2, 1, 0, 2, -3)$.
   (a) (5 points) Show that F is not a one-one map around a neighborhood of P.
   (b) (5 points) Let $(x, y) = \varphi(u, v, w)$ satisfy the equation $F(u, v, w, x, y) = (0, -1)$ near the point P. Compute $D\varphi(2, 1, 0)$.

5. (10 points) If $z = e^x \cos y$, while x and y are implicit functions of t defined by the equations

$$x^3 + e^x - t^2 - t = 1, \quad yt^2 + y^2 t - t + y = 0,$$

   then compute $\frac{dz}{dt}$ at $t = 0$.

（背面仍有題目,請繼續作答）

本試題是否可以使用計算機： □可使用 ， ☑不可使用 （請命題老師勾選）

6. (10 points) Consider the infinite integral with parameter t

$$\int_0^\infty e^{-tx}\frac{\sin x}{x}dx, \quad \text{for } t \geq 0,$$

where we interpret the integrand to be 1 for $x = 0$. Show that for each $t \geq 0$ the infinite integral converges and the convergence is uniform.

7. Let $(u, v) = (e^x \cos y, e^x \sin y)$ be a transformation mapping points in the xy-plane to points in the uv-plane and $R_{xy} = \{0 \leq x \leq 1, 0 \leq y \leq \frac{\pi}{2}\}$.

   (a) (5 points) Show that the transformation defines a one-to-one mapping of the rectangle R_{xy} onto a region R_{uv} of the uv-plane. Draw the picture of R_{uv}.

   (b) (10 points) Express (but do NOT integrate) the double integral

$$\int\int_{R_{xy}} \frac{e^{2x}}{1 + e^{4x}\cos^2 y \sin^2 y} dxdy$$

   as an iterated integral in u, v.